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SIMPLE MODULES AND HEREDITARY RINGS

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The purpose of this note is to prove that if in a semi-primary ring Λ, every simple module that is not a projective Λ-module is an injective Λ-module, then Λ is a semi-primary hereditary ring with radical of square zero. In particular, if Λ is a commutative ring, then Λ is a finite direct sum of fields. If Λ is a commutative Noetherian ring then if every simple module that is not a projective module, is an injective module, then for every maximal ideal M in Λ we obtain $\mathrm{Ext}^1(\Lambda/M, \Lambda/M) = 0$. **The technique of localization now implies that** $\mathrm{gl.dim}\,\Lambda = 0$.

1. We say that Λ is a semi-primary ring if its Jacobson radical N is a nilpotent ideal, and $\Gamma = \Lambda/N$ is a semi-simple Artinian ring.

Throughout this note all modules (ideals) are presumed to be left modules (ideals) unless otherwise stated. For any idempotent e in Λ we denote by Ne the ideal $N \cap \Lambda e$.

We discuss first semi-primary rings Λ with radical N of square zero for which every simple module that is not a projective module is an injective module. We shall study the nonsemi-simple case, i.e., $N \neq 0$.

Under this assumption N becomes naturally a Γ-module.

Let e, e' be primitive idempotents in Λ for which $eNe' \neq 0$. In particular $Ne' \neq 0$. From the exact sequence $0 \rightarrow Ne' \rightarrow \Lambda e' \rightarrow S' \rightarrow 0$, it follows that S' is not a projective module since $\Lambda e'$ is indecomposable. Since S' is a simple module it follows that S' is an injective module.

Next consider the simple module $\Lambda e/Ne = S$. Since $eNe' \neq 0$, since Ne' is a Γ-module, and since on N the Γ-module structure and the Λ-module structure coincide, Ne' contains a direct summand isomorphic with S. This gives rise to an exact sequence $0 \rightarrow S \rightarrow \Lambda e' \rightarrow K \rightarrow 0$ with $K \neq 0$. If S were injective this sequence would split, and this contradicts the indecomposability of $\Lambda e'$. Therefore S is a projective module.

Hence Ne' is a direct sum of projective modules, therefore Ne' is a projective module. The exact sequence $0 \rightarrow Ne' \rightarrow \Lambda e' \rightarrow S' \rightarrow 0$ now implies $l.\mathrm{p.dim}\, S' \leqq 1$, and since S' is not a projective module, then $l.\mathrm{p.dim}\, S' = 1$.

Hence $l.\mathrm{p.dim}_\Lambda \Gamma = 1$, and this implies that Λ is an hereditary ring (i.e., $l.\mathrm{gl.dim}\, \Lambda = 1$) [1].

Conversely, assume that $l.\mathrm{gl.dim}\, \Lambda = 1$. Every ideal in Λ is the direct sum of $N_1, \cdots, N_t$ where N_1 is contained in the radical, and the others (if any) are components of Λ, i.e., $N_i = \Lambda e_i$ where $e_2, \cdots, e_t$ are primitive orthogonal idempotents in Λ [4].

Let $\Gamma e'$ be any simple Λ-module. Since $N_1 \subset N$, N_1 is a Γ-module. Since on N the Γ-module structure coincides with the Λ-module structure, it easily follows that there exists a nonzero map of N_1 onto $\Gamma e'$ if and only if $\Gamma e'$ (up to isomorphism) is a direct summand of N_1. This in particular implies that $\Gamma e'$ is a projective Λ-module, since then $\Gamma e'$ is isomorphic to an ideal. If $\Gamma e'$ is not a projective Λ-module, it follows that $\mathrm{Hom}_\Lambda(N_1, \Gamma e') = 0$. As a consequence, every map from an ideal in Λ into $\Gamma e'$, extends to a map of Λ into $\Gamma e'$, hence $\Gamma e'$ is an injective Λ-module.

This proves:

THEOREM A. *Let Λ be a semi-primary ring with radical of square zero. Then every simple Λ-module that is not a projective Λ-module is an injective Λ-module if and only if Λ is a hereditary ring.*

If Λ is a semi-primary ring with radical N and $N^2 \neq 0$, then a simple module is projective if and only if it is isomorphic to a component, hence if $\Lambda e/Ne$ is a projective module $Ne = 0$, and the idempotent e, when reduced mod N^2 (i.e., in Λ/N^2) will still give rise to a projective module. If $\Lambda e/Ne$ is an injective module e will give rise to an injective Λ/N^2-module. This will follow from the following two lemmas:

LEMMA 1. *Let e, e' be primitive idempotents in Λ. Then Λe is isomorphic to $\Lambda e'$ if and only if* $\mathrm{Hom}_\Lambda(\Lambda e', \Lambda e/Ne) \neq 0$.

Proof. If Λe is isomorphic to $\Lambda e'$ then obviously

$$\mathrm{Hom}_\Lambda(\Lambda e', \Lambda e/Ne) \neq 0 .$$

Conversely, let $f: \Lambda e' \to \Lambda e/Ne$ be a nonzero map. Since $\Lambda e/Ne$ is a simple module f is an epimorphism. Denote by π the canonical projection $\pi: \Lambda e \to \Lambda e/Ne$ then since $\Lambda e'$ is a projective module there exists a map $g: \Lambda e' \to \Lambda e$ such that $f = \pi \circ g$. Since $\pi(Ne) = 0$, it follows that g is an epimorphism. Since Λe is a projective module and $\Lambda e'$ an indecomposable module g is an isomorphism.

LEMMA 2. *Let S be an injective simple Λ-module and I an ideal that is contained in the radical. Then* $\mathrm{Hom}_\Lambda(I, S) = 0$.

Proof. Let f be a nonzero map of I into S. Since S is an injective Λ module it follows that f extends to a map of Λ onto S, $f: \Lambda \to S$, but this implies that $f(N) = 0$. Since $f(I) \subset f(N)$ this is a contradiction. Therefore every map of I into S is the zero map.

THEOREM B. *Let Λ be a semi-primary ring then the following are equivalent:*

(i) *Λ is an hereditary ring with radical of square zero.*

(ii) *Every simple module that is not a projective Λ-module is an injective Λ-module.*

Proof. That (i) implies (ii) follows from Theorem A.

(ii) $\Rightarrow$ (i): Let $e_1, \cdots, e_t$ be a complete set of orthogonal idempotents, i.e., each e_i is a primitive idempotent, and

$$\Lambda = \Lambda e_1 + \cdots + \Lambda e_t .$$

Set $S_i = \Lambda e_i / N e_i$. We denote by $\bar{e}_1, \cdots, \bar{e}_t$ the images of $e_1, \cdots, e_t$ in Λ/N^2 under the canonical epimorphism $\Lambda \to \Lambda/N^2$. Then $S_1, \cdots, S_t$ may be viewed as simple Λ/N^2-modules, and every simple Λ/N^2-module is necessarily isomorphic with some S_i. If S_j is Λ-projective then $Ne_j = 0$, and necessarily S_j is Λ/N^2-projective. If S_j is Λ-injective then we claim that S_j is Λ/N^2-injective. It suffices to prove that for any ideal I' in Λ/N^2, and any Λ/N^2-map f from I' to S_j, f extends to a map of Λ/N^2 into S_j. Since I' is a direct sum of ideals $I_1, \cdots, I'_r$, $I'_1 \subset N/N^2$ and the others (if any) are components of Λ/N^2, we will be done if we prove that $\mathrm{Hom}_{\Lambda/N^2}(I'', S_j) = 0$ whenever $I'' \subset N/N^2$. Let I be the inverse image of I'' under the homomorphism $\Lambda \to \Lambda/N^2$, then $\mathrm{Hom}_\Lambda(I, S_j) = 0$ since $I \subset N$ (Lemma 2). If we denote by h the map $I \to I''$ (restriction of the canonical projection) and if f is any map of I'' into S_j then if f is not the zero map, $f \circ h$ from I into S_j is a nonzero Λ-map of I into S_j. This contradiction implies that S_j is an injective Λ/N^2-module.

By Theorem A it now follows, since Λ/N^2 is a semi-primary ring with radical of square zero, that $l.\mathrm{gl.dim}\, \Lambda/N^2 \leqq 1$. This necessarily implies that $N^2 = 0$ [2].

Remark that if all simple modules are projective modules, or if all simple modules are injective modules, then Λ is a semi-simple ring [1].

Finally, if $N \neq 0$ then there exist a simple projective (injective) module that is not an injective (projective) module.

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