THE SCHUR INDEX FOR PROJECTIVE
REPRESENTATIONS OF FINITE GROUPS

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In this paper the question of determining the absolutely irreducible constituents of an irreducible projective representation of a finite group is considered from the viewpoint of the theory of algebras. New proofs are given for several of the main results of the theory of representations of finite dimensional associative algebras. This theory is applied to determine the center of a simple component of a twisted group algebra modulo its radical and sufficient conditions are given to insure that this center is a normal extention of the base field. The Schur index of an absolutely irreducible projective representation of a finite group is defined as in the theory of linear representations of finite groups. It is shown that every irreducible complex projective representation of a finite group is projectively equivalent to a representation whose associated factor system has values which are roots of unity but whose Schur index over the rationals is 1.

Throughout this paper all modules will be assumed to be finite dimensional unitary left modules. By an algebra over a field K we shall mean a finite dimensional associative algebra in which the identity of K acts as identity. If $\mathfrak{A}$ is an algebra, we denote the radical of $\mathfrak{A}$ by rad $\mathfrak{A}$ and set $\overline{\mathfrak{A}} = \mathfrak{A}/\mathrm{rad}\,\mathfrak{A}$. If E is an extension field of K, we identify $\mathfrak{A}$ as a subalgebra of $\mathfrak{A}^E = E \otimes_K \mathfrak{A}$. If M is an $\mathfrak{A}$-module, we denote by M^E the $\mathfrak{A}^E$-module, $E \otimes_K M$. We refer the reader to [4] for the relevant theory of algebras assumed.

1. **The Schur index.** Let $\mathfrak{A}$ be an algebra over the perfect field K. There exists a finite normal (separable) extension field E of K which is a splitting field for $\mathfrak{A}$, i.e., $\mathfrak{A}^E/\mathrm{rad}\,(\mathfrak{A}^E)$ is a direct sum of complete matrix rings over E. Let $\{x_1, \cdots, x_s\}$ be a basis for $\mathfrak{A}$ over K. Under the usual identification, $\{x_1, \cdots, x_s\}$ is also a basis for $\mathfrak{A}^E$. We denote the Galois group of E over K by $\mathfrak{G}(E \mid K)$. Let N be a left $\mathfrak{A}^E$-module with basis $\{m_1, \cdots, m_n\}$ over E and let the module action be given by $x_k m_i = \sum_j a_{ij}(x_k)m_j$, $k = 1, \cdots, s$, $a_{ij}(x_k) \in E$. Let V be an n-dimensional vector space over E with basis $\{v_1, \cdots, v_n\}$ and let $\sigma \in \mathfrak{G}(E \mid K)$. Under the action $x_k v_i = \sum_j \sigma(a_{ij}(x_k))v_j$, $k = 1, \cdots, s$, V becomes an $\mathfrak{A}^E$-module which we denote by σN. σN is called a conjugate module to N and the representations of $\mathfrak{A}^E$ that these modules afford are said to be conjugate (with respect to the pair of bases $\{x_1, \cdots, x_s\}$, $\{v_1, \cdots, v_s\}$). By the character afforded by a representation

T of a K-algebra $\mathfrak{B}$, we shall mean the function μ from $\mathfrak{B}$ to K defined by $\mu(x) = \text{trace } (T(x))$, $x \in \mathfrak{B}$. If χ is the character of $\mathfrak{A}^E$ afforded by N, we denote by $\sigma\chi$ the character of $\mathfrak{A}^E$ afforded by σN (where $\sigma\chi(x_k) = \sigma(\chi(x_k))$, $k = 1, \cdots, s$). σ and τ will always denote elements of $\mathfrak{G}(E \mid K)$ while χ and ψ will always be characters. $i, j,$ k, n, m, etc., will always denote natural numbers.

Let E^* be an algebraic closure of E. All fields considered will be assumed to be subfields of E^*. Let M be an irreducible $\mathfrak{A}$-module. M is isomorphic to a minimal left ideal of a simple component $\mathfrak{B}$ of $\mathfrak{A}$. $\mathfrak{B}$ is isomorphic to a complete matrix ring $(D)_t$, D a division algebra with center $L, L \supset K$. If r is a natural number, rM is defined to be the direct sum of r copies of M. N will be assumed to be an irreducible $\mathfrak{A}^E$-module which is a composition factor of M^E. χ will denote the character of $\mathfrak{A}^E$ afforded by N. $K(\chi)$ will denote the field generated over K by the values $\chi(x_k)$, $k = 1, \cdots, s$. Let $N^* = E^* \otimes_K N$. We say that N^* is realizable in a subfield F of E^*, $F \supset K$, if there is an $\mathfrak{A}^F$-module W such that $E^* \otimes_K W \cong N^*$. If N^* is realizable in a subfield F of E^*, $F \supset K$, then clearly $F \supset K(\chi)$. We define the Schur index, $m_K(N)$, of N with respect to K to be the minimum of $(F : K(\chi))$ taken over all extension fields F of $K(\chi)$ in which N^* is realizable. $m(D)$ will denote the index of D.

In this section we shall give new proofs of the following two results.

THEOREM 1.1. *The center L of D is isomorphic to $K(\chi)$.*

THEOREM 1.2. (a) $M^E \cong m_K(N)(\sigma_1 N \oplus \cdots \oplus \sigma_t N)$ *where* $\sigma_i \in \mathfrak{G}(E|K)$, $\sigma_1 = 1$, *the* $\{\sigma_i N\}$ *form a complete set of non-isomorphic conjugates of the irreducible $\mathfrak{A}^E$-module N, and* $t = (K(\chi) : K)$.
  (b) $m_K(N) = m(D)$.
  (c) *If K is a finite field,* $m_K(N) = 1$.
  (d) $m_K(N)$ *divides* $(N : E)$.
  (e) $m_K(N)$ *divides* $(J : K(\chi))$, *where J is any finite algebraic extension of K in which N^* is realizable.*
  (f) *If mN is realizable in $K(\chi)$, then $m_K(N)$ divides m.*
  (g) *If K is a finite field or has characteristic zero, then the characters of the nonisomorphic irreducible $\mathfrak{A}$-modules are linearly independent.*

Theorem 1.1 is Satz 3 of [1]. Parts of Theorem 1.2 appear in [1], [2], and [10]. For the case when $\mathfrak{A}$ is the group algebra of a finite group over a field of characteristic zero, several treatments of these theorems have been given. References to the literature in this

regard may be found in [11]. In a previous paper we presented a treatment of Theorems 1.1 and 1.2 for the case when $\mathfrak{A}$ was the group algebra of a finite group over an arbitrary field [5]. The proof that we gave for this special case can be modified to yield new proofs of Theorem 1.1 and Theorem 1.2 when $\mathfrak{A}$ is an arbitrary algebra over a perfect field.

Proof of Theorem 1.1. Let L be the center of D. Extending E if necessary, we may assume that L is embedded in E. Since N is absolutely irreducible we note that the proofs of Theorems 27.8, 30.12(i), and 30.15 of [4] are valid in our context. Thus $\sigma N \cong N$ if and only if $\sigma\chi = \chi$. From the proof of Lemma 1.2 of [5] (see also [4, p. 472]) we see, in particular, that the number t of simple components of $\mathfrak{B}^E$ satisfies:

$$(1) \qquad\qquad t = (K(\chi) : K) .$$

Since K is perfect, $K(\chi)$ and L are both separable extensions of K, so $\mathrm{rad}\,(\mathfrak{A}^L) \cong L \otimes \mathrm{rad}\,\mathfrak{A}$ [4, Th. 69.10]. Thus $\mathfrak{B}^L$ is a component of $\mathfrak{A}^L/\mathrm{rad}\,\mathfrak{A}^L$. The center of $\mathfrak{B}^L$ is isomorphic to $L \otimes_K L$ [7, Th. 1, p. 114]. However, $L \otimes_K L$ is the direct sum of all inequivalent K-composites of L with itself [8, Th. 21, p. 84]. One of these composites is isomorphic to L and so some simple component, $\mathfrak{C}$, of $\mathfrak{B}^L$ is central simple. Let W be the irreducible $\mathfrak{A}^L$-module afforded by a minimal left ideal of $\mathfrak{C}$. $\mathfrak{C}^E$ is a central simple component of $\mathfrak{A}^E/\mathrm{rad}\,(\mathfrak{A}^E)$ and without loss of generality we may assume that N is afforded by a minimal left ideal of $\mathfrak{C}^E$. Since $\mathfrak{C}^E$ has only one simple component, we have by (1) that $L(\chi) \cong L$ and so $K(\chi)$ is embedded in L. Similarly, the center of $K(\chi) \otimes_K \mathfrak{B}$ is isomorphic to $K(\chi) \otimes_K L$. Let $\mathfrak{D}$ be the simple component of $K(\chi) \otimes_K \mathfrak{B}$ such that N is afforded by a minimal left ideal of $\mathfrak{D}^E$. By (1) $\mathfrak{D}^E$ is central simple and so $\mathfrak{D}$ is central simple. Since the center of $\mathfrak{D}$ is also a composite of $K(\chi)$ and L over K, we see that $K(\chi)$ contains a subfield isomorphic to L. Thus $L \cong K(\chi)$.

With the obvious modifications, the proofs of Lemma 1.3 and Theorem 1.4 of [5] prove Theorem 1.2.

2. **Outer tensor products.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be algebras over an arbitrary field K and let $\mathfrak{A} = \mathfrak{A}_1 \otimes_K \mathfrak{A}_2$. Let M_i be an irreducible $\mathfrak{A}_i$-module, $i = 1, 2$. The outer tensor product $M_1 \sharp M_2$ of M_1 and M_2 is the $\mathfrak{A}$-module whose underlying space is $M_1 \otimes_K M_2$ and where the module action is given by

$$(a_1 \otimes a_2) \sum m_j \otimes m'_j = \sum a_1 m_j \otimes a_2 m'_j$$

where $m_j \in M_1$, $m_j' \in M_2$, and $a_i \in \mathfrak{A}_i$, $i = 1, 2$.

In [5, §2] the theory of the Schur index was applied to study the structure of $M_1 \# M_2$ for the case when $\mathfrak{A}_1$ and $\mathfrak{A}_2$ were group algebras of finite groups over an arbitrary field. Although we are not primarily interested in studying the structure of $M_1 \# M_2$ when $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are twisted group algebras, we will show that the results obtained in [5, §2] are valid for arbitrary finite dimensional algebras. Many of these results seem to be well known.

PROPOSITION 2.1. *If M_i is an absolutely irreducible $\mathfrak{A}_i$-module, $i = 1, 2$, then $M_1 \# M_2$ is an absolutely irreducible $\mathfrak{A}$-module.*

Proof. Let M_i afford the matrix representation T_i of $\mathfrak{A}_i$ $i = 1, 2$. Then $M_1 \# M_2$ affords the matrix representation S of $\mathfrak{A}$ where

$$S(a_1 \otimes a_2) = T_1(a_1) \times T_2(a_2)$$

($\times$ denoting Kronecker product), $a_i \in \mathfrak{A}_i$. If T_1, T_2 have degrees r, s, respectively, then S has degree rs. It is sufficient to prove that $S(\mathfrak{A}) = K_{rs}$. By assumption $T_1(\mathfrak{A}_1) = K_r$, $T_2(\mathfrak{A}_2) = K_s$. Let Z be a matrix in K_{rs}. Since $K_r \otimes_K K_s \cong K_{rs}$, $Z = \sum_i A_i \times B_i$, $A_i \in K_r$, $B_i \in K_s$. Then $A_i = T_1(a_i)$, $B_i = T_2(b_i)$, $a_i \in \mathfrak{A}_1$, $b_i \in \mathfrak{A}_2$. Thus

$$Z = \sum_i T_1(a_i) \times T_2(b_i) = \sum_i S(a_i \otimes b_i) \in S(\mathfrak{A}) .$$

Throughout this section K will denote an arbitrary field and $\mathfrak{A}_1$ and $\mathfrak{A}_2$ will be algebras over K such that the centers of $\overline{\mathfrak{A}}_1$ and $\overline{\mathfrak{A}}_2$ are separable over K. All fields considered will be assumed to be subfields of a fixed algebraic closure of E, E being a splitting field for $\mathfrak{A}$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$. M_1 and M_2 will denote irreducible $\mathfrak{A}_1$ and $\mathfrak{A}_2$-modules, respectively. We wish to investigate the case when one or both of the M_i are not absolutely irreducible. The first part of the proof of Theorem 2.2 of [5] generalizes without difficulty to yield the following result.

PROPOSITION 2.2. *If the centers of $\overline{\mathfrak{A}}_1$ and $\overline{\mathfrak{A}}_2$ are separable over K, then $\overline{\mathfrak{A}}_1 \otimes_K \overline{\mathfrak{A}}_2 \cong \overline{\mathfrak{A}}$ and*

$$\mathrm{rad}\ \mathfrak{A} = \mathrm{rad}\ \mathfrak{A}_1 \otimes \mathfrak{A}_2 + \mathfrak{A}_1 \otimes \mathrm{rad}\ \mathfrak{A}_2 .$$

When K has characteristic zero, this result was proved by Ivan [6]. The result is false without the separability condition. For a general discussion of the behavior of the radical under tensor products, see [7, Chapter 5, §14].

As an immediate corollary we have:

COROLLARY 2.3. *If the centers of $\overline{\mathfrak{A}}_1$ and $\overline{\mathfrak{A}}_2$ are separable over K, then $M_1 \sharp M_2$ is completely reducible.*

If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are group algebras, then Theorem 2.2 of [5] shows that $M_1 \sharp M_2$ is the direct sum of irreducible $\mathfrak{A}$-modules of the same dimension with each isomorphism type appearing with the same multiplicity. As we shall see in § 3 this result is false even for twisted group algebras over a field of characteristic zero. This result is true, however, if the centers of $\overline{\mathfrak{A}}_1$ and $\overline{\mathfrak{A}}_2$ are separable over K and the center of the simple component of $\overline{\mathfrak{A}}_i$ affording M_i is a normal extension of K, $i = 1, 2$. In view of the separability condition, in proving this result we may assume that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are simple algebras.

THEOREM 2.4. *Assume that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are simple algebras over the arbitrary field K and let L_i be the center of $\mathfrak{A}_i$, $i = 1, 2$. If L_i is a separable normal extension of K, $i = 1, 2$, then $M_1 \sharp M_2$ is the direct sum of irreducible $\mathfrak{A}$-modules of the same dimension with each isomorphism type appearing with the same multiplicity.*

Proof. Since L_1 and L_2 are separable normal extensions of K, $L_1 \otimes_K L_2$ is isomorphic to a sum of copies of L, where $L = L_1 L_2$ is the composite of L_1 and L_2 [8, Th. 21, p. 84]. Since the center of $\mathfrak{A}_1 \otimes_K \mathfrak{A}_2$ is isomorphic to $L_1 \otimes_K L_2$ we see that

$$\mathfrak{A}_1 \otimes \mathfrak{A}_2 \cong \mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_r ,$$

where the $\mathfrak{B}_i$ are simple algebras over K, each having center L. We have

$$\mathfrak{B}_i \otimes_K L \cong (\mathfrak{B}_i \otimes_L L) \otimes_K L \cong \mathfrak{B}_i \otimes_L (L \otimes_K L)$$
$$\cong \mathfrak{B}_i \otimes_L (L \oplus \cdots \oplus L) \cong \mathfrak{B}_i \oplus \mathfrak{B}_i \oplus \cdots \oplus \mathfrak{B}_i ,$$

where $\mathfrak{B}_i$ is viewed on the right hand side of this equation as an algebra over L. Therefore $\mathfrak{A} \otimes_K L$ is isomorphic to a sum of copies of the $\mathfrak{B}_j$, $j = 1, \cdots, r$. However,

$$\mathfrak{A} \otimes_K L \cong (\mathfrak{A}_1 \otimes_K L) \otimes_L (\mathfrak{A}_2 \otimes_K L)$$

and

$$\mathfrak{A}_i \otimes_K L \cong (\mathfrak{A}_i \otimes_{L_i} L_i) \otimes_K L \cong \mathfrak{A}_i \otimes_{L_i} (L_i \otimes_K L)$$
$$\cong \mathfrak{A}_i \otimes_{L_i} (L \oplus \cdots \oplus L) \cong (\mathfrak{A}_i \otimes_{L_i} L) \oplus \cdots \oplus (\mathfrak{A}_i \otimes_{L_i} L) .$$

Since $\mathfrak{A}_i \otimes_{L_i} L$ is central simple, we see that the $\mathfrak{B}_i$ are isomorphic simple algebras over K. Let $\mathfrak{B}_i \cong (\mathfrak{D})_t$, where $\mathfrak{D}$ is a division algebra with center L. Let $\mathfrak{C}_i$ be a minimal left ideal of $\mathfrak{A}_i$, $i = 1, 2$. Then

$\mathfrak{A} = \mathfrak{A}_1 \otimes_K \mathfrak{A}_2$ is isomorphic to a direct sum of copies of $\mathfrak{C}_1 \otimes_K \mathfrak{C}_2$. As an $\mathfrak{A}_i$-module, M_i is isomorphic to $\mathfrak{C}_i$, $i = 1, 2$, so as an $\mathfrak{A}$-module, $\mathfrak{A}$ is isomorphic to a sum of copies of $M_1 \sharp M_2$. Let N_i be an $\mathfrak{A}$-module isomorphic to a minimal left ideal of $\mathfrak{B}_i$. As an $\mathfrak{A}$-module, we have

$$\mathfrak{A} \cong tN_1 \oplus tN_2 \oplus \cdots \oplus tN_r .$$

Let $M_1 \sharp M_2 \cong \sum_i u_i N_i$ and suppose that $\mathfrak{A} \cong s(M_1 \sharp M_2)$ as an $\mathfrak{A}$-module. Then

$$\sum_i tN_i \cong \sum_i su_i N_i .$$

We conclude that the multiplicity u_i with which N_i appears as a composition factor of $M_1 \sharp M_2$ is t/s. Since the dimension of each N_i is $t[\mathfrak{D} : K]$: the result follows.

If, in addition to the hypotheses of Theorem 2.4, we also assume that K is perfect, then the proof of Theorem 2.2 of [5] is valid in our more general context and we can express the dimension and multiplicity of each irreducible constituent of $M_1 \sharp M_2$ as in that theorem. Theorem 2.3 of [5] is valid in our context but is uninteresting. Suppose that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are simple algebras over K with centers L_1 and L_2 respectively, L_1 and L_2 being separable extensions of K. Then $\mathfrak{A}_1 \otimes_K \mathfrak{A}_2$ is simple if and only if $L_1 \otimes_K L_2$ is a field, and this in turn is equivalent to

$$(L_1 : K)(L_2 : K) = (L_1 L_2 : K) .$$

We also have

$$\text{Hom}_{\mathfrak{A}}(M_1 \sharp M_2, M_1 \sharp M_2) \cong \text{Hom}_{\mathfrak{A}_1}(M_1 \, M_1) \otimes_K \text{Hom}_{\mathfrak{A}_2}(M_2 \, M_2) .$$

Let $D_i = \text{Hom}_{\mathfrak{A}_i}(M_i, M_i)$. Thus $M_1 \sharp M_2$ is irreducible if and only if $D_1 \otimes_K D_2$ is a division algebra. Let $m(D_i)$ denote the index of D_i. If K is perfect, then $m(D_i) = m_K(N_i)$, where N_i is an irreducible $\mathfrak{A}_i^p$-module which is a constituent of M_i^p. Thus, $D_1 \otimes_K D_2$ is a division algebra if and only if

$$m_K(N_1) m_K(N_2) = m_K(N_1 \sharp N_2) \quad \text{and} \quad (L_1 : K)(L_2 : K) = (L_1 L_2 : K) .$$

In particular, condition (b) of Theorem 2.3 of [5] is unnecessary.

Theorem 2.5 and Corollary 2.5 of [5] are false, as stated. Let G_1 be the quaternion group of order 8, G_2 the cyclic group of order 3, and let $K = Q$, the field of rational numbers. Let $\mathfrak{A}_i = Q(G_i)$, $i = 1, 2$. Let M_1 be the irreducible four-dimensional $Q(G_1)$-module and let M_2 be the irreducible two-dimensional $Q(G_2)$-module. The conditions of Theorem 2.5 and Corollary 2.6 of [5] are satisfied but $M_1 \sharp M_2$ is not an irreducible $Q(G_1 \times G_2)$-module. Using the remarks above and

the proof of Theorem 2.5 of [5] we can prove the following result.

PROPOSITION 2.5. *Let K be perfect and let χ_i be the character of N_i, $i = 1, 2$. If*

$$(m_K(N_1)(K(\chi_1) : K),\ m_K(N_2)(K(\chi_2) : K)) = 1 ,$$

then $M_1 \sharp M_2$ is irreducible.

This result generalizes Corollary 1 of [7, p. 179]. An analogue for projective representations of Corollary 2.6 of [5] will be proved in § 3.

3. **Twisted group algebras.** Let $\mathfrak{F}$ be a finite group, K a perfect field. A factor system from $\mathfrak{F}$ to K is a function f from $\mathfrak{F} \times \mathfrak{F}$ to the multiplicative group of K such that $f(1, 1) = 1$, and

$$f(a, b)f(ab, c) = f(a, bc)f(b, c),   \text{for all}   a, b, c \in \mathfrak{F} .$$

The twisted group algebra $K(\mathfrak{F})_f$ is defined to be the algebra over K with basis $\{(g)\}_{g \in \mathfrak{F}}$ and multiplication defined by $(a)(b) = f(a, b)(ab)$ for all $a, b \in \mathfrak{F}$. We get the ordinary group algebra when $f(a, b) = 1$ for all $a, b \in \mathfrak{F}$. Projective representations of $\mathfrak{F}$ with factor system f correspond to representations of $K(\mathfrak{F})_f$.

In discussing splitting field questions for projective representations it seems to be necessary to restrict oneself to one fixed factor system. For example, if $\mathfrak{F}$ is cyclic of order two, $\mathfrak{F} = \langle a \rangle$, then for each integer n, other than zero, there exists a factor system of $\mathfrak{F}$ with $f(a, a) = n$. Thus there does not exist any algebraic number field in which every irreducible projective representation of $\mathfrak{F}$ is absolutely irreducible. However, if one works with a fixed factor system f, the results of § 1 yield an adequate splitting field theory.

We show that Theorem 2.4 is not true without the restrictions on the normality of the centers of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Let $\mathfrak{F}$ be cyclic of order 3, $\mathfrak{F} = \langle b \rangle$. Let f be the rational factor system defined by

$$f(b, b) = f(1, b) = f(b, 1) = f(1, b^2) = f(b^2, 1) = f(1, 1) = 1 ,$$
$$f(b^2, b) = f(b, b^2) = f(b^2, b^2) = 2 .$$

$Q(\mathfrak{F})_f \cong Q(\sqrt[3]{2})$. Thus $Q(\mathfrak{F})_f$ has an irreducible 3-dimensional rational module M. Let $\mathfrak{A}_1 = \mathfrak{A}_2 = Q(\mathfrak{F})_f$. Since $Q(\sqrt[3]{2}) \otimes_Q Q(\sqrt[3]{2})$ is the sum of all inequivalent Q-composites of $Q(\sqrt[3]{2})$ with itself [8, Th. 21, p. 84], we see that

$$Q(\sqrt[3]{2}) \otimes_Q Q(\sqrt[3]{2}) \cong Q(\sqrt[3]{2}) \oplus Q(\omega, \sqrt[3]{2}) ,$$

ω a primitive cube root of 1. Thus $M \sharp M$ is the direct sum of a three dimensional and a six dimensional irreducible $Q(\mathfrak{F})_f \otimes_Q Q(\mathfrak{F})_f$-module.

In view of the above example, we need conditions on K and f to insure that $K(\chi)$ is normal over K, where χ is a character. Let E be a splitting field for $K(\mathfrak{F})_f$. f may be viewed as a factor system in E and $E \otimes_K K(\mathfrak{F})_f \cong E(\mathfrak{F})_f$. Let M be an irreducible $K(\mathfrak{F})_f$-module and let $\mathfrak{B}$ be the simple component of $K(\mathfrak{F})_f/\mathrm{rad}\,(K(\mathfrak{F})_f$ such that M is isomorphic to a minimal left ideal of $\mathfrak{B}$. Let N be an irreducible $E(\mathfrak{F})_f$-module which is a constituent of M^E. Let χ be the character of N. The center of $\mathfrak{B}$ is isomorphic to $K(\chi)$, the field generated over K by the values $\{\chi((g))\}_{g \in \mathfrak{F}}$.

Let $g \in \mathfrak{F}$ and set $\lambda(g) = \prod_{i=1}^{t} f(g, g^i)$, where $t = |g|$ is the order of g. Let T be the representation of $E(\mathfrak{F})_f$ afforded by N. Then

$$[T((g))]^t = \lambda(g)T(1) = \lambda(g)I \, ,$$

I the identity matrix of degree r, $r =$ degree T, so $\chi(g)$ is the sum of $|g|$-th roots of $\lambda(g)$. If m is the exponent of $\mathfrak{F}$ and K contains all m-th roots of unity, the $K(\chi)$ is a subfield of an abelian extension of K and so $K(\chi)$ is normal over K with abelian Galois group. If the characteristic of K does not divide m, $K(\chi)$ is a contained in a Kummer extension of K.

If $f(x, y)$ is an v-th root of unity for all $x, y \in \mathfrak{F}$, then $K(\chi)$ is a subfield of $K(\sqrt[s]{1})$, where $s = mv$. Again we have that $K(\chi)$ is a normal extension of K with abelian Galois group. We have proved:

PROPOSITION 3.1. *Assume that one of the following holds:*
(a) *K contains all m-th roots of unity, m the exponent of $\mathfrak{F}$.*
(b) *$f(x, y)$ is a root of unity for all $x, y \in \mathfrak{F}$.*
Then the center of $\mathfrak{B}$ is a finite normal extension of K with abelian Galois group.

REMARKS 1. If $\mathfrak{F}$ is abelian of exponent m, the characteristic of K not dividing m, and K contains a primitive m-th root of unity, then the center of $K(\mathfrak{F})_f$ is determined in [15, § 6]. In this case the center of $\mathfrak{B}$ is a Kummer extension of K.

2. Suppose that K is an arbitrary field of characteristic p, $p > 0$, and f is a factor system of $\mathfrak{F}$ in K such that $f(x, y)$ is a root of unity in K for all $x, y \in \mathfrak{F}$. Let E be an algebraic closure of K and let N be an irreducible representation of $E(\mathfrak{F})_f$ with character χ. Then $K(\chi)$ is a normal extension of K. The proofs in § 1 of [5] now carry over unchanged and we can conclude that $m_K(N) = 1$.

3. Using Proposition 2.5 and Theorem 1.2 we can prove an an-

alogue of Corollary 2.6 of [5]. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be finite groups of relatively prime order and let K be a finite field or a field of characteristic zero. Let $\mathfrak{A}_i$ be a twisted group algebra of $\mathfrak{F}_i$ over K and let M_i be an irreducible $\mathfrak{A}_i$-module, $i = 1, 2$. Let $\mathfrak{B}_i$ denote the simple component of $\overline{\mathfrak{A}}_i$ such that M_i is isomorphic to a minimal left ideal of $\mathfrak{B}_i$, i = 1, 2. If $\mathfrak{B}_i$ is central, $i = 1, 2$, then $M_1 \sharp M_2$ is irreducible. The result is immediate from Theorem 1.2(c) and Proposition 2.5 if K is a finite field. Suppose that K has characteristic zero and let E be a splitting field for $\mathfrak{A}_1, \mathfrak{A}_2$, and $\mathfrak{A} = \mathfrak{A}_1 \otimes_K \mathfrak{A}_2$. Let N_i be an irreducible $\mathfrak{A}_i$-module, $i = 1, 2$, such that N_i is a constituent of M_i^E. Then $m_K(N_i)$ divides $(N_i : E)$, $i = 1, 2$, by Theorem 1.2(d). But $(N_i : E)$ divides the order of $\mathfrak{F}_i$, $i = 1, 2$ [4, Th. 53.16] and so $(m_K(N_1),\ m_K(N_2)) = 1$.

4. **The Schur index of a projective representation.** Let $\mathfrak{F}$ be a finite group and let K be a field. By a linear representation of $\mathfrak{F}$ in K we shall mean a homomorphism of $\mathfrak{F}$ into $\mathrm{GL}(r, K)$ for some r. A projective representation of $\mathfrak{F}$ in K is a map T from $\mathfrak{F}$ into $\mathrm{GL}(r, K)$ for some r, such that $T(x)T(y) = f(x, y)T(xy)$, $T(1) = 1_r$, where f is a factor system of $\mathfrak{F}$. Let B an algebraically closed field. Let T be an irreducible projective representation of $\mathfrak{F}$ in B with factor system f. Let r be the degree of T and let W be a projective representation of $\mathfrak{F}$ in B of degree r with factor system h. W is said to be projectively equivalent to T in B if there exists $A \in \mathrm{GL}(r, B)$ and a function ρ from $\mathfrak{F}$ to B^*, the multiplicative group of B, such that

$$A^{-1}W(x)A = \rho(x)T(x) \qquad \text{for all} \quad x \in \mathfrak{F}\ .$$

If there is a function π from $\mathfrak{F}$ to B^* such that

$$f(a, b) = \pi(a)\pi(b)\pi(ab)^{-1}h(a, b) \qquad \text{for all} \quad a, b \in \mathfrak{F}\ ,$$

we say that f and h are equivalent. Equivalent projective representations have equivalent factor systems and equivalent factor systems lie in the same equivalence class in $H^2(\mathfrak{F}, B^*)$ (trivial action). If the values of $f(a, b)$, $h(a, b)$, $\pi(a)$ and $\pi(b)$, lie in a subfield K of B for all a, $b \in \mathfrak{F}$, we say that f is equivalent to h in K. Let the equivalence class of f have order k in $H^2(\mathfrak{F}, B^*)$. If the values of f are k-th roots of unity we say that f is a normalized factor system. Every projective representation of $\mathfrak{F}$ is projectively equivalent to one with a normalized factor system [4, p. 360].

Let S be a projective representation of $\mathfrak{F}$ in B of degree r. We say that S is linearly equivalent to T in B if there exists $D \in \mathrm{GL}(r, B)$ such that

$$D^{-1}T(x)D = S(x) \qquad \text{for all} \quad x \in \mathfrak{F}\ .$$

Linearly equivalent projective representations have identical factor systems.

Let F be a subfield of B such that $f(x, y) \in F$ for all x, $y \in \mathfrak{F}$. We say that T is realizable in F if there exists a linearly equivalent projective representation S of $\mathfrak{F}$ in B such that the entries of $S(x)$ lie in F for all $x \in \mathfrak{F}$. Let $\chi(g) = \text{trace } T(g)$, $g \in \mathfrak{F}$. In particular, if T is realizable in F, then F contains the values $\chi(g)$, $g \in \mathfrak{F}$, as well as the values $f(x, y)$, $x, y \in \mathfrak{F}$. Let K be a subfield of B. We define the projective Schur index $m_K(T)$ of T over K to be minimum $(F: K(f, \chi))$, where $K(f, x)$ is the field generated over K by the values $f(x, y)$, $\chi(g)$ for all $x, y, g \in \mathfrak{F}$, the minimum being taken over all fields F, $F \supseteq K(f, \chi)$, in which T is realizable. We note that if T is a linear representation of $\mathfrak{F}$, then $m_K(T)$ is just the usual Schur index.

Let L denote the field $K(f, \chi)$. Let $\mathfrak{A}$ be a simple component of $L(\mathfrak{F})_f / \text{rad} (L(\mathfrak{F})_f)$ such that T is afforded by a minimal left ideal of $B \otimes_L \mathfrak{A}$. Assume that L is perfect. Applying the results of §1 to $L(\mathfrak{F})_f$ we see that $\mathfrak{A}$ is central simple and $m_K(T) = m(D)$, where $m(D)$ is the index of the division algebra component D of $\mathfrak{A}$.

We will be interested in the case when the values of f are roots of unity. If B has characteristic different from zero, it follows from the second remark after Proposition 3.1 that $m_K(T) = 1$. For this reason we restrict our attention to fields of characteristic zero. *Throughout the rest of this paper B will be assumed to be an algebraically closed field of characteristic zero.* We note that if K is any subfield of B and f is any factor system of $\mathfrak{F}$ in K, then $K(\mathfrak{F})_f$ is semi-simple [15, Th. 4.1].

Let T be an irreducible projective representation of $\mathfrak{F}$ in B with factor system f. Let $\mathfrak{B}$ be the character group of the multiplicative cyclic group generated by f. Following Reynolds [12], we construct an f-covering group $\mathfrak{F}^*$ of $\mathfrak{F}$ as follows: For $x, y \in \mathfrak{F}$ let $b_{x,y} \in \mathfrak{B}$ be the character such that $b_{x,y}(f^i) = f(x, y)^i$. Let $\mathfrak{F}^*$ be the set of ordered pairs (b, x), $b \in \mathfrak{B}$, $x \in \mathfrak{F}$, with multiplication defined by

$$(b, x)(c, y) = (bcb_{x,y}, xy) .$$

Let $\mathfrak{B}^*$ consist of the pairs $(b, 1)$. Then $\mathfrak{F} \cong \mathfrak{F}^*/\mathfrak{B}^*$. We define a linear representation T^* of $\mathfrak{F}^*$ by

$$T^*(b, x) = b(f)T(x) .$$

We say that T^* linearizes T.

PROPOSITION 4.1. *Let the values of f be roots of unity and let $T, f, T^*, \mathfrak{F}^*$, be as above. Let K be an arbitrary subfield of the complex numbers. Then $m_K(T) = m_K(T^*)$.*

Proof. Let ψ, χ be the characters of T^*, T respectively. Then $\psi(b, x) = b(f)\chi(x)$ for all $x \in \mathfrak{F}$. Thus $K(f, \chi) = K(\psi)$. Clearly T^* is realizable in a field $F, F \supset K(\psi)$ if and only if T is realizable there. This proves the assertion.

It is frequently possible to determine the projective Schur indices from the properties of $\mathfrak{F}^*$. For example, suppose that $\mathfrak{F}$ is a nilpotent group of odd (even) order and the values of f are v-th roots of unity, where v is odd (possibly even). Then $\mathfrak{F}^*$ is nilpotent of odd (even) order and so $m_F(T) = 1$ (1 or possibly 2) [13], [14, p. 161].

PROPOSITION 4.2. *Let $\mathfrak{F}$ have exponent m and let f be a normalized factor system of order k in $H^2(\mathfrak{F}, B^*)$. Let $t = km$ and let $E = Q(\sqrt[t]{1})$. Then $E(\mathfrak{F})_f$ is a direct sum of complete matrix rings over E.*

Proof. Let S be an irreducible complex projective representation of $\mathfrak{F}$ with factor system f. It is sufficient to prove that S is realizable in E. Let $\mathfrak{F}^*$ be the f-covering group of $\mathfrak{F}$, and let S^* be the linearization of S to $\mathfrak{F}^*$. Then S^* is realizable in $Q(\sqrt[s]{1})$, s the exponent of $\mathfrak{F}^*$ [4, Th. 41.1]. But s divides km and so S is realizable in E.

A theorem of Kuo [9, Th. 5.1] asserts that, with the notation as above, km divides the order n of $\mathfrak{F}$. This shows that the field obtained in Proposition 4.2 is a subfield of $Q(\sqrt[n]{1})$. Kuo's proof of her result uses the cohomology theory of groups to reduce to the case when $\mathfrak{F}$ is a p-group [1, Th. 10.1, p. 259], and then uses further cohomological information to prove the result in this case. We note that when $\mathfrak{F}$ is a p-group this result can also be proved by character theory. If $\mathfrak{F}$ is a finite p-group, then so also is $H^2(\mathfrak{F}, B^*)$ [4, Th. 53.3].

PROPOSITION 4.3. (*Kuo*). *Let $\mathfrak{F}$ be a finite group of order p^t and exponent p^s. Let p^r be the exponent of $H^2(\mathfrak{F}, B^*)$. Then p^{r+s} divides p^t.*

Proof. Let f be a factor system of $\mathfrak{F}$ in B of order p^r in $H^2(\mathfrak{F}, B^*)$. Let $\mathfrak{F}^*$ be an f-covering group of $\mathfrak{F}$ as defined in Proposition 4.1. Then $\mathfrak{F}^*$ has a central subgroup $\mathfrak{H}$ of order p^r and $\mathfrak{F}^*/\mathfrak{H} \cong \mathfrak{F}$ has a cyclic subgroup of order p^s. Thus $\mathfrak{F}^*$ has an abelian subgroup $\mathfrak{M}$ of order p^u, $u = r + s$. It is sufficient to prove that $p^r \leq [\mathfrak{F}^* : \mathfrak{M}]$. By Frobenius reciprocity it is sufficient to prove that $\mathfrak{F}^*$ has an absolutely irreducible representation of degree greater than

or equal to p^r [4, Exercise 8, p. 279]. Let S be an irreducible pro-jective representation of $\mathfrak{F}$ in B with factor system f (afforded by an irreducible $B(\mathfrak{F})_f$-module). Let p^v be the degree of S. Taking deter-minants in the equation

$$S(x)S(y) = f(x, y)S(xy) , \qquad x, y \in \mathfrak{F}$$

we see that the order p^r of f in $H^2(\mathfrak{F}, B^*)$ divides p^v. Let S^* be a linearization of S to $\mathfrak{F}^*$. Since S^* is an absolutely irreducible repre-sentation of $\mathfrak{F}^*$ of degree greater than or equal to p^r, we are done.

Proposition 4.2 is false if E is replaced by $Q(\sqrt[p]{1})$. The simplest example of this is when $\mathfrak{F} = Z_2 \times Z_2$ and $Q(\mathfrak{F})_f$ is the rational quater-nion algebra, Q the field of rationals. We note also that, in view of Kuo's theorem, Proposition 4.2 is a strengthening of a theorem of Reynolds [12, Th. 5].

We consider next the question of how the Schur index behaves under projective equivalence. It is possible for T and S to be projec-tively equivalent irreducible projective representations of $\mathfrak{F}$, both having normalized factor systems, but having different Schur indices over the rationals. Let $\mathfrak{F} = Z_2 \times Z_2$ and let f, g be normalized factor systems of $\mathfrak{F}$ so that $Q(\mathfrak{F})_f \cong (Q)_2$, the algebra of 2×2 matrices over the rationals Q, while $Q(\mathfrak{F})_g \cong Q'$, the rational Quaternion division algebra.

$$B(\mathfrak{F})_f \cong B(\mathfrak{F})_g \cong (B)_2 .$$

Let T, S be irreducible projective representations of $\mathfrak{F}$ in B with factor systems f, g respectively. Then $m_Q(T) = 1$, while $m_Q(S) = 2$. The f, g-covering groups of $\mathfrak{F}$ are respectively the dihedral and quaternion groups of order 8.

Suppose that T and S are projectively equivalent irreducible projective representations of $\mathfrak{F}$ in B with associated factor systems f and g respectively. The possible values of the Schur index over Q are bounded; by Theorem 1.2(b) we have $m_Q(T) \leq \sqrt{n}$, where n is the order of $\mathfrak{F}$. Within these bounds we can have various possibilities for $m_Q(S)$. We show next that S can always be found such that $m_Q(S) = 1$.

PROPOSITION 4.4. *Let T be an irreducible projective representation in B of the finite group $\mathfrak{F}$. Then T is projectively equivalent in B to a projective representation S of $\mathfrak{F}$ such that $m_Q(S) = 1$. S may be chosen to have a factor system whose values are $|\mathfrak{F}|$-th roots of unity.*

Proof. Let f be the factor system associated with T and let χ be the projective character of $\mathfrak{F}$ afforded by T.

$$m_Q(T) = \text{minimum } (F : Q(f, \chi)),$$

the minimum being taken over all fields F in which T is realizable. We will construct S by suitably enlarging $Q(f, \chi)$. We may clearly assume that f is normalized. If $|\mathfrak{F}| < 3$, the result is trivial since S may be taken to be an ordinary representation. Let x and y be nonidentity elements of $\mathfrak{F}$ with $x \neq y$. Let ζ be a primitive $|\mathfrak{F}|$-th root of unity and define a function ρ from $\mathfrak{F}$ to B^* by $\rho(a) = 1$, $a \neq x$, and $\rho(x) = \zeta f(x, y)^{-1}$. Let $S(a) = \rho(a) T(a)$ for all $a \in \mathfrak{F}$. Then S is projectively equivalent to T. Let g be the factor system associated with S. Then $g(x, y) = \zeta$. Let $E = Q(\zeta)$. By Proposition 4.2, $E(\mathfrak{F})_f$ is the direct sum of complete matrix rings over E. By Theorem 1.2(b), T is realizable in E so we may assume that the entries of $T(a)$ lie in E for all $a \in \mathfrak{F}$. We see that S is realizable in E. Let ψ be the projective character of $\mathfrak{F}$ afforded by S. It remains to show that $Q(g, \psi) = E$. However, this follows immediately from $g(x, y) = \zeta$. The factor system g of S have values which are $|\mathfrak{F}|$-th roots of unity.

The author wishes to thank the referee for his many helpful suggestions.

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