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ON VISUAL HULLS

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The concept of visual hull has been introduced by G. H. Meisters and S. Ulam. In the following article we study a few of the problems arising from this notion and, in particular, establish (Theorem 3) a conjecture of W. A. Beyer and S. Ulam.

Let C be a set in R^n and $1 \leqq j \leqq n-1$. Then the j^{th} visual hull $H_j(C)$ of C is defined to be the largest set whose j^{th} projections are contained in those of C. Alternatively, $H_j(C)$ is the set of points x in R^n such that each $(n-j)$-flat through x contains a point of C. Let G_j^n denote the Grassmannian of j-subspaces in R^n with $\mu_j(G_j^n) = 1$ for the usual measure μ_j associated with G_j^n regarded as a metric 0_n-factorspace. (For further information about μ_j compare, for example, [3]). The j^{th} virtual hull $V_j(C)$ of C is defined to be the set of points $x \in R^n$ such that almost all (with respect to μ_{n-j}) $(n-j)$-flats through x contain a point of C. Thus, if $n = 3, j = 2, H_2(C)(V_2(C))$ corresponds to those points in R^3 which are photographically indistinguishable (with probability one) from C. A j^{th} minimal hull of C in R^n is a minimal set in R^n whose j^{th} projections coincide with those of C. In [2] the announced purpose of the paper was to disprove the conjecture that $H_j(C) - C$ is connected to C, i.e., $\not\exists$ disjoint open sets U, V such that $U \supset H_j(C) - C \neq \emptyset$ and $V \supset C \neq \emptyset$. To this we remark that a simple counterexample can be obtained by considering the closed set C formed by removing the relative interiors of alternate sides of a regular hexagon inscribed in a plane circle with centre a. The first visual hull $H_1(C)$ is then $C \cup \{a\}$.

2. Visual hulls of unions of polytopes.

THEOREM 1. *Let $A_1, \cdots, A_{j+1}$ be spherically convex, closed subsets (not necessarily nonempty) of the sphere S^{n-1}, such that each $(n-j-1)$-subsphere of S^{n-1} has a nonempty intersection with $\bigcup_{i=1}^{j+1} A_i$. Then $A_1 \cap \cdots \cap A_{j+1} \neq \emptyset$. (so, that, in particular, each set A_i is nonempty).*

REMARK. S^{n-1} is the unit sphere of R^n and an $(n-j-1)$-subsphere of S^{n-1} is the intersection of an $n-j$ subspace with S^{n-1}. A set $C \subset S^{n-1}$ is spherically convex if C is contained in an open hemisphere of S^{n-1} and, if $x, y \in C$ then C contains the minor arc on the 1-subsphere determined by x, y and 0 (the centre of S^{n-1}).

Proof. The case $n = 1$ is trivial. We assume inductively that the result is true for all $n' < n$ and it remains to prove the result for $j+1$ sets on S^{n-1}. Assume on the contrary that there exist spherically convex closed subsets $A_1, \cdots, A_{j+1} \subset S^{n-1}$ such that

$$T \cap (A_1 \cup \cdots \cup A_{j+1}) \neq \emptyset$$

for each $(n-j-1)$-subsphere T of S^{n-1}, and $A_1 \cap \cdots \cap A_{j+1} = \emptyset$. Let $A = A_1 \cap \cdots \cap A_j$. Then A, A_{j+1} are disjoint spherically convex closed subsets of S^{n-1}, and there exists an $(n-2)$-subsphere S' of S^{n-1} which separates A and A_{j+1} and such that $S' \cap A = \emptyset$, $S' \cap A_{j+1} = \emptyset$. Set $A'_i = A_i \cap S'$ $(1 \leqq i \leqq j)$. Then each A'_i is a spherically convex closed subset of S' and, since $A_{j+1} \cap S' = \emptyset$, each $(n-j-1)$-subsphere of S' has a nonempty intersection with $A'_1 \cup \cdots \cup A'_j$. Hence by the inductive assumption $A'_1 \cap \cdots \cap A'_j = A \cap S' \neq \emptyset$; contradiction.

THEOREM 2. *In R^n let $C_1, \cdots, C_{j+1}$ be $j+1$ compact convex sets. If $x \in H_j(\bigcup_{i=1}^{j+1} C_i)$ then either $x \in \bigcup_{i=1}^{j+1} C_i$ or there exists a halfline l emanating from x such that $l \cap C_i \neq \emptyset$, $1 \leqq i \leqq j+1$.*

COROLLARY. *In R^n let $C_1, \cdots, C_{j+1}$ be compact convex sets. Then asufficient condition for $H_j(\bigcup_{i=1}^{j+1} C_i) = \bigcup_{i=1}^{j+1} C_i$ is that the sets do not have a common transversal.*

Proof. On S^{n-1} define $j+1$ spherically convex closed subsets $A_1, \cdots, A_{j+1}$ so that $u \in A_i$ if $u \in S^{n-1}$ and the half line $\{x + \lambda u \mid \lambda \geqq 0\}$ meets C_i. Then, as $x \in H_j(\bigcup_{i=1}^{j+1} C_i)$ each $(n-j-1)$-subsphere of S^{n-1} has a nonempty intersection with $\bigcup_{i=1}^{j+1} A_i$. And so, by Theorem 1, there exists $u \in \bigcap_{i=1}^{j+1} A_i$, i.e., the halfline $\{x + \lambda u \mid \lambda \geqq 0\}$ meets each of $C_1, \cdots, C_{j+1}$.

THEOREM 3. *In R^n let $C_1, \cdots, C_{j+1}$ be nonempty compact convex sets. Then the number of components of $H_j(\bigcup_{i=1}^{j+1} C_i)$ is at most $j+1$ with equality if and only if $C_1, \cdots, C_{j+1}$ are pairwise disjoint.*

Proof. By Theorem 2, if $x \in H_j(\bigcup_{i=1}^{j+1} C_i) - \bigcup_{i=1}^{j+1} C_i$, then there exists a halfline $l = \{x + \lambda u \mid \lambda \geqq 0\}$ such that l meets each of

$$C_1, \cdots, C_{j+1}\,.$$

Then $x + \alpha_k u \in C_k$ for some $\alpha_k > 0$. We set $\alpha = \min\{\alpha_k \mid 1 \leqq k \leqq j+1\}$ and want to show that $x + \lambda u \in H_j(\bigcup_{i=1}^{j+1} C_i)$ for all λ with $0 \leqq \lambda \leqq \alpha$. Set $y = x + \lambda u$ and let P be an $(n-j)$-subspace. As $x \in H_j(\bigcup_{i=1}^{j+1} C_i)$ there exists i such that the $(n-j)$-flat $x + P$ meets C_i at v, say. Set $z = x + \alpha_i u \in C_i$. Then, as y lies between x and z on l, there exists μ, $0 \leqq \mu \leqq 1$, such that $y = \mu x + (1-\mu)z$. Then the $(n-j)$-flat $y + P$ through y contains the point $\mu v + (1-\mu)z$ of C_i. As P

was arbitrary we conclude that $y \in H_j(\bigcup_{i=1}^{j+1} C_i)$ and hence that $x + \lambda u \in H_j(\bigcup_{i=1}^{j+1} C_i)$ for $0 \leqq \lambda \leqq \alpha$. Hence, if $x \in H_j(\bigcup_{i=1}^{j+1} C_i)$ then x is connected, via a line segment in $H_j(\bigcup_{i=1}^{j+1} C_i)$, to at least one of the sets C_i. Hence $H_j(\bigcup_{i=1}^{j+1} C_i)$ has at most $j + 1$ components with equality only if the C_i's are disjoint. If the sets $C_1, \cdots, C_{j+1}$ are pairwise disjoint then in order to show that $H_j(\bigcup_{i=1}^{j+1} C_i)$ has exactly $j + 1$ components it is enough to show that for each k, $1 \leqq k \leqq j + 1$, there exist disjoint open sets U_k, V_k such that $U_k \cup V_k \supset H_j(\bigcup_{i=1}^{j+1} C_i)$ and $U_k \supset C_k$, $V_k \supset \{C_1 \cup \cdots \cup C_{k-1} \cup C_{k+1} \cup \cdots \cup C_{j+1}\}$. We suppose, without loss of generality, that $k = 1$. For $i = 2, \cdots, j + 1$ let H_i denote a hyperplane which strictly separates C_1 from C_i, and let H_i^0 be the open halfspace bounded by H_i and containing C_1. We can assume that the H_i's are in general position. Set $U_1 = \bigcap_{i=2}^{j+1} H_i^0$, $V_1 = R^n - \bar{U}_1$. Then U_1 and V_1 are disjoint open sets, $C_1 \subset U_1$, $\bigcup_{i=2}^{j+1} C_i \subset V_1$. It remains to show that $H_j(\bigcup_{i=1}^{j+1} C_i) \subset U_1 \cup V_1$, and it is enough to show that $(\bar{U}_1 \cap \bar{V}_1) \cap H_j(\bigcup_{i=1}^{j+1} C_i) = \varnothing$. Since the H_i's are in general position, their intersection $\bigcap_{i=2}^{j+1} H_i$ is an $(n - j)$-dimensional flat L. Let I be the j-dimensional subspace orthogonal to L. If M is any subset of R^n we denote by $\mathrm{proj}_I M$ the set of all points $x \in I$ for which the flat L_x, which is parallel to L and contains x, has a nonempty intersection with M. $\mathrm{proj}_I U_1$ and $\mathrm{proj}_I V_1$ are two open sets in I with common boundary $\mathrm{proj}_I (\bar{U}_1 \cap \bar{V}_1)$. As $\mathrm{proj}_I C_1 \subset \mathrm{proj}_I U_1$, $\mathrm{proj}_I \bigcup_{i=2}^{j+1} C_i \subset \mathrm{proj}_I V_1$ it follows that $(\mathrm{proj}_I (\bar{U}_1 \cap \bar{V}_1)) \cap (\mathrm{proj}_I \bigcup_{i=1}^{j+1} C_i) = \varnothing$. Now, if z is an arbitrary point in $\bar{U}_1 \cap \bar{V}_1$ it follows that $L_z \cap (\bigcup_{i=1}^{j+1} C_i) = \varnothing$, and since $\dim L_z = n - j$, we find, by the definition of H_j, that z does not belong to $H_j(\bigcup_{i=1}^{j+1} C_i)$. Therefore $(\bar{U}_1 \cap \bar{V}_1) \cap H_j(\bigcup_{i=1}^{j+1} C_i) = \varnothing$.

REMARKS. The proof of Theorem 3 also shows that any component of $H_j(\bigcup_{i=1}^{j+1} C_i)$ has the property that any two points of it can be joined by a broken line in it, consisting of at most 3 segments. Hence it is natural to ask: When are these components convex? (supposing now that the C_i's are disjoint). In [1] W. A. Beyer has shown an example of three (nondisjoint) polytopes C_i in R^3 such that $H_2(C_1 \cup C_2 \cup C_3)$ is not a polyhedron. We don't know whether a similar construction would be possible with disjoint polytopes. Let us mention here a few more technical terms. If M is any subset of R^n, we denote by $\mathrm{aff}\, M$ the affine hull of M and by $\mathrm{conv}\, M$ the convex hull of M. $\mathrm{relint}\, M$ means the interior of M with respect to the natural topology in $\mathrm{aff}\, M$. By the dimension $\dim M$ of M we understand the algebraic dimension of the flat $\mathrm{aff}\, M$. A polytope is the convex hull of some finite set. If $P \subset E^n$ is a convex set we denote by $\mathrm{ext}\, P$ the set of extreme points of P and by $\exp P$ the set of its exposed points. For an exact definition of these terms the reader may compare, for example, the introductory chapters of [4].

THEOREM 4. (i) *In* R^n *let* C_1, C_2 *be compact convex sets. Then* $H_1(C_1 \cup C_2)$ *is the union of at most two convex components which are polytopes whenever* C_1 *and* C_2 *are polytopes.*

(ii) *There exist in* R^3 *three disjoint polytopes such that one of the components of the second visual hull of their union is not convex.*

LEMMA 1. *Let* C_1, C_2 *be n-dimensional polytopes in* R^n. *If* $a \notin H_1(C_1 \cup C_2)$ *there exists a hyperplane* H *such that*

(1) $a \notin H$, H *separates* a *from* C_1

(2) $H \cap C_i = \emptyset$ *or* H *supports* C_i $(i = 1, 2)$

(3) $\operatorname{aff}(H \cap (C_1 \cup C_2)) = H$.

Proof of Lemma 1. The case $n = 1$ is trivial, and we assume $n \geqq 2$. If there exists a hyperplane P through a which does not meet $C_1 \cup C_2$ and does not separate C_1 and C_2 then $\operatorname{conv}(C_1 \cup C_2)$ is an n-dimensional polytope not containing a, and the lemma follows from standard results on polytopes. Hence it can be supposed that there is a hyperplane H for which (1) and also (2′): H separates C_1 and C_2 holds. We choose H in the set $\mathfrak{H}$ of hyperplanes for which (1) and (2′) holds. We assume that $h = \dim \operatorname{aff} T$ is maximal, where $T = H \cap (C_1 \cup C_2)$. Obviously $h \geqq 0$. If $h < n - 1$, let $F \subset H$ be an $(n-2)$-dimensional hyperplane in H containing T, and denote by π: $R^n \rightarrow E$ the projection along F onto a 2-dimensional flat E orthogonal to F. It is easy to see that there is a line L in E such that: (α): the singleton $\pi(T)$ is contained in L. (β): $\pi(a) \notin L$, L separates $\pi(a)$ from the polygon $\pi(C_1)(\gamma)$: L separates $\pi(C_1)$ and $\pi(C_2)$.

$$(\delta)\ \operatorname{aff}(L \cap (\pi(C_1) \cup \pi(C_2)) = L\ .$$

(Notice that the conditions $(\alpha) - (\gamma)$ are fulfilled by $\pi(H)$). The hyperplane $\pi^{-1}(L)$ of E^n intersects $C_1 \cup C_2$ in a set S with $\dim \operatorname{aff} S = h + 1$. Since $S \in \mathfrak{H}$ this contradicts the maximality of h. Hence the lemma is established.

Proof of Theorem 4. (i) We first prove the result when C_1, C_2 are n-dimensional polytopes. If $C_1 \cap C_2 \neq \emptyset$ then

$$H_1(C_1 \cup C_2) = \operatorname{conv}(C_1 \cup C_2)\ ,$$

which is a polytope. We suppose therefore that $C_1 \cap C_2 = \emptyset$. Let $\{H_i\}_{i=1}^m$ be the finite set of those hyperplanes which do not contain an interior of C_j $(j = 1, 2)$ and for which $\dim(H_i \cap (C_1 \cup C_2)) = n - 1$. By C_j^* we denote the (finite) intersection of those closed half spaces which contain C_j and whose bounding hyperplane is amongst $\{H_i\}_{i=1}^m$, $j = 1, 2$. Then C_j^* is polyhedral and, since C_1, C_2 are compact, C_j^* is a polytope,

$j = 1, 2$. We show that $H_1(C_1 \cup C_2) = C_1^* \cup C_2^*$. Suppose that $x^* \notin C_1^* \cup C_2^*$. Then there exist closed halfspaces H_1^*, H_2^* with bounding hyperplanes H_1, H_2 amongst $\{H_i\}_{i=1}^m$ such that $x^* \notin H_1^* \supset C_1$, $x^* \notin H_2^* \supset C_2$. If

$$x^* \in H_1(C_1 \cup C_2),\ H_1 \quad \text{and} \quad H_2$$

must separate C_1 and C_2. Consider H_1 and the two disjoint compact sets $H_1 \cap C_1, H_1 \cap C_2$ in H_1. There exists an $n-2$ dimensional flat L in H_1 which strictly separates $H_1 \cap C_1$ and $H_1 \cap C_2$. By slightly rotating H_1 about L in the appropriate direction we obtain a hyperplane H_1' which strictly separates C_1 and C_2 as well as x^* and C_1. Similarly we can obtain a hyperplane H_2' which strictly separates C_1 and C_2, and x^* and C_2. We may suppose that H_1', H_2' are not parallel and so $H_1' \cap H_2'$ is an $n-2$ flat. Suppose, without loss of generality, that $H_1' = \{x \mid \langle x, \xi\rangle = \alpha > 0\}$, $H_2 = \{x \mid \langle x, \eta\rangle = \beta > 0\}$. Then

$$C_1 \subset \{x \mid \langle x, \xi\rangle > \alpha\} \cap \{x \mid \langle x, \eta\rangle > \beta\}$$
$$C_2 \subset \{x \mid \langle x, \eta\rangle < \alpha\} \cap \{x \mid \langle x, \eta\rangle < \beta\} .$$

Consider the hyperplane $H\colon \{x \mid \langle x, \lambda\xi + (1-\lambda)\eta\rangle = 0\}$, where $\lambda\alpha + (1-\lambda)\beta = 0$ and $0 < \lambda < 1$. Then $x^* \in H$ and, using the above inequalities, $C_i \cap H = \emptyset$, $i = 1, 2$. Hence x^* is not in $H_1(C_1 \cup C_2)$, and we have $H_1(C_1 \cup C_2) \subset C_1^* \cup C_2^*$. Conversely, if $x^* \in C_1^* \cup C_2^* - H_1(C_1 \cup C_2)$, suppose without loss of generality that $x^* \in C_1^*$. Then, by Lemma 1, there exists a hyperplane H amongst $\{H_i\}_{i=1}^m$ which does not contain x^* and which separates x^* from C_1. Then, if H^* donotes the closed halfspace containing C_1 whose bounding hyperplane is H, $x^* \notin H^*$ and so $x^* \in C_1^*$; contradiction. And so $H_1(C_1 \cup C_2) = C_1^* \cup C_2^*$, which is the union of two polytopes. If C_1, C_2 are compact convex sets we choose decreasing sequences $\{P_1^n\}_{n=1}^\infty$, $\{P_2^n\}_{n=1}^\infty$ of polytopes such that $C_i = \bigcap_{n=1}^\infty P_i^n$, $i = 1, 2$. Then, using the above notation,

$$H_1(C_1 \cup C_2) = \bigcap_{n=1}^\infty P_1^{n*} \cap \bigcap_{n=1}^\infty P_2^{n*} .$$

(ii) Let W be the cube $\{x = (x_1, x_2, x_3) \mid -1 \leqq x_i \leqq 1, i = 1, 2, 3\}$ in R^3, and denote by W_i the facet of W defined by $x_i = 1$. Set $C_1 = W_1$, $C_2 = 2W_2$, $C_3 = 3W_3$. Let $B_i (1 \leqq i \leqq 3)$ be the components of $H_2(\bigcup_{i=1}^3 C_i)$, where the indices are chosen such that, for all i, $C_i \subset B_i$. Clearly $(0, 0, 0) \in B_1$ as does, of course, the point $(1, -1, -1) \in B_1 \cap C_1$. However we show that the line segment $m\colon \{x = \lambda(1, -1, -1) \mid 0 < \lambda < 1\}$ is not in B_1. Now $C_1 \cup C_2$ is contained in the halfspace $\{x \mid \langle x, (0, 1, 1)\rangle \geqq 0\}$ whose bounding hyperplane P passes through the points $(0, 0, 0)$, $(1, -1, 1)$ and $(-1, -1, 1)$; $P \cap \operatorname{aff} W_1$ is a line in direction $(0, -1, 1)$. If $y \in m$, then $y = \mu(1, -1, -1)$ for some μ, $0 < \mu < 1$. Consider the line $l = y + \{\lambda(0, -1, 1) \mid \lambda \text{ real}\}$. If $z = (z_1, z_2, z_3) \in l$ then $z_1 = \mu < 1$, i.e., $z \notin C_1$. Also $\langle z, (0, 1, 1)\rangle = -2\mu < 0$ which means that $z \notin C_1 \cup C_2$. Therefore l does not meet $C_1 \cup C_2 \cup C_3$, m does not belong to B_1, and B_1 is not convex.

In [6] V. L. Klee proved that if all j^{th} projections of a compact convex body C in R^n (j fixed $\geqq 2$) are polytopes, then C is a polytope. As a partial analogue to this for unions of two convex bodies we prove

THEOREM 5. *Let C_1, C_2 be two disjoint compact convex bodies in R^n such that each j^{th} projection of $C_1 \cup C_2$ (j fixed $\geqq 2$) is the union of two polytopes. Then* (i) $\operatorname{ext}(C_i) = \exp(C_i)$ *and* $\operatorname{ext}(C_i)$ *is countable* ($i = 1, 2$) *but* (ii) $\operatorname{ext}(C_i)$ *is not necessarily finite.*

Proof. Let a be an extreme point of C_1 and we suppose, without loss of generality, that $a = 0$, the origin of R^n. Then, to prove (i) it is enough to prove that the convex cone K of outward normals to C_1 at 0 is n-dimensional. We assume that $\dim K \leqq n - 1$ so that K is contained in an $(n - 1)$-subspace P_1, and seek a contradiction. Let P_2 be an $(n - 1)$-subspace which supports C_1 at 0. Of course $P_1 \neq P_2$. We can choose an $(n - 1)$-subspace P_3 so that there exists a translate of P_3 which strictly separates C_1 and C_2 and such that the normal to P_3 at 0 intersects P_1 only at 0. Then $P_2 \cap P_3$ is a subspace of dimension at least $n - 2$ and we choose an $n - j$ subspace Q in $P_2 \cap P_3$. The orthogonal complement S of Q in R^n is a j-dimensional subspace which meets P_1 in a $(j - 1)$-subspace. The projection of $C_1 \cup C_2$ onto S is the union of two polytopes. Further, as $P_3 \cap C_2 = \emptyset$, 0 is at positive distance from $\operatorname{proj} C_2$. As 0 is an extreme point of $\operatorname{proj} C_1$, it follows that 0 is a locally polyhedral extreme point for $\operatorname{proj} C_1$. Hence, in S, the cone of outward normals to $\operatorname{proj} C_1$ at 0 is j-dimensional. Further, any $(j - 1)$-plane H of support in S to $\operatorname{proj} C_1$ at 0 can be extended to an $(n - 1)$-plane of support $H + Q$ in R^n to C_1 at 0. Also, the outward normals to these planes form a j-dimensional convex cone lying in S. Hence $j = \dim(K \cap S) = \dim(P_1 \cap S) = j - 1$; contradiction. And so (i) is proved.

To prove (ii) we construct an example in R^3 of two convex bodies C_1, C_2, both of which have a countable infinity of extreme points but, nevertheless, each 2-projection of $C_1 \cup C_2$ is the union of two convex polygons. Let $l = \{x \mid x_1 = x_2 = 0, -1 \leqq x_3 \leqq 1\}$ be a line segment and $S = \{x \mid (x_1 - 1)^2 + x_2^2 = 1, x_3 = 0\}$ a plane circle. By T we denote the set of those points on S with x_2-coordinate $^{\pm}(1/n)$ for $n = 1, 2, \cdots$. We take $C_1 = \operatorname{conv}\{l \cup T\}$, which is a compact convex body in R^3 with extreme points $T \cup \{(0, 0, -1), (0, 0, 1)\}$. It is easily seen that there is precisely one 2-projection of C_1 which is not a convex polygon, and that is in the direction $(0, 0, 1)$. Further the only limit point of extreme points of this projection is $(0, 0, 0)$. Define C_2 as a disjoint copy of

C_1 formed by placing C_2 above C_1 in such a way that their respective major lines pierce the centres of their respective circles. From above, every 2-projection of $C_1 \cup C_2$ is the union of two convex polygons and and both C_1 and C_2 are compact bodies with a countable infinity of extreme points.

3. Visual hulls of more general sets. The following problem can be formulated.

Is the visual (virtual) (minimal) hull of a borel (analytic) set in R^n necessarily borel (analytic)?

The answer is affirmative (Theorem 6) for virtual hulls and negative (Theorem 7) for minimal hulls. Whilst it is not true (Theorem 8) that the j^{th} visual hull of a borel set is necessarily borel, we have been unable to decide whether or not the j^{th} visual hull of a borel or of an analytic set is always analytic, except in the cases covered by Theorem 9. It is possible also that the j^{th} visual hull of a convex borel (analytic) set is a borel (analytic) set, and we include some partial results (Theorem 9) in this direction. As before we denote by G_j^n the Grassmannian of j-subspaces of R^n and by μ_j the invariant (with respect to 0_n acting in the usual way on G_j^n) measure normalised so that $\mu_j(G_j^n) = 1$.

LEMMA 2. *Let A be an analytic set in R^n and denote by A^* the set of those j-subspaces in G_j^n which meet A. Then*

(i) *A^* is an analytic set in G_j^n and hence A^* is μ_j measurable.*

(ii) *If $\mu_j(A^*) > a$ then there exists a compact subset A' of A such that $\mu_j(A'^*) > a$.*

(iii) *If $A_1 \subset A_2 \subset \cdots$ is an increasing sequence of analytic sets in R^n then $\mu_j(\bigcup_{i=1}^\infty A_i)^* = \lim_{i\to\infty} \mu_j(A_i^*)$.*

(iv) *If $A_1 \supset A_2 \supset \cdots$ is a decreasing sequence of analytic sets in R^n then $\mu_j(\bigcap_{i=1}^\infty A_i)^* = \lim_{i\to\infty} \mu_j(A_i^*)$.*

Proof. (i) Let I be the set of irrational numbers in $[0, 1]$ and, if $i = (i_1, \cdots, i_n, \cdots)$ is a typical member of I expressed as a continued fraction, set $i \mid n = (i_1, \cdots, i_n)$. Then, as A is analytic, it can be represented as $A = \sum_{i\in I} \bigcap_{n=1}^\infty A(i \mid n)$ where the sets $A(i \mid n)$ form, for each fixed i, a decreasing sequence of compact subsets of R^n. Then $A^* = \sum_{i\in I} \bigcap_{n=1}^\infty A^*(i \mid n)$. As each $A^*(i \mid n)$ is a compact subset of G_j^n, we conclude that A^* is an analytic set.

(ii) If $\mu_j(A^*) > a + \delta$ with $\delta > 0$, then we can choose m_1, $1 \leqq m_1 < \infty$, such that if I_1 denotes the set of irrational numbers

$$i = (i_1 \cdots i_n \cdots)$$

with $1 \leqq i_1 \leqq m_1$ and $A_1^* = \sum_{i\in I_1} \bigcap_{n=1}^\infty A^*(i \mid n)$ then $\mu_j(A_1^*) > a + \delta$.

Proceeding by induction we may define natural numbers m_p, $1 \leqq p < \infty$, such that if I_q denotes the subset of those irrationals i with $1 \leqq i_p \leqq m_p$ for $p = 1, \cdots q$, and $A_q^* = \sum_{i \in I_q} \bigcap_{n=1}^{\infty} A^*(i \mid n)$ then $\mu_j(A_q^*) > a + \delta$. Let I' be the compact subset of $[0, 1]$ defined as the set of those irrational numbers i for which $1 \leqq i_p \leqq m_p$ for $p = 1, 2, \cdots$, and

$$A'^* = \sum_{i \in I'} \bigcap_{n=1}^{\infty} A^*(i \mid n) .$$

Then $\bigcap_{q=1}^{\infty} A_q^* = A'^*$ and so $\mu_j(A'^*) \geqq a + \delta > a$. Also

$$A' = \sum_{i \in I'} \bigcap_{n=1}^{\infty} A(i \mid n)$$

is a compact subset of A, as I' is a compact subset of I.

(iii) $\mu_j(\bigcup_{i=1}^{\infty} A_i)^* = \mu_j(\bigcup_{i=1}^{\infty} A_i^*) = \lim_{i\to\infty} \mu_j(A_i^*)$.

(iv) Clearly $\mu_j(\bigcap_{i=1}^{\infty} A_i)^* \leqq \lim_{i\to\infty} \mu_j(A_i^*)$. Now set $\mu_j(\bigcap_{i=1}^{\infty} A_i)^* = a$ and suppose $\lim_{i\to\infty} \mu_j(A_i^*) > a + \varepsilon$, for some positive number ε. By (ii) we find a compact set $B_1 \subset A_1$ such that $\mu_j(B_1^*) \geqq \mu_j(A_1^*) - \varepsilon/2$. Now we have $A_2^* = (B_1 \cap A_2)^* \cup (A_2^* - B_1^*)$, where

$$A_2^* - B_1^* = \{F \in G_j^n \mid F \cap A_2 \neq \emptyset , \quad \text{but} \quad F \cap B_1 = \emptyset\} .$$

Since $A_2^* \subset A_1^*$ we derive further $A_2^* \subset (B_1 \cap A_2)^* \cup (A_1^* - B_1^*)$, or $\mu_j(A_2^*) \leqq \mu_j(B_1 \cap A_2)^* + \varepsilon/2$. Since $B_1 \cap A_2$ is analytic there exists, again by (ii), a compact set $B_2 \subset (B_1 \cap A_2)$ such that

$$\mu_j(B_2)^* \geqq \mu_j(B_1 \cap A_2)^* - \varepsilon/4$$

and consequently $\mu_j(B_2)^* \geqq \mu_j(A_2)^* - (\varepsilon/2 + \varepsilon/4)$. Continuing this process we obtain a decreasing sequence $\{B_i\}_{i=1}^{\infty}$ of compact subsets of R^n such that $B_i \subset A_i$, $i = 1, 2, \cdots$, and $\mu_j(B_i^*) \geqq \mu_j(A_i^*) - \sum_{p=1}^{i} \varepsilon/(2^p)$. Then $\bigcap_{i=1}^{\infty} B_i^* = (\bigcap_{i=1}^{\infty} B_i)^* \subset (\bigcap_{i=1}^{\infty} A_i)^*$, and $\mu_j(\bigcap_{i=1}^{\infty} B_i^*) = \lim_{i\to\infty} \mu_j(B_i^*) \leqq a$; but also $\lim_{i\to\infty} \mu_j(B_i^*) \geqq \lim_{i\to\infty} \mu_j(A_i^*) - \varepsilon$. Combining the last two inequalities we find $\lim_{i\to\infty} \mu_j(A_i) \leqq a + \varepsilon$, a contradiction.

THEOREM 6. *Let C be a borel (analytic) set in R^n. Then the j^{th} virtual hull $V_j(C)$ is a borel (analytic) set.*

Proof. Suppose first that C is a borel set in R^n, and we need to show that $V_j(C)$ is a borel set. If D is a subset of R^n and $x \in R^n$, let $D[x, n-j]$ denote the set of those $n-j$ subspaces F in G_{n-j}^n such that $(x + F) \cap D \neq \emptyset$. If $0 < \lambda < 1$ let $D(n-j, \lambda)$ be the set of all x in R^n such that $\mu_{n-j}(D[x, n-j]) > \lambda$. Let B denote the largest family of subsets of R^n such that $D \in B$ if (i) D is a borel set in R^n. (ii) $D(n-j, \lambda)$ is a borel set for all λ, $0 < \lambda < 1$. We shall prove that B coincides with the family of borel subsets of R^n, and it is enough to show that B contains the open sets and is closed under the operations of increasing union and decreasing intersection. If D is an open subset of R^n, then it is easy to see that $D(n-j, \lambda)$ is open for all λ, $0 < \lambda < 1$, and so B contains all the open sets. Now suppose that $\{E_i\}_{i=1}^{\infty}$ is an increasing sequence of sets in B and set $E = \bigcup_{i=1}^{\infty} E_i$. We want to show that for each λ, $0 < \lambda < 1$, the equality $E(n-j, \lambda) = \bigcup_{i=1}^{\infty} E_i(n-j, \lambda)$ holds. In order to do this we observe the following equivalences: $x \in E(n-j, \lambda) \leftrightarrow \mu_{n-j}(E[x, n-j]) > \lambda \leftrightarrow \lim_{i\to\infty} \lambda_{n-j}(E_i[x, n-j]) > \lambda \leftrightarrow x \in \bigcup_{i=1}^{\infty} E_i(n-j, \lambda)$. Here the first equivalence holds by definition, the second one follows directly from Lemma 2, (iii), if we observe that this lemma remains true if M^* denotes, for each $M \subset R^n$, the set $M[x, n-j]$ ($x \in R^n$ fixed). (The lemma itself is stated for the special case where x is the origin of R^n.) The last equivalence again follows immediately from the definitions, we only have to observe that the sequence $\{E_i\}_{i=1}^{\infty}$ is increasing. Now suppose that $\{H_i\}_{i=1}^{\infty}$ is a decreasing sequence of subsets of B and set $H = \bigcap_{i=1}^{\infty} H_i$. Suppose λ fixed, $0 < \lambda < 1$, and let m be a natural number such that $\lambda + 1/m < 1$. Then, using (iv) of Lemma 2, we find by an argument analogous to the one above, $H(n-j, \lambda) = \bigcup_{p=m}^{\infty} \bigcap_{i=1}^{\infty} H_i(n-j, \lambda + 1/p)$. Hence $H(n-j, \lambda)$ is a borel set, and $H \in B$. Therefore, B is the family of borel subsets of R^n and so, in particular, $C \in B$. Further $V_j(C) = \bigcap_{p=2}^{\infty} C(n-j, 1-(1/p))$ and so $V_j(C)$ is a borel set.

To show that $V_j(A)$ is analytic whenever A is analytic, we use the well known result that there exists an $F_{\sigma\delta}$ set K in R^{n+1} such that A is the orthogonal projection $\operatorname{proj} K$ of K into R^n (see, for example, [8]). Call an $(n-j+1)$-subspace H of R^{n+1} upright if H has the form $\{\hat{H} + \lambda(0, \cdots, 0, 1) \mid -\infty < \lambda < \infty\}$ where $\hat{H} \in G_{n-j}^n$. Let U_{j+1} be the set of upright $(n-j+1)$-subspaces in R^{n+1} with the measure μ' induced by μ_{n-j} in the obvious manner. We can define $U_{j+1}(C)$ of a set C in R^{n+1} as the set of all those points x in R^{n+1} such that almost all (with respect to μ') upright $(n-j+1)$-flats through x meet C. As above, it can been shown that $U_{j+1}(C)$ is a borel set whenever C is a borel set. Clearly $\operatorname{proj} U_{j+1}(K) = V_j(A)$ and, since the projection of a borel set is analytic, we conclude that $V_j(A)$ is an analytic subset of R^n.

THEOREM 7. *Let C be an open convex subset of R^n. Then assuming the continuum hypothesis, C contains a minimal j^{th} hull D such that every analytic subset of D is countable.*[1]

Proof. We assume the continuum hypothesis and let Ω be the

[1] As the referee pointed out, Theorem 7 may be a special case of a much more general theorem on effective constructions.

first uncountable ordinal. Let $\{A_\xi\}_{\xi<\Omega}$ be an enumeration of the analytic subsets of R^n of $(n-j)$-dimensional measure zero; let $\{H_\xi\}_{\xi<\Omega}$ be an enumeration of the $(n-j)$-flats which meet C. Let F be a fixed $(n-j)$-subspace of R^n and denote by α a fixed set, which is not a point of R^n. We now choose a set $E = \{M_\xi\}_{\xi<\Omega}$ and a collection of translates $\{F_\xi\}_{\xi<\Omega}$ of F inductively as follows. Take $M_1 \in (H_1 - A_1) \cap C$ and let F_1 be a translate of F through M_1. Suppose now that $M_{\xi'}$, $F_{\xi'}$ have been defined for all $\xi' < \xi$, where ξ is some ordinal proceeding Ω. If H_ξ is a translate of F we take $F_\xi = H_\xi$ and consider two possibilities:

(a) If $\exists \xi' < \xi$ such that $M_{\xi'} \in H_\xi$ then we take $M_\xi = \alpha$.

(b) If $\exists \xi' < \xi$ such that $M_{\xi'} \in H_\xi$ we choose M_ξ in the set $(H_\xi - (\bigcup_{\xi'<\xi} H_{\xi'} \cup \bigcup_{\xi'<\xi} A_{\xi'})) \cap C$. Such a choice is possible as $H_\xi \cap C$ has positive $(n-j)$-dimensional measure whereas $H_\xi \cap (\bigcup_{\xi'<\xi} H_{\xi'} \cup \bigcup_{\xi'<\xi} A_{\xi'})$ has zero $(n-j)$-dimensional measure, being a countable union of sets of measure zero. If H_ξ is not a translate of F we find, by similar arguments, that the set $(H_\xi - (\bigcup_{\xi'<\xi} H_{\xi'} \cup \bigcup_{\xi'<\xi} A_{\xi'} \cup \bigcup_{\xi'<\xi} F_{\xi'})) \cap C$ is not empty. We choose M_ξ in this set and let F_ξ be the translate of F through M_ξ. We claim that the set $D = E - \alpha$ is a j^{th} minimal hull for C which meets each analytic subset in at most a countable number of points. To show that all j^{th} projections of D coincide with those of C, it is enough to show that the j^{th} visual hull of D contains C. Let x be a point of C and let P be an $(n-j)$-flat through x. Then P is amongst $\{H_\xi\}_{\xi<\Omega}$, say $P = H_{\xi'}$. If $M_{\xi'} \neq \alpha$ then $M_{\xi'} \in D \cap H_{\xi'}$. If $M_{\xi'} = \alpha$ then $\exists M_{\xi''}$, $\xi'' < \xi'$, such that $M_{\xi''} \in D \cap H_{\xi'}$. In either case P meets D and so $x \in H_j(D)$.

If D is not minimal then there exists M_ξ, $\xi < \Omega$, such that

$$H_j(D - M_\xi) = C\,.$$

But, projecting C and $D - M_\xi$ onto the orthogonal complement of F we see that by construction $\operatorname{proj} C \cap \operatorname{proj} F_\xi \neq \varnothing$, but $\operatorname{proj}(D - M_\xi) \cap \operatorname{proj} F_\xi = \varnothing$. Hence D is a j^{th} minimal hull for C. Finally, suppose that B is an uncountable analytic subset of D. If B has positive j-dimensional measure then it is possible to find an uncountable analytic subset of B of zero j-dimensional measure. Hence it can be supposed that B has zero j-dimensional measure and so $B = A_\xi$ for some $\xi < \Omega$. But $A_\xi = A_\xi \cap D \subset \bigcup_{\xi'<\xi} M_{\xi'}$, which is countable; contradiction.

Of course, if G is an open or compact set in R^n then $H_j(G)$ will accordingly be an open or compact set. Apart from these cases it does not seem entirely trivial to determine the nature of $H_j(G)$ for a given subset G of R^n. Here we prove the following

THEOREM 8. (i) *There exists, in the plane R^2, a borel set C such that $H_1(C)$ is analytic but not borel.*

(ii) *If D is an F_σ-subset of R^n then $H_j(D)$ is the complement of an analytic set.*

REMARKS. We note that by (i) if C is analytic then $H_1(C)$ is not necessarily the complement of an analytic set. To disprove the statement that whenever A is analytic then $H_j(A)$ is analytic, it would be enough, using (ii), to find an F_σ-subset D of R^n such that $H_j(D)$ is not borel. (Notice that, a subset, M of R^n is borel if and only if M and $R^n - M$ are both analytic. Compare, for example, [5]).

Proof. (i) As already observed, every analytic set in R^1 can be represented as the projection into R^1 of some $F_{\sigma\delta}$ set in R^2. Let A be an analytic subset of R^1 such that A is not a borel set and let B be an $F_{\sigma\delta}$ set in R^2 such that $\operatorname{proj} B = A$. Take C to be the union of B and the "y-axis" $(R^1)^\perp$. Then it is easily seen that $H_1(C)$ is the union of all lines which are parallel to $(R^1)^\perp$ and contain a point of C. However this is not a borel set as $H_1(C) \cap R^1 = A \cup \{(0, 0)\}$ is not a borel set.

(ii) We define a complete separable metric space Ω, whose points are the $(n-j)$-flats of R^n, as follows. For each $(n-j)$-flat F in R^n let y be the nearest point of F to 0 and set $F \cap (S^{n-1} + y) = \hat{F}$. Then the distance $\rho(F, F')$ of two $(n-j)$-flats in Ω is defined as the Hausdorff distance of $\hat{F}, \hat{F}'$ in R^n. Let $D \subset R^n$ be an F_σ set, say $D = \bigcup_{i=1}^\infty D_i$ with $D_i \subset D_{i+1}$, each D_i compact, $i = 1, 2, \cdots$. Let D_i^*, $i = 1, 2 \cdots$ denote the closed subsets of Ω such that $F \in D_i^*$ if F meets D_i in R^n. Similarly defined, relative to D, is D^*. Then $D^* = \bigcup_{i=1}^\infty D_i^*$ and so D^* is an F_σ subset of Ω. Hence $\Omega - D^*$ is a G_δ set and so, in particular, $\Omega - D^*$ is an analytic subset of Ω. Set

$$\Omega - D^* = \sum_{i \in I} \bigcap_{p=1}^{\infty} A(i \mid p) ,$$

where the $A(i \mid p)$, $p = 1, 2, \cdots$, form a decreasing sequence of compact subsets of Ω, for each $i \in I$. Set

$$B_m = \{x \mid x \in R^n, \ -m \leqq x_i \leqq m, \ i = 1, \cdots, n\} .$$

Let $K_m(i \mid p)$ be the closed subset of B_m such that $x \in K_m(i \mid p)$ if x is contained in an $(n-j)$-flat F with $F \in A(i \mid p)$. Similarly, we define $K_m \subset B_m$ relative to $\Omega - D^*$. Then $K_m = \sum_{i \in I} \bigcap_{p=1}^\infty K_m(i \mid p)$ is an analytic subset of R^n and so, therefore, is $K = \bigcup_{m=1}^\infty K_m$. We claim that $H_j(D) = R^n - K$. If $x \in K$ then $x \in K_m$ for some m and so x is contained in some $(n-j)$-flat F which is contained (in Ω) in some set $\bigcap_{p=1}^\infty A(i \mid p)$. Hence $F \in \Omega - D^*$ which means that F does not meet D; i.e., $x \notin H_j(D)$. Therefore $R^n - K \supset H_j(D)$. Conversely if $x \notin H_j(D)$ then there exists an $(n-j)$-flat F through x such that F does not meet D. Hence $F \in \Omega -$

D^* and so $F \in \bigcap_{p=1}^{\infty} A(i \mid p)$ for some $i \in I$. Hence $x \in \bigcap_{p=1}^{\infty} K_m(i \mid p)$ for some positive integer m, i.e., $x \in K$. Therefore $R^n - K \subset H_j(D)$ and so $H_j(D) = R^n - K$ is the complement of the analytic set K.

DEFINITION. An irregular point x of some closed convex set C in R^3 is an extreme point x of C such that x lies in two distinct 1-faces l_1, l_2 of C, with neither of l_1, l_2 being contained in a 2-face of C. Let C be a closed subset of a simple closed curve in the plane OXY. We say that a set $B \subset C \times (-\infty, \infty)$ is vertically convex if every line which is perpendicular to OXY meets B in a (possibly empty) line segment. We shall make use of the following immediate corollary to a theorem of K. Kunugui [7].

LEMMA 3. (*Kunugui*) *Let B be a vertically convex borel set in $C \times (-\infty, \infty)$. Then the projection of B into C is a borel set.*

As an immediate consequence of Lemma 3, we have

LEMMA 4. *Let B be a vertically convex borel subset of some vertically convex closed subset D in $C \times (-\infty, \infty)$. Then the set $D \cap \{(\text{proj.}\, B) \times (-\infty, \infty)\}$ is a vertically convex borel set.*

In [9] the authors have derived properties of visual hulls for the class of convex sets. Our contribution in this direction is

THEOREM 9. (i) *If C is a convex borel (analytic) set in R^3 then $H_2(C)$ is a borel (analytic) set.*

(ii) *If C is a convex borel (analytic) set in R^3 and $\bar{C}$ does not have irregular points then $H_1(C)$ is a borel (analytic) set.*

Proof. (i) We first show that if C is a convex borel (analytic) set in R^2 then $H_1(C)$ is a borel (analytic) set. If $\dim C = 1$ then the result is trivial and so it can be supposed that $\dim C = 2$. Note that $C^0 \subset H_1(C) \subset \bar{C}$. Let the 1-faces of $\bar{C}$ be $\{F_i\}_{i=1}^{\infty}$. Then

$$H_1(C) \cap (\bar{C} - \bigcup_{i=1}^{\infty} F_i) = C - \bigcup_{i=1}^{\infty} F_i\ ,$$

which is a borel set. Let $\{F_{i_\nu}\}_{\nu=1}^{\infty}$ be the 1-faces of $\bar{C}$ which meet C. Then relint $F_{i_\nu} \subset H_1(C) \cap F_{i_\nu}$, $\nu = 1, 2, \cdots$. The two endpoints of F_{i_ν} may, or may not, be in $H_1(C)$. Nevertheless, $H_1(C)$ differs from the borel set $(C - \bigcup_{i=1}^{\infty} F_i) \cup \bigcup_{\nu=1}^{\infty}$ relint F_{i_ν} by at most a countable number of points. And so $H_1(C)$ is a borel set. Similarly, if C is a convex analytic set in R^2, then $H_1(C)$ is an analytic set. Suppose now that C is a convex borel set in R^3. If $\dim C \leqq 2$ then $H_2(C) = C$, and so it can be supposed that $\dim C = 3$. Let $\{F_i\}_{i=1}^{\infty}$ be an enumeration of the 2-faces of $\bar{C}$. Then each F_i is closed and $H_2(C) \cap (\bar{C} - \bigcup_{i=1}^{\infty} F_i) = C \cap (\bar{C} - \bigcup_{i=1}^{\infty} F_i)$, which is a borel set. As $H_2(C) \subset \bar{C}$, it is now enough to show that $H_2(C) \cap F_i$ is a borel set for $i = 1, 2, \cdots$. Let $H_1'(C \cap F_i)$ denote the first visual hull of $C \cap F_i$ relative to aff F_i. Then, from above, $H_1'(C \cap F_i)$ is a borel set. Let $\{F_{i_j}\}_{j=1}^{\infty}$ be an enumeration of the 1-faces of F_i. Then $H_2(C) \cap (F_i - \bigcup_{j=1}^{\infty} F_{i_j}) = H_1'(C \cap F_i) - \bigcup_{j=1}^{\infty} F_{i_j}$ which is a borel set K_i, say. Let $\{F_{i_{j_\nu}}\}_{\nu=1}^{\infty}$ be the 1-faces of F_i which meet C and have the property that the only plane of support to $\bar{C}$ which contains $F_{i_{j_\nu}}$ is aff F_i. Then relint $F_{i_{j_\nu}} \subset H_2(C)$ and the end points of $F_{i_{j_\nu}}$ may or may not be in $H_2(C)$. Hence $H_2(C) \cap F_i$ differs from the borel set $K_i \cup (\bigcup_{\nu=1}^{\infty} \text{relint } F_{i_{j_\nu}}) \cup (\bigcup_{j=1}^{\infty} (F_{i_j} \cap C))$ by at most a countable number of points. Therefore $H_2(C) \cap F_i$ is a borel set, and so, therefore, is $H_2(C)$. Similarly, it can be shown that if C is a convex analytic set in R^3 then $H_2(C)$ is an analytic set.

(ii) Again we shall prove the result for convex borel sets, and indicate at the end the modifications required for convex analytic sets. Let $\{r_i\}_{i=1}^{\infty}$ be an enumeration of the rational numbers and let P_{ik} denote the 2-flat $\{x \mid x_k = r_i\}$ $k = 1, 2, 3$; $i = 1, 2, \cdots$. For each i, j, k, let $B(i, j, k)$ denote the closed set formed by the point set union of all maximal line segments in $\bar{C} - C^0$ which meet both both P_{ik} and P_{jk}. Let $\{G_m\}_{m=1}^{\infty}$ be the 2-faces of $\bar{C}$. If a 2-face G_m of $\bar{C}$ meets $B(i, j, k)$ then G_m meets $C_i (C_i = (\bar{C} - C^0) \cap P_{ik})$ and $C_j (C_j = (\bar{C} - C^0) \cap P_{jk})$ in line segments 1_{im} and 1_{jm} respectively. Let $1_m^1, 1_m^2$ denote the (at most) two maximal line segments in G_m such that each segment contains an endpoint of 1_{im} and 1_{jm} but 1_m^1 and 1_m^2 do not intersect except possibly at end points. Set $C^* = (\bar{C} - C^0) \cap P$, where P is a plane parallel to P_{ik} and lying strictly between P_{ik} and P_{jk}. Then G_m cuts C^* in an interval I_m. Let 1_m denote the subinterval of I_m with endpoints $1_m^1 \cap C^*, 1_m^2 \cap C^*$, and let 1_m^0 be the relative interior of 1_m. Then

$$C' = B(i, j, k) \cap \left(C^* - \bigcup_{m=1}^{\infty} 1_m^0\right)$$

is a closed subset of C^*. If $x \in C'$, let $\hat{x}$ denote the unique maximal line segment in $B(i, j, k)$ which passes through x and meets C_1 and C_2. Let X denote the closed set formed by the point set union of the line segments $\hat{x}$, $x \in C'$, and set $Q(i, j, k) = \{y \mid y \in X, \exists x \in C', \hat{x} \cap C \neq \emptyset, y \in \hat{x}\}$. We now show that $Q(i, j, k)$ is a borel set. Every point y of X can be given a coordinate vector $y = \langle x, h \rangle$, where $y \in \hat{x}$ and h is the height, relative to the j^{th} coordinate, of y above C^*. Because $\bar{C}$ does not have irregular points, the number of points y in X which receive two different coordinate vectors is countable. Let Φ be the mapping $X \to C^* \times (-\infty, \infty)$ defined by taking $\Phi\langle x, h \rangle = (x, h)$, $x \in C'$. Then K is a borel subset of X if and only if $\Phi(K)$ is a borel subset of the closed set $\Phi(X)$. Hence $\Phi(C \cap X)$ is a vertically convex borel subset of $C' \times (-\infty, \infty)$. Hence the set $D = X \cap \{\text{proj}\, \Phi(C \cap X) \times (-\infty, \infty)\}$ is a convex borel set and so $Q(i, j, k) = \Phi^{-1}(D)$ is a borel set. Hence the set $R(i, j, k) = Q(i, j, k) - \bigcup_{m=1}^{\infty} G_m$ is a borel set. Consider now the set $S = \bigcup_{i,j,k} R(i, j, k)$ and consider the borel set T defined as the point set union of all 1-faces of $\bar{C}$ which are not contained in some 2-face of $\bar{C}$. We assert that the set $H_1^1(C) = H_1(C) \cap (T - \bigcup_{m=1}^{\infty} G_m)$ equals S. For if $y \in H_1^1(C)$ then, because $\bar{C}$ does not have any irregular points, there exists a unique 1-face l, not contained in $\bigcup_{m=1}^{\infty} G_m$, such that $y \in l$. Then $y \in H_1(C)$ if and only if $l \cap C = \emptyset$, which happens if and only if $l \subset Q(i, j, k)$ or in other words $y \in R(i, j, k)$ for some i, j, k. Hence $H_1^1(C) = S$. Let V denote the borel set of exposed points of $\bar{C}$ and $H_1^2(C) = V \cap H_1(C)$, $H_1^3(C) = \bigcup_{m=1}^{\infty} (H_1(C) \cap (G_m - V))$. Now $H_1(C) = H_1^1(C) \cup H_1^2(C) \cup H_1^3(C)$. $H_1^1(C) = S$ is a borel set and, since $H_1^2(C) = V \cap C$, $H_1^2(C)$ is a borel set. Hence it is enough to show that $H_1(C) \cap (G_m - V)$ is a borel set for all m. Now let $\{G_{m_\nu}\}_{\nu=1}^{\infty}$ be those 2-faces of $\bar{C}$ which meet C. Then relint $G_{m_\nu} \subset H_1^3(C)$ for all ν. Let $\{G_{m_\nu n}\}_{n=1}^{\infty}$ be the 1-faces of G_{m_ν}. Then either relint $G_{m_\nu n} \subset H_1^3(C)$ or relint $G_{m_\nu n} \cap H_1^3(C) = \emptyset$. Then the endpoints of $G_{m_\nu n}$ may or may not be in $H_1^3(C)$. Let H_{m_ν} be the countable set of those endpoints of $\{G_{m_\nu n}\}_{\nu=1}^{\infty}$ which lie in $H_1^3(C)$ and let $\{G_{m_\nu n_\mu}\}_{\mu=1}^{\infty}$ be the 1-faces of G_{m_ν} whose relative interiors are contained in $H_1^3(C)$. We have $G_{m_\nu} \cap H_1^3(C) =$ relint $G_{m_\nu} \cup (\bigcup_{\mu=1}^{\infty}$ relint $G_{m_\nu n_\mu}) \cup H_{m_\nu}$, which is a borel set. If, on the other hand, a 2-face of $\bar{C}$ does not meet C, its intersection with $H_1^3(C)$ is empty. Therefore $H_1^3(C) \cap G_m$ is a borel set for all m, and $H_1(C)$ is a borel set.

For the case when C is an analytic set, say $C = \sum_{i \in I} \bigcap_{n=1}^{\infty} C(i \mid n)$ in the usual representation, the only modification required to the above proof is to show that the set $Q(i, j, k)$ is an analytic set. With the previous notation, $Q(i \mid n) = \{y \mid y \in X, \exists x \in C', \hat{x} \cap C(i \mid n) \neq \emptyset, y \in \hat{x}\}$. Then $Q(i \mid n)$ is a closed set and $Q(i, j, k) = \sum_{i \in I} \bigcap_{n=1}^{\infty} Q(i \mid n)$. Therefore $Q(i, j, k)$ is an analytic set.

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