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GROUPS ADMITTING A FIXED-POINT-FREE AUTOMORPHISM OF ORDER 2^n

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Let G be a finite solvable group which admits a fixed-point-free automorphism of order 2^n. The main result of this paper is that the nilpotent length of G is at most $2n-2$ for $n \geqq 2$. This is an improvement on earlier results in that no assumptions are made regarding the Sylow subgroups of G.

Suppose G is a finite solvable group which admits a fixed-point-free automorphism of order p^n where p is a prime. Then it is known that the nilpotent length of G is at most n provided that $p \neq 2$ ([8], [10], [6]). This result also holds for $p = 2$ if the Sylow q-subgroups of G are abelian for all Mersenne primes q ([8], [10]). The purpose of the present paper is to obtain an upper bound on the nilpotent length in the case $p = 2$ without imposing any restrictions on the Sylow subgroups of G. Our result is

THEOREM 1.1. *If G is a finite group admitting a fixed-point-free automorphism of order 2^n, then G is solvable and has nilpotent length at most* $\mathrm{Max}\,\{2n-2, n\}$.

Here it should be noted that if G admits a 2-group as a fixed-point-free operator group then G must have odd order and thus must be solvable from [2].

The usual methods employed to prove results about solvable groups admitting a fixed-point-free automorphism of order p^n are so similar to the methods used by Hall and Higman [7] to find upper bounds on the p-length that it seems natural to ask whether both types of results might follow from some general theorem about linear groups. If $p = 2$ this can be done and the theorem is the following:

THEOREM 1.2. *Let G be a finite solvable linear group over a field K such that the order of $F_1(G)$ is divisible by neither 2 nor the characteristic of K. Assume that g is an element of order 2^n in G such that the minimal polynomial of g has degree $< 2^n$. Then g^{2n-1} must belong to $F_2(G)$.*

Here $F_1(G)$ is the greatest normal nilpotent subgroup of G and $F_2(G) = F_1(G \bmod F_1(G))$. In addition to implying Theorem 1.1, Theorem 1.2 also immediately implies Theorem B of [4] which in turn implies that $l_2(G) \leqq \mathrm{Max}\,\{2e_2(G)-2, e_2(G)\}$ for any solvable group G ([4], [5]).

2. Preliminary results. For the rest of this paper we adopt the convention that all groups referred to are assumed to be finite. If G is a linear group operating on V and U is a G-invariant subspace, then $\{G \mid U\}$ denotes the restriction of G to U. If g is an element of a linear group such that the minimal polynomial of g has degree less than the order of g, then g is said to be exceptional. The rest of the notation used agrees with that of [2].

Before proceeding to the proof of Theorem 1.2, some preliminary results are needed.

LEMMA 2.1. *Let Q be an extra-special q-group which is operated upon by an automorphism g of order p^n where p is a prime distinct from q. Assume that $[Q', g] = 1$ and let K be an algebraically closed field of characteristic different from q. Then, if M is any irreducible $K - Q\langle g\rangle$ module which represents Q faithfully, it follows that M is an irreducible $K - Q$ module.*

This follows from either [1, Th. 1.30] or [7, Lemma 2.2.3] depending on whether the characteristic of K differs from or is equal to p, respectively. Next we need a generalization of Theorem 2.5.4 of [7].

THEOREM 2.2. *Suppose that*

(i) *Q is an extra-special q-group which admits an automorphism g of order p^n where p is a prime distinct from q.*

(ii) $[Q', g] = 1$.

(iii) *K is a field of characteristic different from q.*

(iv) *M is a faithful, irreducible $K - Q\langle g\rangle$ module.*

(v) *g is exceptional on M.*

Then the following must hold:

(a) $p^n - 1 = q^d$.

(b) *If Q_1/Q' is a subgroup of Q/Q' that is transformed faithfully and irreducibly by $\langle g\rangle$, then $|Q_1/Q'| = q^{2d}$ and $[Q, g] \leqq Q_1$.*

(c) *The minimal polynomial of g on M has degree $p^n - 1$.*

Proof. First we show that K may be taken to be algebraically closed. Let L be an algebraically closed extension of K and let N be an irreducible $L - Q\langle g\rangle$ submodule of $M \otimes_K L$. Now if c generates Q', then, since $c \in Z(Q\langle g\rangle)$, c has no nonzero fixed vectors in M. It immediately follows from this that c is not the identity on N. Since any nontrivial normal subgroup of $Q\langle g\rangle$ must contain c, this implies that N is a faithful $L - Q\langle g\rangle$ module.

Thus in proving the theorem we may as well assume that K is algebraically closed. The lemma now implies that M is an irreducible $K - Q$ module. If char $(K) = p$, then the theorem follows from

Theorems 2.5.1. and 2.5.4 of [7]. Hence we now suppose that char $(K) \neq p$.

Q/Q' is the direct product of groups transformed irreducibly by g. Thus there is a subgroup Q_1/Q' such that g transforms Q_1/Q' irreducibly according to some automorphism of order p^n. Now if Q_1 were abelian, then, since $g^{p^{n-1}}$ does not centralize Q_1 and M is a completely reducible $K - Q_1$ module, it would follow easily that the minimal polynomial of g would have degree p^n. Hence Q_1 is not abelian and so must be extra-special. This implies that $|Q_1| = q^{2d+1}$ for some d.

Now if N is an irreducible $K - Q_1\langle g\rangle$ submodule of M, N must faithfully represent Q_1 since c is represented by a scalar matrix. Hence N is an irreducible $K - Q_1$ module.

Since g is exceptional, there is at least one p^n-th root of unity in K which is not an eigenvalue of g. The argument given in [10, pp. 706–707] now implies that $p^n - 1 = q^d$ and exactly $(p^n - 1)$ p^n-th roots of unity occur as eigenvalues of g. Thus it only remains to show that $[Q, g] \leqq Q_1$ to complete the proof of the theorem. If $Q_1 = Q$, this is trivial. Therefore assume that $Q \neq Q_1$. Then if $Q_2 = C_Q(Q_1)$ we find that Q_2 admits g and Q is the central product of Q_1 and Q_2.

We now use the construction given in [7, p. 21] to construct linear groups H_1, H_2 where $H_i = Q_i\langle g_i\rangle$ and g_i is a p-element which transforms Q_i in the same way as g. In the Kronecker product of H_1 and H_2, the product of Q_1 and Q_2 becomes identified with Q. Since M is an irreducible $K - Q$ module, it follows that $g_1 \otimes g_2$ differs from g only by a scalar factor. Since g is of order p^n, we find that

$$g = \alpha(g_1 \otimes g_2)$$

where $\alpha^{p^n} = 1$. Now if $[Q_2, g] \neq 1$, then g_2 has at least two distinct eigenvalues β, γ. But g_1 has $p^n - 1$ distinct eigenvalues. Thus if λ is any p^n-th root of unity then at least one of $\lambda/\alpha\beta$ and $\lambda/\alpha\gamma$ must be an eigenvalue of g_1. But this would imply that λ would be an eigenvalue of g. Since g is exceptional, we must have that $[Q_2, g] = 1$.

COROLLARY 2.3. *Under the hypothesis of the theorem let V be Q/Q' written additively and consider V as a $GF(q) - \langle g\rangle$ module. Then the minimal polynomial of g on V is of degree at most $2d + 1$.*

Proof. This follows immediately from (b).

THEOREM 2.4. *Let $G = PQ$ be a linear group over a field K where Q is a q-group normal in G $(q \neq 2)$ and P is cyclic of order $2^n > 2$ generated by an element g such that $[Q, g^{2^{n-1}}] \neq 1$. Assume that char $(K) \neq q$ and that the minimal polynomial of g is of degree at most 3. Then we must have $q = 3$ and $n = 2$.*

Proof. Extending K affects neither hypothesis nor conclusion so we may as well assume that K is algebraically closed. Now let S be a subgroup of Q which is minimal with respect to being normalized by g but not centralized by h where $h = g^{2^{n-1}}$. Then S is a special q-group.

If V is the space on which G operates, then $V = V_1 \oplus V_2 \oplus \cdots$ where the V_i are the homogeneous $K - S$ submodules of V. Without loss of generality we may assume that $[S, h]$ is not the identity on V_1. But if g^{2^m} is the first power of g fixing V_1, then the minimal polynomial of g has degree at least 2^m times the degree of the minimal polynomial of $\{g^{2^m} \mid V_1\}$. This implies that g must fix V_1

Now let U be an irreducible $K - PS$ submodule of V_1. $[S, h]$ is not the identity on U but $Z\{S \mid U\}$ must be cyclic generated by a scalar matrix. Thus we conclude that $\{S \mid U\}$ is an extra special q-group whose center is centralized by $\{g \mid U\}$. From Theorem 2.2 we now obtain that $2^n = q^d + 1$ and the minimal polynomial of $\{g \mid U\}$ has degree $2^n - 1$. This implies that $n = 2$ and $q = 3$.

3. **Proof of Theorem 1.2.** Neither the hypothesis nor the conclusion of the theorem is affected by extending the field K. Thus we may assume without loss of generality that K is algebraically closed. Now if $n = 1$, then, since g is exceptional, g would have to be a scalar matrix which would imply that $g \in Z(G)$. Hence we assume that $n > 1$ and let $h = g^{2^{n-2}}$.

If Q is any normal nilpotent subgroup of G, then char $(K) \nmid |Q|$ and so V, the space on which G operates, is a completely reducible $K - Q$ module. Therefore $V = V_1 \oplus V_2 \oplus \cdots$ where the V_i are the homogeneous $K - Q$ submodules. G must permute the V_i since $Q \lhd G$. Now if h^2 did not fix each V_i, then it would follow that the minimal polynomial of g would be of degree 2^n which is a contradiction. Let H be the set of all elements in G which fix each minimal characteristic $K - Q$ submodule of V for each normal nilpotent subgroup Q in G. Clearly $H \lhd G$. Hence $F_i(H) \leqq F_i(G)$ for $i = 1, 2$. Also we have shown that $h^2 \in H$.

It follows from [4, Lemmas 3.2 and 3.3] that $[Q, H] = 1$ if Q is any normal abelian subgroup of G and that $F_1(H)$ is of class 2. $F_1(H) = Q_1 \times Q_2 \times \cdots$ where Q_i is the Sylow q_i-subgroup of $F_1(H)$ and q_i is an odd prime. Since Q_i is of class at most 2, Q_i is a regular q_i-group. Then the elements of order at most q_i form a subgroup R_i in Q_i. If $R = R_1 \times R_2 \times \cdots$, then $C_H(R) \leqq F_1(H)$ [9, Hilfssatz 1.5].

The proof now divides into two parts. First we will show that h^2 induces the identity automorphism on any $2'$-subgroup of $F_2(H)/F_1(H)$. In the second part we consider how h^2 operates on a 2-subgroup of $F_2(H)/F_1(H)$.

Part I. Suppose that p is an odd prime which divides

$$|F_2(H)/F_1(H)| .$$

It is easy to show that there is a Sylow p-subgroup P of $F_2(H)$ which is normalized by g. We now proceed to prove that

$$[P, h^2] \leqq F_1(H) .$$

To do this we first note that, since $P \nleqq F_1(H)$, $C_P(O_{P'}(F_1(H))) = P \cap F_1(H)$. Now let $N = P \cap F_1(H)$ and suppose that $[P, h^2] \nleqq N$.

Since $C_P(O_{P'}(F_1(H))) = N$, there is a $q_i \neq p$ such that $[h^2, P, R_i] \neq 1$. Now let U be a minimal characteristic $K - R_i$ submodule of V on which $[h^2, P, R_i]$ is not the identity. Let $q = q_i$, $S = \{P \mid U\}$, and $Q = \{R_j \mid U\}$. h^2 must fix U but cannot be a scalar matrix on U since $\{[h^2, P, R_i] \mid U\} \neq 1$. Let $g^{2^{n-m}}$ be the first power of g to fix U and let g_1 be the restriction of $g^{2^{n-m}}$ to U. But if g_1 were not exceptional then g could not be exceptional. Hence g_1 is exceptional and so m must be > 1. Now let $h_1 = g_1^{2^{m-2}}$.

Then $[h_1^2, S, Q] \neq 1$. Since U is the sum of isomorphic, irreducible $K - Q$ modules, $Z(Q)$ must be cyclic generated by a scalar matrix. Therefore $[Z(Q), S\langle g_1\rangle] = 1$ and, since Q is a homomorphic image of a class 2 group of exponent q, Q must be an extra-special q-group.

Next let U_1 be an irreducible $K - Q\langle g_1\rangle$ submodule of U. Lemma 2.1 implies that U_1 is an irreducible $K - Q$ module and so U is the sum of $K - Q$ modules isomorphic to U_1. From Theorem 2.2 we obtain that $2^m - 1 = q^d$ and $[Q: C_Q(g_1)] = q^{2d}$. Then q must be a Mersenne prime and $d = 1$.

Now let W be Q/Q' written additively and consider W as a $GF(q) - S\langle g_1\rangle$ module. The minimal polynomial of g_1 on W has degree at most 3 from Corollary 2.3. Since $[h_1^2, S]$ is not the identity on W, Theorem 2.4 now implies that $m = 2$ and $p = 3$ which contradicts

$$p \neq q = 2^m - 1 .$$

Thus we have shown that h^2 induces the identity automorphism on any $2'$-subgroup of $F_2(H)/F_1(H)$.

Part II. The 2-subgroups of $F_2(H)/F_1(H)$ have to be handled differently and we apply the method of [4, pp. 1224–1228]. Accordingly, let $V = V_{i1} \oplus V_{i2} \oplus \cdots$ where the V_{ij} are the homogeneous $K - R_i$ submodules of V. For each i and j, let

$$C_{ij} = \{x \mid x \in H \text{ and } \{[R_i\, x] \mid V_{ij}\} = 1\} .$$

Next let H_1 be the intersection of all the C_{ij} which contain h^2. If h^2 belongs to no C_{ij} then set H_1 equal to H. In any event $H_1 \lhd H$,

$h^2 \in H_1$, and g normalizes H_1.

Now choose P to be a Sylow 2-subgroup of $F_2(H_1)$ such that $P\langle g\rangle$ is a 2-group. If $x \in P$, we now assert that $[h^2, x] = [h, x]^2$. The proof of this is identical with the proof of Lemma 3.4 in [4] and, for this reason, is omitted.

Now from the above we see that $[h^2, P] \leqq D(P)$. This combined with our result proved in Part I implies that $[h^2, F_2(H_1)] \leqq D(F_2(H_1)$ mod $F_1(H_1))$. But this implies that $h^2 \in F_2(H_1)$. Since $F_2(H_1) \leqq F_2(H)$ and $F_2(H) \leqq F_2(G)$, this completes the proof of the theorem.

4. **Proof of Theorem 1.1.** Let σ denote the fixed-point-free automorphism of order 2^n. If $n \leqq 2$, then the result is a known one [3]. Consequently, we assume that $n \geqq 3$ and proceed by induction on the order of G.

Now if G has two distinct minimal σ-admissible normal subgroups H_1 and H_2, then by induction, $(G/H_1) \times (G/H_2)$ has nilpotent length at most $2n - 2$. Since G is isomorphic to a subgroup of $(G/H_1) \times (G/H_2)$, the theorem would follow immediately.

Therefore we may assume that G has a unique minimal σ-admissible normal subgroup. This implies that $F_1(G)$ is a p-group for some p. Then we may consider $H = \langle\sigma\rangle G/F_1(G)$ as a linear group operating on V where V is $F_1(G)/D(F_1(G))$ written additively. Now p cannot be 2 and $(\sigma - 1)$ must be nonsingular on V. Thus σ must be exceptional and we obtain from Theorem 1.2 that $\sigma^{2^{n-1}} \in F_2(H)$.

This implies that $\sigma^{2^{n-1}}$ centralizes $F_3(G)/F_2(G)$ which in turn implies that $\sigma^{2^{n-1}}$ centralizes $G/F_2(G)$ [8, Lemma 4]. Thus, by induction, the nilpotent length of $G/F_2(G)$ is at most Max $\{2n - 4, n - 1\}$. Since we are assuming that $n \geqq 3$, this implies that G has nilpotent length at most $2n - 2$.

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