

OUTER GALOIS THEORY FOR SEPARABLE ALGEBRAS

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Let G be a finite group of automorphisms of a ring A which has identity element. Let C be the center of A , let Γ be the subring of G -invariant elements of A , and assume that C is a separable extension of $C \cap \Gamma$. In the first section of this paper, it is shown that every finite group of automorphisms of A over Γ is faithfully represented as a group of automorphisms of C by restriction if, and only if, $A = C \otimes_{C \cap \Gamma} \Gamma$. Moreover, suppose that $A = C \otimes_{C \cap \Gamma} \Gamma$ and Ω is a subring of A such that $\Gamma \subseteq \Omega \subseteq A$. Then there exists a finite group H of automorphisms of A such that Ω is the subring of H -invariant elements of A if, and only if, $C \cap \Omega$ is a separable extension of $C \cap \Gamma$ and $\Omega = (C \cap \Omega) \otimes_{C \cap \Gamma} \Gamma$.

Let R be a commutative ring with identity element; and assume now that A is a separable algebra over R and G is a finite group of automorphisms of the R -algebra A . In the second section of this paper, it is shown that C is the centralizer of Γ in A if, and only if, $A = C \otimes_{C \cap \Gamma} \Gamma$. Moreover, suppose that $A = C \otimes_{C \cap \Gamma} \Gamma$ and Ω is a subalgebra of A such that $\Gamma \subseteq \Omega \subseteq A$. Then there exists a finite group H of automorphisms of A such that Ω is the subalgebra of H -invariant elements of A if, and only if, Ω is a separable algebra over R .

These results are obtained without the assumption of no non-trivial idempotent elements of C , which is required for the Kanzaki-DeMeyer Galois theory of separable algebras. Moreover, these results extend the Villamayor-Zelinsky Galois theory of commutative rings in the same way that the results of Kanzaki and DeMeyer extend the Chase-Harrison-Rosenberg Galois theory of commutative rings.

1. Galois theory. Throughout this paper, ring will mean ring with identity element and subring of a ring will mean subring which contains the identity element of the ring. Let Γ be a subring of a ring A . Call A a projective Frobenius extension of Γ if A is a finitely generated, projective right Γ -module and there is a (Γ, A) -bimodule isomorphism of A onto $\text{Hom}_{\Gamma}(A, \Gamma)$. Call A a separable extension of Γ if the (A, A) -bimodule epimorphism of $A \otimes_{\Gamma} A$ onto A , which is determined by the ring multiplication in A , splits. Equivalently, A is a separable extension of Γ if there exist a positive integer n and elements x_i, y_i of A , for $1 \leq i \leq n$, such that $\sum_{i=1}^n x_i y_i = 1$ and $\sum_{i=1}^n a x_i \otimes y_i = \sum_{i=1}^n x_i \otimes y_i a$ in $A \otimes_{\Gamma} A$ for every $a \in A$. Also, let M be a left A -module and let N be a Γ -submodule of M . A canonical A -module homomorphism ϕ of $A \otimes_{\Gamma} N$ into M is determined by the

correspondence of $a \cdot x$ to $a \otimes x$ for $a \in A$ and $x \in N$. It will be convenient to write $M = A \otimes_r N$ when ϕ is an isomorphism.

Let G be a finite group of automorphisms of a ring A , and let Γ be the subring of G -invariant elements of A . Call A a Galois extension of Γ relative to G if there exist a positive integer n and elements x_i, y_i of A , $1 \leq i \leq n$, such that $\sum_{i=1}^n x_i \cdot \sigma(y_i) = \delta_{1,\sigma}$ for all $\sigma \in G$. If A is a Galois extension of Γ relative to G , then A is a separable extension of Γ by [8, Proposition 1.3]. Let C be the center of A . If A is a Galois extension of Γ relative to G and C is the centralizer of Γ in A , call A an outer Galois extension of Γ relative to G . A generalization of the concept of outer Galois extension is that of outer semi-Galois extension given in [7, Definition 2.4]. A will be called an outer semi-Galois extension of Γ if, in addition to the assumptions stated at the beginning of this paragraph, A is a separable extension of Γ and C is the centralizer of Γ in A . Finally, we note that, if S is a G -stable subring of A ; then a homomorphism of G onto a finite group \bar{G} of automorphisms of S is obtained by restricting each element of G to S , and $S \cap \Gamma$ is the subring of \bar{G} -invariant elements of S .

For the remainder of the paper, let G be a finite group of automorphisms of a ring A , let Γ be the subring of G -invariant elements of A , and let C be the center of A .

THEOREM 1.1. *If S is a G -stable subring of C such that S is a separable extension of $S \cap \Gamma$, then the following statements are equivalent.*

- (i) $C = S \otimes_{S \cap \Gamma} (C \cap \Gamma)$ and $A = C \otimes_{C \cap \Gamma} \Gamma$.
- (ii) $A = S \otimes_{S \cap \Gamma} \Gamma$.
- (iii) *An isomorphism of the group of all automorphisms of A over Γ for which S is stable onto the group of all automorphisms of S over $S \cap \Gamma$ is obtained by restricting each automorphism of A to S .*
- (iv) *Every finite group of automorphisms of A over Γ for which S is stable is faithfully represented as a group of automorphisms of S by restriction.*

Proof. It is evident that statement (i) implies statement (ii). If $A = S \otimes_{S \cap \Gamma} \Gamma$, then every automorphism of S over $S \cap \Gamma$ may be extended to an automorphism of A over Γ and the identity map on A is the only automorphism of A over Γ which restricts to the identity map on S . With these observations it is easily verified that statement (ii) implies statement (iii). Clearly statement (iii) implies statement (iv), and it only remains to verify that statement (iv) implies statement (i). Since S is a commutative ring and a separable

extension of $S \cap \Gamma$, S is an outer semi-Galois extension of $S \cap \Gamma$. Let e be an idempotent element of S and σ be an element of G such that $\sigma(e \cdot a) = e \cdot a$ for all $a \in S$. An automorphism $\bar{\sigma}$ of A over Γ is defined by the rule $\bar{\sigma}(a) = \sigma(e \cdot a) + (1 - e) \cdot a$ for $a \in A$. Let H be the group of automorphisms of A over Γ which is generated by $\bar{\sigma}$. Since G is a finite group, σ has finite order. Therefore $\bar{\sigma}$ has finite order and H is a finite group. Moreover, each element of S is H -invariant; and H is faithfully represented as a group of automorphisms of S by restriction, only if $\bar{\sigma}(a) = a$ and, hence, $\sigma(e \cdot a) = e \cdot a$ for all $a \in A$. Since C is stable for any group of automorphisms of A , the following lemma may be applied with $T = C$ to establish that statement (iv) implies statement (i).

LEMMA 1.2. *Let S, T be G -stable subrings of A such that $S \subseteq T$ and S is an outer semi-Galois extension of $S \cap \Gamma$. Assume that whenever e is a central idempotent of S and σ is an element of G such that $\sigma(e \cdot a) = e \cdot a$ for all $a \in S$, then $\sigma(e \cdot a) = e \cdot a$ for all $a \in A$. Then $T = S \otimes_{S \cap \Gamma} (T \cap \Gamma)$ and $A = T \otimes_{T \cap \Gamma} \Gamma$.*

Proof. By hypothesis, S is a separable extension of $S \cap \Gamma$. Let n be a positive integer and let x_i, y_i be elements of S for $1 \leq i \leq n$, such that $\sum_{i=1}^n x_i y_i = 1$ and $\sum_{i=1}^n a x_i \otimes y_i = \sum_{i=1}^n x_i \otimes y_i a$ in $S \otimes_{S \cap \Gamma} S$ for every $a \in S$. Setting $e_\sigma = \sum_{i=1}^n x_i \cdot \sigma(y_i)$, $a \cdot e_\sigma = e_\sigma \cdot \sigma(a)$ for $a \in S$ and $\sigma \in G$. Therefore e_σ is an element of the centralizer of $S \cap \Gamma$ in S , which is the center of S , for $\sigma \in G$. Moreover

$$e_\sigma^2 = \sum_{i=1}^n e_\sigma \cdot x_i \cdot \sigma(y_i) = \sum_{i=1}^n x_i \cdot e_\sigma \cdot \sigma(y_i) = \sum_{i=1}^n x_i \cdot y_i \cdot e_\sigma = e_\sigma$$

for $\sigma \in G$. Thus $\{\sigma(e_\tau) \mid \sigma, \tau \in G\}$ is a finite set of central idempotents in S , and it generates a finite, G -stable subalgebra E of the Boolean algebra of all central idempotents in S . Letting M be the set of minimal elements in E ; M is a finite, G -stable set of pairwise orthogonal idempotents such that $\sum_{e \in M} e = 1$. For $e \in M$ and $\sigma \in G$, let $e\sigma$ denote the mapping $a \mapsto e \cdot \sigma(a)$, $a \in A$; and let

$$N = \{e\sigma \mid e \in M, \sigma \in G\}.$$

The mapping $\alpha \mapsto \sigma \circ \alpha$, $\alpha \in N$, is a permutation on N for each $\sigma \in G$. Consequently, letting γ be the sum of the distinct elements of N ; γ is a left $S \cap \Gamma$ -module endomorphism of A , the image of which must be contained in Γ . Since S and T are G -stable, γ must map S into $S \cap \Gamma$ and T into $T \cap \Gamma$. If, for $e \in M$ and $\sigma \in G$,

$$\sum_{i=1}^n x_i \cdot e \cdot \sigma(y_i) = e \cdot e_\sigma = e_\sigma \cdot \sigma(e)$$

is not zero; then $e = e \cdot e_\sigma = e_\sigma \cdot \sigma(e) = \sigma(e)$ since e and $\sigma(e)$ are minimal elements of E , $\sigma(e \cdot a) = e_\sigma \cdot \sigma(e \cdot a) = e \cdot a \cdot e_\sigma = e \cdot a$ for all $a \in S$, and by hypothesis $\sigma(e \cdot a) = e \cdot a$ for all $a \in A$. Therefore $\sum_{i=1}^n x_i \cdot \gamma(y_i a) = \sum_{e \in M} e \cdot a = a$ for all $a \in A$. It is now readily verified that the canonical left S -module homomorphism of $S \otimes_{S \cap \Gamma} \Gamma$ into A has an inverse which is the mapping $a \mapsto \sum_{i=1}^n x_i \otimes \gamma(y_i a)$, $a \in A$; and the canonical left S -module homomorphism of $S \otimes_{S \cap \Gamma} (T \cap \Gamma)$ into T has an inverse which is the mapping $a \mapsto \sum_{i=1}^n x_i \otimes \gamma(y_i a)$, $a \in T$. Thus $A = S \otimes_{S \cap \Gamma} \Gamma$ and $T = S \otimes_{S \cap \Gamma} (T \cap \Gamma)$. Since $S \otimes_{S \cap \Gamma} \Gamma$ is naturally isomorphic to $S \otimes_{S \cap \Gamma} (T \cap \Gamma) \otimes_{T \cap \Gamma} \Gamma$, $A = T \otimes_{T \cap \Gamma} \Gamma$.

If S has no central idempotents other than 0 and 1, then the hypotheses of Lemma 1.2 are equivalent to the requirements that S and T be G -stable subrings of A such that $S \subseteq T$, S be an outer semi-Galois extension of $S \cap \Gamma$, and G be faithfully represented as a group of automorphisms of S by restriction. The following example, however, shows that in general the conclusion of Lemma 1.2 cannot be obtained if only these latter conditions are assumed.

EXAMPLE 1.3. Let A be the ring of all complex 3×3 matrices

of the form $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & g \end{pmatrix}$, and set $\sigma \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & g \end{pmatrix} = \begin{pmatrix} a & -b & 0 \\ -c & d & 0 \\ 0 & 0 & \bar{g} \end{pmatrix}$ and $\tau \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & g \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} & 0 \\ \bar{c} & \bar{d} & 0 \\ 0 & 0 & \bar{g} \end{pmatrix}$. σ and τ are automorphisms of A , and they generate a sub-

group G of order four in the group of all automorphisms of A . The subring Γ of G -invariant elements of A consists of all real, diagonal 3×3 matrices, and the center C of A consists of all complex, diagonal 3×3 matrices of the form $\text{diag} \{a, a, b\}$. Take $S = T = C$. C is a commutative G -stable subring of A and G is faithfully represented as a group of automorphisms of C by restriction. Moreover it may be verified that C is a Galois extension of $C \cap \Gamma$ with respect to the group H of automorphisms of C generated by the restriction of τ to

C , but $A \neq C \otimes_{C \cap \Gamma} \Gamma$. In fact, setting $\phi \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & g \end{pmatrix} = \begin{pmatrix} a & -b & 0 \\ -c & d & 0 \\ 0 & 0 & g \end{pmatrix}$, ϕ is a nontrivial automorphism of A over Γ which restricts to the identity map on C .

The remaining results of this section are directed toward developing a Galois theory for a ring A which satisfies any of the four equivalent statements of Theorem 1.1.

LEMMA 1.4. *If S is a G -stable subring of C such that S is a separable extension of $S \cap \Gamma$ and $A = S \otimes_{S \cap \Gamma} \Gamma$, then A is an outer semi-Galois extension of Γ and Γ is a (Γ, Γ) -bimodule direct summand of A .*

Proof. Let S be a G -stable subring of C such that S is a separable extension of $S \cap \Gamma$ and $A = S \otimes_{S \cap \Gamma} \Gamma$. Then one may readily verify that A is a separable extension of Γ and the centralizer of Γ in A is C . Therefore A is an outer semi-Galois extension of Γ . Furthermore, since S is a commutative ring, S is an outer semi-Galois extension of $S \cap \Gamma$; and, by [7, Th. 3.2], S is a projective Frobenius extension of $S \cap \Gamma$. In particular, S is a finitely generated, projective module over $S \cap \Gamma$; and it follows from [1, Proposition A.3] and [9, Proposition 1] that $S \cap \Gamma$ is an $S \cap \Gamma$ -module direct summand of S . Therefore Γ is a (Γ, Γ) -bimodule direct summand of $A = S \otimes_{S \cap \Gamma} \Gamma$.

LEMMA 1.5. *Let \bar{G} be a finite group of automorphisms of a commutative ring S , let R be the subring of \bar{G} -invariant elements of S , and assume that S is a separable extension of R . For an intermediate ring $T, R \subseteq T \subseteq S$, the following statements are equivalent.*

- (i) *There exists a finite group H of automorphisms of S such that T is the subring of H -invariant elements of S .*
- (ii) *S is a projective Frobenius extension of T .*
- (iii) *T is a separable extension of R .*

Proof. Apply Lemma 1.4 with $A = S = C$ and $\Gamma = R$ to establish that S is an outer semi-Galois extension of R and R is an R -module direct summand of S . The equivalence of statements (i) and (ii) follows from [7, Th. 3.3]. But it is a consequence of [7, Th. 2.3] and [10, 3.15] that S is a weakly Galois R -algebra, and the equivalence of statements (i) and (iii) follows from [10, Th. 3.8].

THEOREM 1.6. *Let S be a G -stable subring of C such that S is a separable extension of $S \cap \Gamma$ and $A = S \otimes_{S \cap \Gamma} \Gamma$; and let Ω be a subring of A such that $\Gamma \subseteq \Omega \subseteq A$. There exists a finite group H of automorphisms of A such that S is H -stable and Ω is the subring of H -invariant elements in A if, and only if, $S \cap \Omega$ is a separable extension of $S \cap \Gamma$ and $\Omega = (S \cap \Omega) \otimes_{S \cap \Gamma} \Gamma$.*

Proof. Suppose H is a finite group of automorphisms of A such that S is H -stable and Ω is the subring of H -invariant elements in A . Then $S \cap \Omega$ is the subring of H -invariant elements in S , and

$S \cap \Omega$ is a separable extension of $S \cap \Gamma$ by Lemma 1.5. Also by Lemma 1.5, S is a finitely generated, projective module over $S \cap \Omega$; and it follows from [1, Proposition A.3] and [7, Lemma 1.6] that S is a faithfully flat module over $S \cap \Omega$. Since $A = S \otimes_{S \cap \Gamma} \Gamma$ and $\Gamma \subset \Omega$, every group of automorphisms of A over Ω for which S is stable is faithfully represented as a group of automorphisms of S by restriction. Therefore $A = S \otimes_{S \cap \Omega} \Omega$ by Theorem 1.1. But $A = S \otimes_{S \cap \Omega} (S \cap \Omega) \otimes_{S \cap \Gamma} \Gamma$ also. Since S is a faithfully flat module over $S \cap \Omega$, $\Omega = (S \cap \Omega) \otimes_{S \cap \Gamma} \Gamma$.

Conversely, suppose $S \cap \Omega$ is a separable extension of $S \cap \Gamma$ and $\Omega = (S \cap \Omega) \otimes_{S \cap \Gamma} \Gamma$. By Lemma 1.5, there exists a finite group H of automorphisms of S such that $S \cap \Omega$ is the subring of H -invariant elements in S . Since $A = S \otimes_{S \cap \Gamma} \Gamma$, there is a unique extension of H to a group of automorphisms of A over Γ . Let Ω' be the subring of H -invariant elements in A . $\Gamma \subseteq \Omega'$; and, by the first part of this proof, $\Omega' = (S \cap \Omega') \otimes_{S \cap \Gamma} \Gamma$. But $S \cap \Omega'$ is the subring of H -invariant elements in S , so $S \cap \Omega' = S \cap \Omega$ and

$$\Omega' = (S \cap \Omega') \otimes_{S \cap \Gamma} \Gamma = (S \cap \Omega) \otimes_{S \cap \Gamma} \Gamma = \Omega .$$

If S is a G -stable subring of C such that S is a separable extension of $S \cap \Gamma$ and $A = S \otimes_{S \cap \Gamma} \Gamma$; then $C = S \otimes_{S \cap \Gamma} (C \cap \Gamma)$ and $A = C \otimes_{C \cap \Gamma} \Gamma$ by Theorem 1.1, and C is a separable extension of $C \cap \Gamma$ by [2, Corollary 1.6]. Since C is stable for any group of automorphisms of A , S may be replaced by C in the preceding considerations. The following corollary is stated for comparison with Lemma 1.5.

COROLLARY 1.7. *Assume that C is a separable extension of $C \cap \Gamma$ and every finite group of automorphisms of A over Γ is faithfully represented as a group of automorphisms of C by restriction. For a subring Ω of A such that $\Gamma \subseteq \Omega \subseteq A$, the following statements are equivalent.*

- (i) *There exists a finite group H of automorphisms of A such that Ω is the subring of H -invariant elements of A .*
- (ii) *A is a projective Frobenius extension of Ω .*
- (iii) *$C \cap \Omega$ is a separable extension of $C \cap \Gamma$ and*

$$\Omega = (C \cap \Omega) \otimes_{C \cap \Gamma} \Gamma .$$

Proof. Since every finite group of automorphisms of A over Γ is faithfully represented as a group of automorphisms of C by restriction, $A = C \otimes_{C \cap \Gamma} \Gamma$ by Theorem 1.1. Therefore A is an outer semi-Galois extension of Γ and Γ is a (Γ, Γ) -bimodule direct summand of A by Lemma 1.4. Statements (i) and (ii) are equivalent by [7, Th. 3.3], and statements (i) and (iii) are equivalent by Theorem 1.6.

2. Separable algebras. In this section, let A be an algebra over a commutative ring R and let G be a finite group of automorphisms of the R -algebra A . Let Γ be the subalgebra of G -invariant elements of A and let C be the center of A . The results of the preceding section may be sharpened if A is a separable algebra over R . Thus let A be a separable algebra over R . Then A is a separable extension of C and C is a separable algebra over R by [2, Th. 2.3]. Clearly $R \cdot 1 \subseteq C \cap \Gamma \subseteq \Gamma$; and, consequently, A is a separable extension of Γ and C is a separable extension of $C \cap \Gamma$.

PROPOSITION 2.1. *If A is a separable extension of C and C is the centralizer of Γ in A , then the group of all automorphisms of A over Γ is faithfully represented as a group of automorphisms of C by restriction.*

Proof. Assume that A is a separable extension of C and C is the centralizer of Γ in A ; but suppose that the group of all automorphisms of A over Γ is not faithfully represented as a group of automorphisms of C by restriction, and let η be a nontrivial automorphism of A over Γ which restricts to the identity map on C . Let a be an element of A such that $\eta(a) \neq a$, let m be a maximal ideal of C which contains the set $\{x \in C \mid x \cdot (\eta(a) - a) = 0\}$, and let C_m be the quotient ring of C with respect to the multiplicative system $C - m$. $A \otimes_C C_m$ is a central separable algebra over $C \otimes_C C_m = C_m$ by [2, Corollary 1.6], and $\eta \otimes 1$ is an automorphism of $A \otimes_C C_m$ over C_m . Since C_m is a local ring, $\eta \otimes 1$ is an inner automorphism by [2, Th. 3.6 and the remark which follows it]. Let $w \otimes 1/s$, $w \in A$ and $s \in C - m$, be a unit in $A \otimes_C C_m$ such that $w \cdot \eta(x) \otimes 1/s = xw \otimes 1/s$ for all $x \in A$. A is a finitely generated module over C by [2, Th. 2.1]; so let n be a positive integer and $\{b_i \in A \mid 1 \leq i \leq n\}$ be a set of generators for the C -module A . Since $w \cdot \eta(b_i) \otimes 1/s = b_i \cdot w \otimes 1/s$; there exists $t_i \in C - m$ such that $t_i \cdot (w \cdot \eta(b_i) - b_i \cdot w) = 0$, $1 \leq i \leq n$, by [3, § 2, No. 2, Proposition 4]. Letting $t = \prod_{i=1}^n t_i$, it is easily verified that $t \in C - m$ and $tw \cdot \eta(x) = xtw$ for all $x \in A$. Therefore tw is an element of the centralizer of Γ in A , which is C ; $w \otimes 1/s = tw \otimes 1/(ts)$ is a unit in the center of $A \otimes_C C_m$; and, consequently, $\eta(x) \otimes 1 = x \otimes 1$ for all $x \in A$. In particular $\eta(a) \otimes 1 = a \otimes 1$; and, by [3, § 2, No. 2, Proposition 4], there exists $u \in C - m$ such that $u \cdot (\eta(a) - a) = 0$. But such an element u cannot exist by the choice of m , and the proposition follows from this contradiction.

COROLLARY 2.2. *If A is a separable algebra over R , then the following statements are equivalent.*

- (i) C is the centralizer of Γ in A .

- (ii) A is an outer semi-Galois extension of Γ .
- (iii) $A = C \otimes_{C \cap \Gamma} \Gamma$.

Proof. Assume that A is a separable algebra over R . Then A is a separable extension of Γ , and therefore statements (i) and (ii) are equivalent. Moreover, A is a separable extension of C and C is a separable extension of $C \cap \Gamma$. It follows from Proposition 2.1 and Theorem 1.1 that statement (i) implies statement (iii). Clearly statement (iii) implies statement (i).

THEOREM 2.3. *Let A be a separable algebra over R such that $A = C \otimes_{C \cap \Gamma} \Gamma$; and let Ω be a subalgebra of A such that $\Gamma \subseteq \Omega \subseteq A$. There exists a finite group H of automorphisms of A such that Ω is the subalgebra of H -invariant elements of A if, and only if, Ω is a separable algebra over R .*

Proof. A is an outer semi-Galois extension of Γ and Γ is a (Γ, Γ) -bimodule direct summand of A by Lemma 1.4. Since A is a separable algebra over R , A is a (A, A) -bimodule direct summand of $A \otimes_R A$; and thus Γ is a (Γ, Γ) -bimodule direct summand of $A \otimes_R A$. As a (Γ, Γ) -bimodule, $A \otimes_R A$ is a left module over the enveloping algebra $\Gamma^e = \Gamma \otimes_R \Gamma^0$ of Γ ; and for any left Γ^e -module X there is a natural isomorphism of $\text{Hom}_{\Gamma^e}(A \otimes_R A, X)$ onto $\text{Hom}_{(\Gamma, \Gamma)}(A, \text{Hom}(A, X))$. But A is a projective Frobenius extension of Γ by [7, Th. 3.2]; and, therefore, A is projective as either a left or right Γ -module. Consequently, $A \otimes_R A$ must be a projective left Γ^e -module. Therefore Γ is a projective left Γ^e -module, and it follows that Γ is a separable algebra over R .

By Theorem 1.6, there exists a finite group H of automorphisms of A such that Ω is the subring of H -invariant elements of A if, and only if, $C \cap \Omega$ is a separable extension of $C \cap \Gamma$ and $\Omega = (C \cap \Omega) \otimes_{C \cap \Gamma} \Gamma$. But if $C \cap \Omega$ is a separable extension of $C \cap \Gamma$ and $\Omega = (C \cap \Omega) \otimes_{C \cap \Gamma} \Gamma$, then one may readily verify that Ω is a separable algebra over R . Conversely, suppose Ω is a separable algebra over R . Since $R \cdot 1 \subseteq C \cap \Gamma$, Γ and Ω are separable extensions of $C \cap \Gamma$; and, since C is the centralizer of Γ in A , $C \cap \Gamma$ is the center of Γ while $C \cap \Omega$ is both the centralizer of Γ in Ω and the center of Ω . But then $C \cap \Omega$ is a separable extension of $C \cap \Gamma$ by [2, Th. 2.3], and $\Omega = (C \cap \Omega) \otimes_{C \cap \Gamma} \Gamma$ by [2, Th. 3.1].

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