PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 1, 1970

BOUNDS FOR THE SOLUTIONS OF A CERTAIN CLASS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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This paper is a study of boundedness and other properties of the solutions of nonlinear partial differential equations of the form

$$\Delta u = P(x_1, x_2, \cdots, x_n)f(u) \tag{1.1}$$

where $P(x_1, x_2, \cdots, x_n)$ is positive, and $u(x_1, x_2, \cdots x_n)$ is to be defined in some region of Euclidean n-space, and $\Delta u = \sum_{i=1}^{n} \partial^2 u/\partial x_i^2$ is the Laplacian of u. In particular, we consider the case $f(u) = e^u$.

Our principal result is concerned with the nonexistence of entire solutions. An entire solution $u = u(x_1, x_2, \cdots, x_n)$ will be defined as a solution which though continuous for $0 \leqq r < \infty$ is twice continuously differentiable for $0 < r < \infty$. Other results are concerned with the general form of and explicit bounds for solutions.

In the literature on the subject [3, 4, 5, 8, 9, 11, 12] conditions have been given on $f(u)$ in order that the equation

$$\Delta u = f(u) \tag{1.2}$$

or, more generally, the differential inequality

$$\Delta u \geqq f(u) \tag{1.3}$$

will have no solutions $u = u(x_1, x_2, \cdots, x_n)$ having two continuous derivatives for all finite values of $x_1, x_2, \cdots, x_n$. The most general conditions which exclude such solutions, obtained by Keller [5], are: $f(u) > 0$, $f'(u) \geqq 0$ for $-\infty < u < \infty$ and

$$\int_0^\infty \left[\int_0^u f(t)dt\right]^{-1/2} du < \infty .$$

For $n = 2$ Redheffer [10] showed that the monotonicity of $f(u)$ may be dispensed with.

In § 2 we shall consider a more general question for the equation

$$\Delta u = P(x, y)e^u,\ P(x, y) > 0,\ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} . \tag{1.4}$$

While the coefficient $P(x, y)$ will be assumed to be positive and twice continuously differentiable for $0 < r < \infty$, $P(x, y)$ will be permitted to vanish or to become singular in a manner specified in the statement of the Theorem 2.1. If $P(x, y)$ has such a singularity it will, of course, be reflected in the singular behaviour of the solutions of (1.4). We shall thus give conditions on $P(x, y)$ which exclude entire solutions of (1.4). An example of such a solution is $u = r$ which solves equation (1.4) with $P(x, y) = e^{-r}/r$.

For $n = 2$ it is well known that the function

$$u(z, \bar{z}) = \log\frac{|f'(z)|}{1 - |f(z)|^2} \tag{1.5}$$

is a solution of

$$\varDelta u = 4e^{2u} \tag{1.6}$$

if $f(z)$ is an analytic function satisfying $|f(z)| < 1$ and $|f(z)| \neq 0$ in the domain considered. In § 3 we show, conversely, that every solution of (1.6) is essentially of this form. This converse result is necessary if it desired to use (1.5) and the theory of bounded analytic functions to obtain general properties of the regular solutions of (1.6). If the solution $u(z, \bar{z})$ of (1.6) is regular in a disk $|z| < R$, Theorem 3.1 leads to a bound for u in this disk. If $|f(z)| < 1$ in $|z| < R$ then, by Schwarz' lemma $|f'(z)|/1 - |f(z)|^2 \leqq R/R^2 - |z|^2$. Hence, a solution of (1.6) which is regular for $|z| < R$ is subject to the inequality.

$$u(z, \bar{z}) \leqq \log\frac{R}{R^2 - |z|^2} .$$

For $z = 0$, this leads, in particular, to the well known fact that the equation (1.6) can not have twice continuously differentiable solutions.

In § 4 comparison theorems are proved and explicit bounds are obtained for the solutions of

$$\varDelta u = P(r)f(u) \tag{1.7}$$

or, more generally

$$\varDelta u \geqq P(r)f(u) . \tag{1.8}$$

The behaviour of these solutions at an isolated singularity is investigated.

2. Entire solutions.

The main result is:

THEOREM 2.1. *Let*

$$\iint_{r<r_0} P(x, y)dxdy = O(r_0) \qquad \text{(for small } r_0) \tag{2.1}$$

and

$$\int_0^r t\sigma(t)dt = O(r^\varepsilon)\ , \qquad \varepsilon > 0 \tag{2.2}$$

where

$$\sigma(r) = \frac{1}{2\pi}\int_0^{2\pi} \Delta(\log P)d\theta\ . \tag{2.3}$$

If either

$$\int^\infty e^{(1-\beta)g(r)} r^{c-1} (\log r)^{1-3\beta} dr = \infty \tag{2.4}$$

or

$$\int^\infty e^{(1-\beta)g(r)} r^{(1-2\beta)+\varepsilon^2-\varepsilon/2} (\log r)^{-\beta-\varepsilon} dr = \infty \tag{2.4$'$}$$

where

(i) *c is a constant such that $c = (2-\varepsilon)(1-\beta)$ where $1/2 < \beta < 1$ and $\varepsilon > 0$ but small. And*

(ii) *the function $g(r)$ is a solution of*

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dg}{dr}\right) = \frac{1}{2\pi}\int_0^{2\pi} \Delta\,(\log P)d\theta$$

such that $rg'(r) \to 0$ as $r \to 0$.

Then (1.4) cannot have a solution which is twice continuously differentiable for $0 < r < \infty$ and continuous for $0 \leqq r < \infty$.

That such solutions of (1.4) may exist for certain $P(x, y)$ is shown by the example $u = r^n$, $n \geqq 2$ where $P(x, y) = n^2 r^{n-2} e^{-r^n}$.

Proof. If we set

$$u = v - \log P \tag{2.5}$$

equation (1.4) becomes

$$\Delta v = e^v + \Delta(\log P)\ . \tag{2.6}$$

We introduce the notation

$$\omega(r) = \frac{1}{2\pi}\int_0^{2\pi} v(r, \theta)d\theta\ . \tag{2.7}$$

By Green's formula for the circle $|z| \leqq r < R$

$$\iint_{|z|\leqq r} \Delta v\, dxdy = \int_{|z|=r} \frac{\partial v}{\partial n} ds$$

where n is the exterior normal. On account of $\partial/\partial n = \partial/\partial r$ it follows that

$$\int_0^r\int_0^{2\pi} \Delta v r d\theta dr = \int_0^{2\pi} \frac{\partial v}{\partial r} r d\theta = r\frac{\partial}{\partial r}\int_0^{2\pi} v(r, \theta) d\theta .$$

With the help of (2.6) and (2.7), this yields

$$r\frac{d}{dr}\omega(r) = \frac{1}{2\pi}\int_0^r\int_0^{2\pi}(e^v + \Delta(\log P)) r d\theta dr . \tag{2.8}$$

$\omega(r)$ is single valued and twice continuously differentiable for $r < R$. Because of (2.3) and (2.5), (2.8) is equivalent to

$$\frac{r d\omega(r)}{dr} = \frac{1}{2\pi}\int_0^r\int_0^{2\pi} P(x, y) e^u r d\theta dr + \int_0^r t\sigma(t) dt . \tag{2.9}$$

Since u is continuous, it follows from assumption (2.1) and (2.2) that

$$r\omega'(r) \longrightarrow 0 \tag{2.10}$$

as $r \to 0$.

Differentiating (2.8) with respect to r and using (2.3), we obtain

$$\frac{1}{r}\frac{d}{dr}\Big(d\frac{d\omega}{dr}\Big) = \sigma(r) + \frac{1}{2\pi}\int_0^{2\pi} e^v d\theta . \tag{2.11}$$

Since e^ξ is convex for all ξ, the right hand side of (2.11) can be estimated by

$$\frac{1}{2\pi}\int_0^{2\pi} e^{v(r,\theta)} d\theta \geqq e^{1/2\pi}\int_0^{2\pi \int_0^{2\pi} v(r,\theta) d\theta} = e^{\omega(r)} .$$

Hence (2.11) yields

$$\frac{d}{dr}\Big(r\frac{d\omega}{dr}\Big) \geqq r\sigma(r) + re^{\omega(r)} . \tag{2.12}$$

We now set

$$\omega(r) = g(r) + f(r) \tag{2.13}$$

where $g(r)$ is a solution of

$$\frac{d}{dr}\Big(r\frac{dg}{dr}\Big) = r\sigma(r)$$

which is continuous at the origin; that is, we compute $g(r)$ from

$$r\frac{d}{dr}(g(r)) = \int_0^r t\sigma(t) dt . \tag{2.14}$$

Because of our assumption on the behaviour of $\sigma(r)$ at $r = 0$, $g(r)$ will be continuous at $r = 0$. Inequality (2.12) then takes the from

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) \geqq r\tau(r)e^{f} \tag{2.15}$$

where $\tau(r) = e^{g(r)}$. Introducing the new independent variable by $\rho = \log r$ and setting

$$F = f + 2\rho \tag{2.16}$$

inequality (2.15) yields

$$\ddot{F} \geqq \tau(\rho)e^{F} \tag{2.17}$$

where dot denotes the differentiation with respect to ρ. Since the right hand side of (2.17) is always positive $F(\rho)$ is convex in ρ therefore, $\omega(r)$ is convex in $\log r$.

Now suppose (1.4) and, therefore, also (2.17) has entire solutions.

We observe that $\dot{F}(\rho)$ must be positive for all ρ in $(-\infty, \infty)$. Indeed, from (2.16), we get, $\dot{F}(\rho) = 2 + e^{\rho}(df(e^{\rho}))/dr$. Since by (2.14) and the assumption (2.2), $g'(r) = O(r^{\varepsilon-1})$ we have, $\lim_{r\to 0} rg'(r) = 0$. Hence, by (2.10) and (2.13) $\lim_{r\to 0} r\omega'(r) = \lim_{r\to 0} rf'(r) = 0$. It follows, therefore, that $\lim_{\rho\to-\infty} \dot{F}(\rho) = 2$. But, by (2.17) $F(\rho)$ is convex in ρ and we have, consequently,

$$\dot{F}(\rho) \geqq 2 \tag{2.18}$$

throughout $(-\infty, \infty)$. It, therefore, follows that $F(\rho)$ is ultimately positive. We choose ρ_0 large enough so that $F(\rho) > 0$ for $\rho > \rho_0$ and set

$$\phi = F\dot{F}\,. \tag{2.19}$$

Differentiating with respect to ρ and using (2.17) we have

$$\dot{\phi}\phi^{-\gamma} \geqq \tau F^{1-\gamma}e^{F}\dot{F}^{-\gamma} + F^{-\gamma}\dot{F}^{2-\gamma} \tag{2.20}$$

where γ is a constant to be chosen later.

Using the inequality [Hardy-Littlewood-Polya] $A + B \geqq (A/\alpha)^{\alpha}(B/\beta)^{\beta}$ where $\alpha + \beta = 1$, $0 \leqq \alpha$, $\beta \leqq 1$. the inequality (2.20) yields

$$\dot{\phi}\phi^{-\gamma} \geqq \tau^{1-\beta}(1-\beta)^{\beta-1}\beta^{-\beta}e^{(1-\beta)F}F^{1-\beta-\gamma}\dot{F}^{2\beta-\gamma}\,. \tag{2.21}$$

Now we consider two cases:

Case I. Let $2\beta - \gamma = 0$, $1/2 < \beta < 1$. Then the inequality (2.21) becomes

$$\dot{\phi}\phi^{-2\beta} \geqq C_1\tau^{1-\beta}e^{(1-\beta)F}F^{1-3\beta} \tag{2.22}$$

where $c_1 = (1-\beta)^{\beta-1}\beta^{-\beta}$. Since $\dot{F} \geqq 2$ we have $F \geqq (2-\varepsilon)\rho$ if ρ is sufficiently large. Moreover, since $e^{(1-\beta)F}F^{(1-3\beta)}$ is increasing for $F > 3\beta - 1/1 - \beta$, inequality (2.22) yields

$$\dot{\phi}\phi^{-2\beta} \geqq c_2\tau^{1-\beta}\rho^{1-3\beta}e^{c\rho}$$

provided $(2-\varepsilon)\rho > 3\beta - 1/1 - \beta$, $c_2 = c_1(2-\varepsilon)^{1-3\beta}$ and $c = (2-\varepsilon)(1-\beta)$. Integration of (2.22) gives

$$(2.23)\qquad \frac{1}{2\beta-1}\left[\frac{1}{\phi^{2\beta-1}(\rho_0)} - \frac{1}{\phi^{2\beta-1}(\rho)}\right] \geqq c_2\int_{e^{\rho_0}}^{e^{\rho}} e^{(1-\beta)g(r)}r^{c-1}(\log r)^{1-3\beta}dr\ .$$

Since F is convex and increasing in ρ, $\phi^{1-2\beta}(\rho)$ tends to zero as $\rho \to \infty$. Hence, the left hand side of (2.23) is bounded as $\rho \to \infty$. This contradicts the assumption (2.4).

Hence the inequality (2.17) and also (1.4) does not have entire solutions.

Case II. Let $2\beta - \gamma > 0$, $1/2 < \beta < 1$. The inequality (2.21) becomes in this case

$$\dot{\phi}\phi^{-\gamma} \geqq c_1\tau^{1-\beta}F^{1-\beta-\gamma}e^{(1-\beta)F}2^{2\beta-\gamma}$$

where we have used (2.18). But since

$$F^{1-\beta-\gamma}e^{(1-\beta)F} > e^{(1-\beta)(2-\varepsilon)\rho}\{(2-\varepsilon)\rho\}^{1-\beta-\gamma}$$

provided $(2-\varepsilon)\rho > (\gamma+\beta-1)(1-\beta)^{-1}$, we have

$$\dot{\phi}\phi^{-\gamma} \geqq c_1 2^{(2\beta-\gamma)}\tau^{1-\beta}e^{(1-\beta)(2-\varepsilon)\rho}[\rho(2-\varepsilon)]^{1-\beta-\gamma}\ .$$

Choose $\gamma = 1+\varepsilon$, $\varepsilon > 0$. Then $\beta > (1+\varepsilon)/2$. Therefore, integration with respect to ρ gives

$$(2.24)\qquad \frac{1}{\varepsilon}\left[\frac{1}{\phi^{\varepsilon}(\rho_0)} - \frac{1}{\phi^{\varepsilon}(\rho)}\right] \geqq c_3\int^{\rho} e^{(1-\beta)g(r)}r^{(1-2\beta)+(\varepsilon^2-\varepsilon/2)}(\log r)^{-\beta-\varepsilon}dr$$

where $c_3 = c_1(2-\varepsilon)^{-\beta-\varepsilon}$.

If it were true that $u = u(x, y)$ is entire, the left-hand side of (2.24) would remain bounded as $\rho \to \infty$. Since by (2.4)$'$ the right hand side of (2.24) is unbounded, this leads to a contradiction.

This completes the proof of Theorem 2.1.

3. General solution. Let $u(x, y)$ be of class C^2 in the region D of x, y-plane and satisfy (1.6). Introducing the new independent variables $z = x + iy$ and $\bar{z} = x - iy$ equation (1.6) becomes

$$(3.1)\qquad u_{z\bar{z}} = e^{2u}$$

where $\partial/\partial z = 1/2(\partial/\partial x - i(\partial/\partial y))$ and $\partial/\partial \bar{z} = 1/2(\partial/\partial x + i(\partial/\partial y))$. How we prove

THEOREM 3.1. *Every solution of* (1.6) *which is twice continuously differentiable in a given region D can be written in the form*

$$u(z, \bar{z}) = \log \frac{|f'(z)|}{1 - |f(z)|^2}$$

where $f(z)$ is analytic in D such that $|f'(z)| \neq 0$ and $|f(z)| < 1$.

Proof. According to an observation which goes back to Bieberbach [1] a regular solution of (1.6) can be associated with an analytic function of z in the following manner: We set

$$Q = u_{zz} - u_z^2$$

where u is a solution of (1.6) or, equivalently, of (3.1) and we compute $Q_{\bar{z}}$. We have, with the help of (3.1), $Q_{\bar{z}} = 0$. Thus, Q is found to satisfy the Cauchy-Riemann equations. Since Q is continuous, it must therefore be regular analytic function $\omega(z)$.

If we set

$$\psi = \bar{e}^{u} \tag{3.2}$$

and observe that

$$\psi_{zz} = \bar{e}^{u}(u_z^2 - u_{zz})$$

we find that ψ is a solution of the linear differential equation

$$\psi_{zz} + \omega(z)\psi = 0\,. \tag{3.3}$$

Since $\omega(z)$ is analytic in z the general solution of (3.3) contains the analytic solutions of the equation

$$F''(z) + \omega(z)F(z) = 0 \tag{3.3$'$}$$

because, for an analytic F, we have $F'(z) = \partial F/\partial z$. The general solution of (3.3) can, therefore, be written in the form

$$\psi = A^*\psi_1(z) + B^*\psi_2(z)$$

where ψ_1 and ψ_2 are two linearly independent (analytic) solutions of (3.3)$'$ which may be assumed to be normalized by

$$\psi_1\psi_2' - \psi_2\psi_1' = 1 \tag{3.4}$$

and A^* and B^* are constants with respect to $\partial/\partial z$ — differentiation used in (3.3) i.e., $\partial A^*/\partial z = \partial B^*/\partial z = 0$. Since these are Cauchy-

Riemann equations for functions in $\bar{z}$ we have $A^* = \overline{A(z)}$, $B^* = \overline{B(z)}$ where A and B are analytic. The general solution of (3.3) is, therefore, found to be of the form

$$\psi = \overline{A(z)}\psi_1(z) + \overline{B(z)}\psi_2(z) \tag{3.5}$$

where A, B, ψ_1 and ψ_2 are analytic functions in D. In view of (3.2), equation (3.5) can be written

$$\bar{e}^u = \bar{A}(z)\psi_1(z) + \bar{B}(z)\psi_2(z) \, . \tag{3.6}$$

Now the proof of the theorem will follow from the following lemma:

LEMMA 3.1. *Let ψ_1 and ψ_2 be linearly independent solutions of the differential equation* (3.3)$'$ *where $\omega(z)$ is analytic in D. If $A(z)$ and $B(z)$ are analytic in D and if the expression*

$$K(z, \bar{z}) = \bar{A}(z)\psi_1(z) + \bar{B}(z)\psi_2(z) \tag{3.7}$$

is real throughout D but does not vanish identically then $K(z, \bar{z})$ can be written in the form

$$K(z, \bar{z}) = \pm|\sigma(z)|^2 \mp |\tau(z)|^2$$

where $\sigma(z)$ and $\tau(z)$ are two linearly independent solutions of (3.3)$'$ *for which*

$$\tau(z)\sigma'(z) - \sigma(z)\tau'(z) = 1 \, . \tag{3.8}$$

Proof. Since $K(z, \bar{z})$ is real, we have

$$\bar{A}(z)\psi_1(z) + \bar{B}(z)\psi_2(z) = A(z)\overline{\psi_1(z)} + B(z)\overline{\psi_2(z)} \, . \tag{3.9}$$

Differentiation with respect to z and (3.4) give

$$\bar{\psi}_1(z)[\psi_1'(z)A(z) - \psi_1(z)A'(z)] + \bar{\psi}_2(z)[\psi_1'(z)B(z) - B'(z)\psi_1(z)] = -\bar{B}(z) \, .$$

Setting

$$g(z) = \psi_1'(z)A(z) - \psi_1(z)A'(z) \tag{3.10}$$

and

$$h(z) = \psi_1'(z)B(z) - \psi_1(z)B'(z) \tag{3.11}$$

we have

$$\psi_1(z)\bar{g}(z) + \psi_2(z)\bar{h}(z) = -B(z) \, . \tag{3.12}$$

But the left-hand side of (3.12) is a solution of (3.3)$'$; hence $(-B(z))$ satisfies

$$B_{zz} + \omega(z)B = 0$$

where $\omega(z)$ is an analytic function. But since $B(z)$ is analytic in z,

$$B''(z) + \omega(z)B(z) = 0 ,$$

consequently, B is of the form

$$B(z) = \alpha\psi_1(z) + \beta\psi_2(z) \tag{3.13}$$

where α and β are constants. Arguing in the same manner (3.4) and (3.9) give

$$A(z) = \gamma\psi_1(z) + \delta\psi_2(z) \tag{3.14}$$

where γ and δ are constants.

Also from (3.12) and (3.13), $\psi_1(z)/\psi_2(z) = -\overline{((h(z) + \beta)/(g(z) + \alpha))}$. But since $\psi_1(z)/\psi_2(z)$ is analytic in z and, moreover, since ψ_1 and ψ_2 are linearly independent, we must have $\bar{g}(z) + \bar{\alpha} \equiv 0$ and $\bar{h}(z) + \bar{\beta} \equiv 0$, or equivalently

$$(\gamma\psi_1 + \delta\psi_2)\psi_1' - (\gamma\psi_1' + \delta\psi_2')\psi_1 = -\bar{\alpha} \tag{3.15}$$

and

$$(\alpha\psi_1 + \beta\psi_2)\psi_1' - (\alpha\psi_1' + \beta\psi_2')\psi_1 = -\bar{\beta} \tag{3.16}$$

respectively. With the help of (3.12), (3.14), (3.15) and (3.16) the equation (3.7) becomes

$$K(z, \bar{z}) = \gamma\,|\,\psi_1\,|^2 + \beta\,|\,\psi_2\,|^2 + \bar{\alpha}\bar{\psi}_1\psi_2 + \alpha\bar{\psi}_2\psi_1 \,. \tag{3.17}$$

Now let $\sigma(z)$ and $\tau(z)$ be any other solutions of (3.3)$'$ such that $\psi_1(z) = a\sigma(z) + b\tau(z)$ and $\psi_2(z) = c\sigma(z) + d\tau(z)$ where a, b, c and d are constants satisfying

$$ad - bc = 1 \tag{3.18}$$

and

$$b(\gamma\bar{a} + \alpha\bar{c}) + d(\bar{c}\beta + \bar{a}\bar{\alpha}) = 0 \,. \tag{3.19}$$

This is possible if the determinant

$$D = \gamma\,|\,a\,|^2 + \beta\,|\,c\,|^2 + 2Re(a\alpha\bar{c})$$

does not vanish. Evidently this can always be achieved as long as not all numbers α, β and γ are zero. However α, β and γ cannot all be zero since, in view of (3.17) $K(z, \bar{z})$ would then be identically zero, and this case is excluded.

Substituting ψ_1 and ψ_2 in (3.17) and using (3.19) we obtain

$$K(z, \bar{z}) = |\sigma(z)|^2\{\gamma|a|^2 + \beta|c|^2 + \bar{a}c\bar{\alpha} + a\bar{c}\alpha\} \tag{3.20}$$
$$+ |\tau(z)|^2\{\gamma|b|^2 + \beta|d|^2 + \bar{b}d\bar{\alpha} + b\bar{d}\alpha\} .$$

Now we consider the following two cases:

Case I. Let $\beta \neq 0$, $\gamma \neq 0$. We set $a \neq 0$ and $c = 0$ then, with the help of (3.18) and (3.19), (3.20) becomes

$$K(z, \bar{z}) = |\sigma(z)|^2\gamma|a|^2 + |\tau(z)|^2|d|^2\gamma^{-1}(\beta\gamma - |\alpha|^2) .$$

(i) Let $\gamma > 0$, $\beta\gamma - |\alpha|^2 = m$ (m is a positive integer). Hence,

$$K(z, \bar{z}) = |\sigma^*|^2 + |\tau^*|^2$$

where $\sigma^* = \sigma(\gamma|a|^2)^{1/2}$ and $\tau^* = \tau m^{1/2}(\gamma|a|^2)^{-1/2}$ are solutions of (3.3)$'$.

(ii) $\gamma > 0$, $\beta\gamma - |\alpha|^2 = -m$. In this case

$$K(z, \bar{z}) = |\sigma^*|^2 - |\tau^*|^2 .$$

(iii) Let $\gamma < 0$, $\beta\gamma - |\alpha|^2 = m$. Then

$$K(z, \bar{z}) = -|\sigma^*|^2 - |\tau^*|^2$$

(iv) $\gamma < 0$, $\beta\gamma - |\alpha|^2 = -m$. This gives

$$K(z, \bar{z}) = -|\sigma^*|^2 + |\tau^*|^2 .$$

Case II. Let $\beta = 0$, $\gamma = 0$. We set $a, b \neq 0$. With this choice (3.18) and (3.19) reduce (3.20) to

$$K(z, \bar{z}) = -|\sigma_1|^2 + |\tau_1|^2$$

where $|\sigma_1| = |\sigma|(\bar{a}\bar{\alpha})^{1/2}b^{-1/2}$ and $|\tau_1| = a^{-1/2}|\tau|(\bar{\alpha}\bar{\beta})^{1/2}$ and are solutions of (3.3)$'$.

Summing up, we have thus proved that, if the function $K(z, \bar{z})$ is real, it must have either of the three following forms

$$\left.\begin{array}{lr} (1) & K(z, \bar{z}) = |\tau|^2 - |\sigma|^2 \\ (2) & K(z, \bar{z}) = |\tau|^2 + |\sigma|^2 \\ (3) & K(z, \bar{z}) = -|\tau|^2 - |\sigma|^2 \end{array}\right\}(S)$$

where σ and τ are solutions of the differential equation (3.3)$'$ normalized by (3.8). The case $K(z, \bar{z}) = |\sigma|^2 - |\tau|^2$ is evidently not essentially different from case (1). Case (3) can be excluded immediately, since beacuse of (3.6) and (3.7) $K(z, \bar{z})$ must be positive. This also shows that, in case (1), we necessarily must have

$$|\tau(z)| > |\sigma(z)|. \tag{3.21}$$

We now define

$$f(z) = \frac{\sigma(z)}{\tau(z)} \, . \tag{3.22}$$

In view of (3.8) we have

$$f'(z) = \frac{1}{\tau^2(z)} \tag{3.23}$$

and thus $|\sigma|^2 + |\tau|^2 = (1 + |f(z)|^2)/|f'(z)|$ in case (2) and $|\tau|^2 - |\sigma|^2 = (1 - |f(z)|^2)/|f'(z)|$ in case (1). Comparing this with (3.6), (3.7) and (S) we find that $u(z, \bar{z})$ must be either of the forms

$$u(z, \bar{z}) = \log \frac{|f'(z)|}{1 - |f(z)|^2}$$

$$u(z, \bar{z}) = \log \frac{|f'(z)|}{1 + |f(z)|^2}$$

$$u(z, \bar{z}) = \log \frac{1 + |f(z)|^2}{|f'(z)|} \, .$$

Since the last two functions are not solutions of (1.6), these cases are excluded. Hence any real solution of (1.6) must be of the from

$$u(z, \bar{z}) = \log \frac{|f'(z)|}{1 - |f(z)|^2}$$

where because of (3.21) and (3.22) $|f(z)| < 1$ and in view of (3.23) $|f'(z)| \neq 0$.

This completes the proof of Theorem 3.1.

4. Bounds for the solutions of $\Delta_u \geqq P(r)f(u)$.

Let

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

denote the n-dimensional Laplace operator and let D_r and S_r stand for the open sphere $x_1^2 + x_2^2 + \cdots + x_n^2 < r^2$ and its boundary

$$x_1^2 + x_2^2 + \cdots + x_n^2 = r^2$$

respectively. We are concerned here with functions

$$\omega = \omega(Q)(Q \in D_r, \, 0 < r < R)$$

which are of class C^2 in D_r and satisfy the differential equation

$$\Delta\omega = P(r)F(\omega)$$

or, more generally, the differential inequality

$$\Delta\omega \geqq P(r)F(\omega) \, . \tag{4.1}$$

Nehari [6] found explicit bounds for the solutions of the differential equation $\Delta u = F(u)$ or, more generally the differential inequality $\Delta u \geqq F(u)$ which are regular in a disk. We shall obtain here a more general result, which also applies to certain equations of the form (4.1).

LEMMA 4.1. *Let $F(t)$ and $G(t)$ be positive and differentiable functions for $-\infty < t < \infty$ and such that the integrals*

$$\int_{\omega}^{\infty}\frac{dt}{F(t)}\,,\quad \int_{v}^{\infty}\frac{dt}{G(t)}$$

exist, and let $\omega = \omega(x_1, x_2, \cdots, x_n)$ and $v = v(x_1, x_2, \cdots x_n)$ be two functions related by the identity

$$\int_{\omega}^{\infty}\frac{dt}{F(t)} = \int_{v}^{\infty}\frac{dt}{G(t)}\,. \tag{4.2}$$

Then

$$\frac{\Delta\omega}{F(\omega)} \geqq \frac{\Delta v}{G(v)} \tag{4.3}$$

provided $F'(\omega) \geqq G'(v)$.

Proof. We write x for one of the variables $x_1, x_2, \cdots, x_n$ and differentiate (4.2) twice with respect to x. This yields

$$-\frac{v_x}{G(v)} = -\frac{\omega_x}{F(\omega)}$$

$$-\frac{v_{xx}}{G(v)} + \frac{v_x^2 G'(v)}{G^2(v)} = -\frac{\omega_{xx}}{F(\omega)} + \frac{v_x^2 F'(\omega)}{G^2(v)}\,.$$

Summing over all x_n and using the fact that $F'(\omega) \geqq G'(v)$, we get (4.3).
We derive the following corollary.

COROLLARY 5.1. *If $v = v(x_1, x_2, \cdots, x_n)$ is a function satisfying the differential inequality*

$$\Delta v \leqq Pv^k\,, \qquad k > 1 \tag{4.4}$$

where $P = P(x_1, x_2, \cdots, x_n)$ is positive, and if $F(u)$ is such that

$$F'(u)\int_{u}^{\infty}\frac{dt}{F(t)} \leqq \frac{k}{k-1} \tag{4.5}$$

then, the function u defined by

$$\frac{1}{(k-1)v^{k-1}} = \int_{u}^{\infty}\frac{dt}{F(t)} \tag{4.6}$$

is subject to the inequality

$$\Delta u \leqq PF(u) . \tag{4.7}$$

Setting $G(v) = v^k$ in Lemma 4.1, the proof of the Corollary 4.1 is immediate.

As an application of Corollary 4.1, we prove the following result.

THEOREM 4.1. *If the function* $\omega = \omega(x_1, x_2, \cdots, x_n)$ *satisfies the inequality*

$$\Delta\omega \geqq r^2 F(\omega) \tag{4.8}$$

where $F(\omega)$ *is such that* $F'(\omega)\int^{\infty} dt/F(t) \leqq 9/8$ *and* $F'(\omega) \geqq 0$ *then the function* u *defined by*

$$\frac{(r^2 - \rho^2)^2(R^2 - r^2)^2}{20R^4} = \int_u^{\infty} \frac{dt}{F(t)} \qquad 0 < \rho < r < R$$

is such that

$$\omega \leqq u .$$

Proof. Consider the function v defined by

$$v = \frac{1}{(r^2 - \rho^2)^{\alpha}(R^2 - r^2)^{\alpha}} \tag{4.9}$$

where α is a constant to be determined later. Differentiating (4.9) twice with respect to one of the variables $x = x_k$, we obtain

$$\begin{aligned} v_x &= -\frac{2x\alpha}{(r^2 - \rho^2)^{\alpha+1}(R^2 - r^2)^{\alpha}} + \frac{2x\alpha}{(r^2 - \rho^2)^{\alpha}(R^2 - r^2)^{\alpha+1}} \\ v_{xx} &= -\frac{2\alpha}{(R^2 - r^2)^{\alpha}(r^2 - \rho^2)^{\alpha+1}} + \frac{4x^2\alpha(\alpha+1)}{(R^2 - r^2)^{\alpha}(r^2 - \rho^2)^{\alpha+2}} \\ &\quad + \frac{2\alpha}{(r^2 - \rho^2)^{\alpha}(R^2 - r^2)^{\alpha+1}} - \frac{8x^2\alpha^2}{(r^2 - \rho^2)^{\alpha+1}(R^2 - r^2)^{\alpha+1}} \\ &\quad + \frac{4x^2\alpha(\alpha+1)}{(r^2 - \rho^2)^{\alpha}(R^2 - r^2)^{\alpha+2}} . \end{aligned}$$

Summing over all $x = x_k$ and choosing $\alpha \geqq 1/4$ we get,

$$\Delta v \leqq \frac{5}{2} r^2 R^4 v^9 .$$

Now let $v = (2^{1/8}y)/(5^{1/2}R^2)^{1/4}$ then we have

$$\Delta y \leqq r^2 y^9 \tag{4.10}$$

where y is given by

$$y = \left(\frac{R^2 5^{1/2} 2^{-1/2}}{(R^2 - r^2)(r^2 - \rho^2)}\right)^{1/4}.$$

Now applying Corollary 4.1 to (4.10), we obtain,

$$\Delta u \leqq r^2 F(u)$$

when u is defined by

$$\frac{(r^2 - \rho^2)^2 (R^2 - r^2)^2}{20R^4} = \int_u^\infty \frac{dt}{F(t)}.$$

Clearly, $u'(0) = 0$ and $u \to \infty$ as $r \to R$ or $\rho \to r$. The fact that $\omega \leqq u$ now follows from Osserman's lemma [8]. This proves our assertion.

THEOREM 4.2. *Let $f(\omega)$ be positive, nondecreasing, differentiable function in $(-\infty, \infty)$ for which*

$$\int_\omega^\infty \frac{dt}{f(t)} \qquad (\omega > -\infty)$$

exists and

$$f'(\omega)\int_\omega^\infty \frac{dt}{f(t)} \leqq 1 + \lambda \qquad (\lambda > 0)\,. \tag{4.11}$$

If

$$u(r) = \sup_{Q \in S_r} \omega(Q) \tag{G}$$

where $\omega(Q)$ ranges over all functions of class C^2 in D_r which satisfy (4.1). *Then*

$$\frac{C(\lambda)\alpha(R^2 - r^2)^2}{R^2} \leqq \int_{u(r)}^\infty \frac{dt}{f(t)} \tag{4.12}$$

in case $P(r) = \alpha$ $(\alpha > 0)$.

$$\frac{C(\lambda)\beta r^{n/1+\lambda}(R^2 - r^2)^2}{R^2} \leqq \int_{u(r)}^\infty \frac{dt}{f(t)} \tag{4.13}$$

if $P(r) = \beta r^{n/1+\lambda}$ $(\beta > 0)$ and

$$\frac{C(\lambda)\gamma r^{n-2/\lambda}(R^2 - r^2)^2}{R^2} \leqq \int_{u(r)}^\infty \frac{dt}{f(t)} \tag{4.14}$$

if $P(r) = \gamma r^{n-2/\lambda}$ $(\gamma > 0)$

where

$$C(\lambda) = \frac{1}{4n} \qquad (4\lambda \leqq n-2) \tag{4.15}$$

and

$$C(\lambda) = \frac{1}{8(2\lambda+1)} \qquad (4\lambda > n-2)\,. \tag{4.16}$$

The inequalities (4.12), (4.13) *and* (4.14) *are sharp.*

The case $\lambda = 0$ had been considered by the author in [2].

Proof. Consider the function $g = g(r)$ defined by

$$\frac{C(\lambda)(R^2 - r^2)^2}{R^2} = \frac{1}{p(r)}\int_g^\infty \frac{dt}{f(t)} \tag{4.17}$$

where $p(r)$ is positive, monotonically increasing and twice continuously differentiable and C is a positive constant to be chosen later. Denoting by x one of the variables x_k and differentiating twice with respect to x we have

$$-\frac{4cx(R^2 - r^2)}{R^2} = -\frac{g_x}{p(r)f(g)} - \frac{2x}{p^2(r)}\int_g^\infty \frac{dt}{f(t)} \tag{4.18}$$

$$\begin{aligned} -\frac{4c(R^2 - r^2)}{R^2} &= -\frac{8cx^2}{R^2} - \frac{g_{xx}}{p(r)f(g)} + \frac{4x\dot{p}(r)g_x}{p^2(r)f(g)} + \frac{g_x^2 f'(g)}{p(r)f^2(g)} \\ &\quad \frac{2\dot{p}(r)}{p^2(r)}\int_g^\infty \frac{dt}{f(t)} - \frac{4x^2\ddot{p}(r)}{p^2(r)}\int_g^\infty \frac{dt}{f(t)} \\ &\quad \frac{8x^2\dot{p}^2(r)}{p^3(r)}\int_g^\infty \frac{dt}{f(t)} \end{aligned} \tag{4.19}$$

where dot denotes differentiation with respect to r^2. With the help of (4.17) and (4.18), (4.19) becomes

$$\begin{aligned} \frac{g_{xx}}{p(r)f(g)} &= -\frac{8cx^2}{R^2} + \frac{4c(R^2 - r^2)}{R^2} + \frac{16cx^2\dot{p}(r)(R^2 - r^2)}{R^2 p(r)} + \frac{4cx^2}{R^2} \\ &\quad \times f'(g)cp(r)\frac{(R^2 - r^2)^2}{R^2}\left[2 - \frac{\dot{p}(r)(R^2 - r)^2}{p(r)}\right]^2 \\ &\quad - \frac{2}{p^2(r)}(2x^2\ddot{p}(r) + \dot{p}(r))\int_g^\infty \frac{dt}{f(t)}\,. \end{aligned}$$

Summing over all x_k and using (4.11) it reduces to

$$
\begin{aligned}
\frac{\Delta g}{p(r)f(g)} \leqq 4c\left\{n - \frac{r^2}{R^2}(n-2-4\lambda)\right\} - \frac{16(R^2-r^2)cr^2\dot{p}(r)}{R^2}\lambda \\
- \frac{2c(R^2-r^2)^2}{R^2}\left\{\frac{2r^2\ddot{p}(r)+n\dot{p}(r)}{p(r)} - \frac{2r^2\dot{p}^2(r)}{p^2(r)}(1+\lambda)\right\} .
\end{aligned} \tag{4.20}
$$

We now consider the following cases:

Case I. Choose $p(r)$ such that $\dot{p}(r)/p(r)((2r^2\dot{p}(r)/p(r)) - n/(1+\lambda)) = 0$.

(i) If $\dot{p} = 0$ or $p = \alpha$ where α is an arbitrary positive constant then (4.20) becomes

$$
\frac{\Delta g}{\alpha f(g)} \leqq 4c\left\{n - \frac{r^2}{R^2}(n-2-4\lambda)\right\} . \tag{4.21}
$$

If, $4\lambda \leqq n-2$ it follows that $\Delta g \leqq 4nc\alpha f(g)$ and if C is given by (4.15), we have

$$
\Delta g \leqq \alpha f(g) . \tag{4.22}
$$

If $4\lambda > n-2$ the right hand of (4.21) attains maximum for $R = r$ and the value of (4.16) for C again leads to (4.22). Since $\dot{g}(0) = 0$ and increases to ∞ as $r \to R$ the proof of (4.12) will follow from Osserman's lemma [8].

REMARK. If $\alpha = 1$ the left hand inequality (9) of Theorem 1 of Nehari [6] becomes a particular case of this result.

(ii) If $2r^2\dot{p}(r)/p(r) - (n/1+\lambda) = 0$ or $p = r^{n/1+\lambda}\beta$ where β is an arbitrary positive constant then (4.20) gives

$$
\frac{\Delta g}{\beta r^{n/1+\lambda} f(g)} \leqq 4c\left\{n - \frac{r^2}{R^2}(n-2-4\lambda)\right\} .
$$

If C is given by the values (4.15) and (4.16), we have

$$
\Delta g \leqq \beta r^{n/1+\lambda} f(g) .
$$

Now the proof of (4.13) will follow from Osserman's lemma [8].

Case II. Assume $p(r)$ to satisfy

$$
2r^2p(r)(\ddot{p}r) + np(r)\dot{p}(r) - 2r^2(1+\lambda)\dot{p}^2(r) = 0
$$

or $p(r) = \gamma r^{n-2/\lambda}$ where γ is an arbitrary positive constant. Then (4.20) reduces to

$$
\frac{\Delta g}{\gamma r^{n-2/\lambda} f(g)} \leqq 4c\left\{n - \frac{r^2}{R^2}(n-2-4\lambda)\right\} .
$$

Now if C takes the values (4.15) and (4.16) respectively, we have

$$\Delta g \leqq \gamma r^{n-2/\lambda} f(g)$$

and (4.14) is proved with the help of Osserman's lemma [8].

We derive the following corollary:

COROLLARY 4.2. *If ω satisfies the equation*

$$\Delta\omega = \beta r^{n/1+\lambda}\omega^{1+(1/\lambda)} \qquad (\lambda > 0,\ n \geqq 2)$$

where β is an arbitrary constant, then

$$\omega \leqq \left(\frac{\lambda R^2}{c(\lambda)\beta r^{n/1+\lambda}(R^2 - r^2)^2}\right)^{\lambda}. \tag{4.23}$$

Also the behaviour of ω is such that

$$\overline{\lim_{r\to 0}}\left(\frac{\log\omega}{\log 1/r}\right) \leqq \frac{n\lambda}{1+\lambda}\,.$$

Indeed, setting $f(t) = t^{1+(1/\lambda)}$ in (4.13), we have (4.23), where $\omega = u$. Taking logarithm on both sides, we have, from (4.23)

$$\log\omega \leqq \lambda\log\frac{\lambda R^2}{\beta c(\lambda)(R^2 - r^2)^2} + \frac{n\lambda}{1+\lambda}\log\frac{1}{r}\,.$$

Dividing by $\log 1/r$ and letting $r \to 0$

$$\overline{\lim_{r\to 0}}\left(\frac{\log\omega}{\log 1/r}\right) \leqq \frac{n\lambda}{1+\lambda}\,.$$

A similar result could also be proved about the solutions of the equation

$$\Delta\omega = \gamma r^{n-2/\lambda}\omega^{1+(1/\lambda)}\,.$$

The next theorem concerns the lower bounds for the maximum of the solutions of (4.1).

THEOREM 4.3. *Let $f(\omega)$ satisfy the conditions of theorem 4.2 with (4.11) replaced by*

$$f'(\omega)\int_{\omega}^{\infty}\frac{dt}{f(t)} = 1 + \lambda\,, \qquad (\lambda > 0)\,. \tag{4.11$'$}$$

If

$$v(r) = \operatorname*{Sup}_{Q\in S_r}\omega(Q) \tag{G$'$}$$

where $\omega(Q)$ ranges over all functions of class C^2 in D_r and which satisfy (4.1) *then*

(4.24) $$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\kappa(R^2 - r^2)}{2n}$$

if $p(r) = \kappa$ *where* κ *is an arbitrary positive constant,*

(4.25) $$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\delta r^{n-2/\lambda-1}(R^2 - r^2)}{2n} \quad \left(n > 2,\ \lambda > 1,\ n < \frac{4\lambda}{1+\lambda}\right)$$

provided $p(r) = \delta r^{n-2/\lambda-1}$ $(\delta > 0)$.

(4.26) $$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\mu r^{1/\lambda}(R^2 - r^2)}{6} \qquad (n = 3)$$

in case $p(r) = \mu r^{1/\lambda}$ $(\mu > 0)$. *However, in 2-dimensional case*

(4.27) $$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\nu r^l(R^2 - r^2)}{4}$$

where $p(r) = \nu r^l$, ν *and* l *being arbitrary positive constants.*

Proof. Consider the function $h = h(r)$ defined by

(4.28) $$\frac{\rho^2 - r^2}{2n} = \frac{1}{p(r)}\int_h^\infty \frac{dt}{f(t)} \qquad (\rho > R > r)$$

where $p(r)$ is positive, monotonically increasing and twice continuously differentiable. Clearly, h belongs to the class C^2 in D_r. Differentiating (5.28) twice with respect to $x = x_k$ we obtain

$$-\frac{n}{x} = -\frac{h_x}{f(h)p(r)} - \frac{2x\dot{p}(r)}{p^2(r)}\int_h^\infty \frac{dt}{f(t)}$$

(4.29) $$-\frac{1}{n} = -\frac{h_{xx}}{f(h)p(r)} + \frac{4xh_x\dot{p}(r)}{f(h)p^2(r)} + \frac{h_x^2 f'(h)}{p(r)f^2(h)} - \frac{2\dot{p}}{p^2}\int_h^\infty \frac{dt}{f(t)} - \frac{4x^2\ddot{p}(r)}{p^2(r)}\int_h^\infty \frac{dt}{f(t)} + \frac{8x^2\dot{p}(r)}{p^3(r)}\int_h^\infty \frac{dt}{f(t)}\,.$$

Using (4.29) and summing over all x_k, we obtain

$$\frac{\Delta h}{f(h)p(r)} = 1 + \frac{4r^2\dot{p}(r)}{np(r)} + r^2 p(r) f'(h)\left[\frac{\rho^2 - r^2}{n}\frac{\dot{p}}{p} - \frac{1}{n}\right]^2 - \frac{2r^2\ddot{p} + n\dot{p}}{p} \times \frac{\rho^2 - r^2}{n}\,.$$

Since $f' > 0$ we obtain with the help of (4.11)′

(4.30) $$\frac{\Delta h}{f(h)p(r)} \geqq 1 - \frac{4r^2\dot{p}}{np}\lambda - \frac{\rho^2 - r^2}{n}\left[\frac{n\dot{p} + 2r^2\ddot{p}}{p} - (1 + \lambda)\frac{2r^2\dot{p}^2}{p^2}\right].$$

Now we consider the following cases.

Case I. Choose p such that $\dot{p} = 0$ or, $p = \kappa$ where κ is an arbitrary positive constant. Hence (4.30) reduces to

$$\Delta h \geqq \kappa f(h) . \tag{4.31}$$

Consequently $(G)'$ implies

$$h(r) \leqq v(r) .$$

Since we can take ρ arbitrarily close to R, we have

$$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\kappa(R^2 - r^2)}{2n} .$$

Case II. Assume $p(r)$ to be such that

$$n\dot{p}(r)p(r) + 2r^2p(r)\ddot{p}(r) - 2\lambda r^2\dot{p}^2(r) = 0$$

or $p = \delta r^{n-2/\lambda-1}$ where δ is an arbitrary positive constant, $n > 2$, $\lambda > 1$ and such that $n < (4\lambda/1 + \lambda)$. Hence (4.30) becomes

$$\Delta h \geqq \left\{1 - \frac{2\lambda(n-2)}{n(\lambda - 1)}\right\}\delta r^{n-2/\lambda-1}f(h) .$$

Using $(G)'$ and arguing as above, we obtain

$$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\delta r^{n-2/\lambda-1}(R^2 - r^2)}{2n} .$$

Case III. Choose p to satisfy

$$np(r)\dot{p}(r) + 2r^2p(r)\ddot{p}(r) - (1 + \lambda)2r^2\dot{p}^2(r) = 0$$

or $p = \mu r^{1/\lambda}$ where μ is an arbitrary positive constant and $n = 3$. Hence (4.30) gives

$$\Delta h \geqq \frac{\mu}{3} r^{1/\lambda} f(h) .$$

Using the same argument as above, we have

$$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\mu r^{1/\lambda}(R^2 - r^2)}{6} .$$

Case IV. Assume p to be such that $2r^2p\ddot{p} + np\dot{p} - 2r^2\dot{p}^2 = 0$ or $p = \nu r^l$ where ν and l are arbitrary positive constants. Consequently

$$\Delta h \geqq \nu(1 - l\lambda)r^l f(h) .$$

And, as above we conclude

$$\int_v^\infty \frac{dt}{f(t)} \leqq \frac{\nu r^l(R^2 - r^2)}{4} .$$

This completes the proof of the theorem.

We derive the following corollaries:

COROLLARY 4.3. *In case of a function ω regular in D_r and which satisfies the differential equation*

$$\Delta\omega = \delta r^{n-2/\lambda-1}\left\{1 - \frac{2\lambda(n-2)}{n(\lambda+1)}\right\}\omega^{1+(1/\lambda)}$$

where δ is an arbitrary positive constant, $n > 2$, $\lambda > 1$ and such that $n < (4\lambda/1 + \lambda)$ we have

$$\left(\frac{2n\lambda}{\delta r^{n-2/\lambda-1}(R^2 - r^2)}\right)^\lambda \leqq \omega .$$

And also the behaviour of ω is such that

$$\overline{\lim_{r\to 0}}\left(\frac{\log \omega}{\log 1/r}\right) \geqq \lambda\frac{n-2}{\lambda - 1} .$$

Indeed, setting $f(t) = t^{1+(1/\lambda)}$ in (4.25), where $v = \omega$, we obtain

$$\omega^{1/\lambda} \geqq \frac{2n\lambda}{\delta r^{n-2/\lambda-1}(R^2 - r^2)} .$$

Taking logarithm on both sides, we get

$$\log \omega \geqq \lambda \log \frac{2n\lambda}{\delta(R^2 - r^2)} + \lambda\frac{n-2}{\lambda-1}\log\frac{1}{r} .$$

Dividing by $\log 1/r$ and taking the limit

$$\overline{\lim_{r\to 0}}\left(\frac{\log \omega}{\log 1/r}\right) \geqq \lambda\frac{n-2}{\lambda - 1} .$$

COROLLARY 4.4. *If $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$ is a 3-dimensional Laplace operator and ω satisfies the equation*

$$\Delta\omega = \frac{\mu}{3} r^{1/\lambda}\omega^{1+(1/\lambda)}$$

we have

$$\omega \geqq \left(\frac{6}{\mu r^{1/\lambda}(R^2 - r^2)}\right)^\lambda$$

and

$$\varlimsup_{r\to 0}\left(\frac{\log \omega}{\log 1/r}\right) \geqq 1 .$$

COROLLARY 4.5. *If the function ω is regular in D_r and satisfies the differential equation*

$$\Delta\omega = \delta(1 - l\lambda)r^l\omega^{1+(1/\lambda)} \qquad \left(\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$$

we have

$$\left(\frac{4}{\delta r^l(R^2 - r^2)}\right)^\lambda \leqq \omega$$

and also the behaviour of ω is such that

$$\varlimsup_{r\to 0}\left(\frac{\log \omega}{\log 1/r}\right) \geqq l\lambda .$$

The proof of Corollaries 4.4 and 4.5 is exactly the same as that of 4.3.

REFERENCES

1. L. Bieberbach, *$\Delta u = e^u$ und die automorphen Funktionen*, Math. Ann. **77** (1916), 173-212.
2. V. B. Goyal, *Bounds for the solutions of $\Delta u \geqq p(r)f(u)$*, Compositio Mathematica **18** (1967), 162-169.
3. E. K. Haviland, *A note on unrestricted solutions of the differential equation $\Delta u = f(u)$*, J. London Math. Soc. **26** (1951), 210-214.
4. E. Hopf, *On non-linear partial differential equations*, Berkeley Symposium on Partial Differential Equations (Lecture series), 1957, 1-31.
5. J. D. Keller, *On solutions of $\Delta u = f(u)$*, Comm. Pure Appl. Math. **10** (1957), 503-510.
6. Z. Nehari, *Bounds for the solutions of a class of non-linear partial differential equations*, Oroc. Math. Soc. (1963), 829-836.
7. J. Nitsche, *Uber die isolierten Singularitäten der Lösungen von $\Delta u = e^u$*, Math. Zeit. **68** (1957-58), 316-324.
8. R. Osserman, *On the inequality $\Delta u \geqq f(u)$*, Pacific J. Math **7** (1957), 1641-1647.
9. R. Redheffer, *On entire solutions of non-linear equations* (abstract), Bull. Amer. Math. Soc. **62** (1956), 408.
10. ———, *On the inequality $\Delta u \geqq f(u, |\operatorname{grad} u|)$*, J. Math. Anal. Appl. **1** (1960), 277-299.
11. W. Walter, *Überganze Lösungen der Differentialgleichung $\Delta u = f(u)$*, Jahresb. d. Deutsch. Math. Ver. **57** (1955), 94-102.
12. H. Wittich, *Ganze Lösungen der Differentialgleichung $\Delta u = e^u$*, Math. Zeit. **49** (1943-44), 579-582.

Received January 20, 1966, in revised form January 27, 1967, and July 22, 1969.

This work is a portion of the author's doctoral dissertation written under the direction of Professor Z. Nehari at Carnegie Institute of Technology in 1963, and was supported by the United States Air Force under contract No. AF20A.

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