PACIFIC JOURNAL OF MATHEMATICS
Vol. 26, No. 2, 1968

FRACTIONAL INTEGRATION AND INVERSION FORMULAE ASSOCIATED WITH THE GENERALIZED WHITTAKER TRANSFORM

H. M. SRIVASTAVA

In the present note, we invoke the theories of the Mellin transform as well as fractional integration to investigate a solution of the integral equation

$$(*)\qquad \int_0^\infty (xt)^{\sigma-(1/2)}e^{-(1/2)xt}W_{k+(1/2),m}(xt)f(t)dt = W_{k,m}^{(\sigma)}\{f: x\}\ ,\qquad x > 0\ ,$$

which defines a generalized Whittaker transform of the unknown function $f \in L_2(0, \infty)$ to be determined in terms of its image $W_{k,m}^{(\sigma)}\{f: x\}$.

It is shown that under certain constraints **$(*)$ can be reduced to the form of a Laplace integral which is readily solvable by familiar techniques.**

Two well-known generalizations of the classical Laplace transform (cf., e.g., [12])

$$(1)\qquad L[f: x] = \int_0^\infty e^{-xt}f(t)dt\ ,\qquad x > 0\ ,$$

are due to Meijer [6] and Varma [11]. The object of the present note is to investigate a solution of the integral equation

$$(2)\qquad \int_0^\infty (xt)^{\sigma-(1/2)}e^{-(1/2)xt}W_{k+(1/2),m}(xt)f(t)dt = W_{k,m}^{(\sigma)}\{f: x\}\ ,\qquad x > 0\ ,$$

which defines a generalized Whittaker transform [5, p. 23] of the unknown function $f(t) \in L_2(0, \infty)$ to be determined in terms of its image $W_{k,m}^{(\sigma)}\{f(t): x\}$, so that by appropriately specializing the parameter σ our results would readily enable us to invert the integral transforms of Meijer (cf., [1], [7]) and Varma (cf., [8], [9]).

In what follows we shall make a free use of the existing theories of (i) fractional integration due to Kober [4] and Erdélyi [2], and (ii) the Mellin transform detailed in [10, p. 94]. In the familiar notation, the operator of fractional integration that we need in our analysis is defined as follows:—

$$(3)\qquad K_{\zeta,\alpha,n}^{(-)}f(x) = \frac{n}{\Gamma(\alpha)}x^{\zeta}\int_x^\infty (u^n - x^n)^{\alpha-1}u^{-\zeta-n\alpha+n-1}f(u)du\ ,$$

where $f \in L_p(0, \infty)$, $p^{-1} + q^{-1} = 1$, if $1 < p < \infty$, and q^{-1} or $p^{-1} = 0$

according as p or $q = 1; \alpha > 0, n > 0, \zeta > -p^{-1}$.

Confining ourselves to the L_2-space theory, for simplicity of the conditions involved, and invoking Fox's lemma (see [3], p. 458), we can establish the following theorems in the usual manner.

THEOREM 1. *Let $f \in L_2(0, \infty)$ be a solution of the integral equation* (2). *Then*

$$f(x) = L^{-1}[K^{(-)}_{\sigma-k,\sigma+k,1} W^{(\sigma)}_{k,m}\{f: x\}] , \tag{4}$$

provided (i) $x > 0$, (ii) $\sigma + k \geqq 0$ *and* (iii) $1/2 + \sigma - k > 0$.

THEOREM 2. *Let $f(x)$ be a solution of* (1) *that belongs to* $L_2(0, \infty)$. *Then*

$$K^{(-)}_{\sigma+m,\alpha,1} x^{\sigma-m} L[t^{\sigma-m} f(t): x] = W^{(\sigma)}_{-m-\alpha,m}\{f: x\} , \tag{5}$$

provided (i) $x > 0$, (ii) $\alpha \geqq 0$ *and* (iii) $1/2 + \sigma + m > 0$.

It may be of interest to remark here that when $\alpha = -k - m$, (5) is reduced to the interesting relationship

$$W^{(\sigma)}_{k,m}\{f: x\} = K^{(-)}_{\sigma+m,-k-m,1} x^{\sigma-m} L[t^{\sigma-m} f(t): x] , \tag{6}$$

which leads us to the construction of a table of generalized Whittaker transforms from that of the classical Laplace transform, provided $k + m \leqq 0, 1/2 + \sigma + m > 0$ and $x > 0$.

The author wishes to thank Professor Richard F. Arens and the referee for helpful suggestions.

REFERENCES

1. A. Erdélyi, *On a generalization of the Laplace transformation*, Proc. Edinburgh Math. Soc. (2) **10** (1951), 53-55.
2. ———, *On some functional transformations*, Rend. Sem. Mat. Univ. e Politec. Torino **10** (1950-51), 217-234.
3. C. Fox, *An inversion formula for the kernel $K_\nu(x)$*, Proc. Cambridge Philos. Soc. **61** (1965), 457-467.
4. H. Kober, *On fractional integrals and derivatives*, Quart. J. Math. (Oxford Ser.) **11** (1940), 193-211.
5. V. P. Mainra, *A new generalization of the Laplace transform*, Bull. Calcutta Math. Soc. **53** (1961), 23-31.
6. C. S. Meijer, *Eine neue Erweiterung der Laplace-Transformation*, Proc. Kon. Nederl. Akad. Wetensch. **44** (1941), 727-737 and 831-839.
7. K. M. Saksena, *Inversion and representation theorems for a generalization of Laplace transformation*, Nieuw Arch. v Wisk. (3) **6** (1958), 1-9.
8. ———, *Inversion and representation theorems for a generalized Laplace integral*, Pacific J. Math. **8** (1958), 597-607.

9. R. K. Saxena, *An inversion formula for the Varma transform*, Proc. Cambridge Philos. Soc. **62** (1966), 467-471.
10. E. C. Titchmarsh, *An Introduction to the Theory of Fourier Integrals*, The Oxford University Press, London, 1937.
11. R. S. Varma, *On a generalization of Laplace integral*, Proc. Nat. Acad. Sci. India (A) **20** (1951), 209-216.
12. D. V. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton, 1941.

Received November 27, 1967.

WEST VIRGINIA UNIVERSITY
MORGANTOWN, WEST VIRGINIA