PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 2, 1970

LOCAL BEHAVIOUR OF AREA FUNCTIONS OF CONVEX BODIES

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The area function of a convex body K in Euclidean n-space is a particular measure over the field $\mathscr{B}$ of Borel sets of the unit spherical surface. The value of such a function at a Borel set ω is the area of that part of the boundary of K touched by support planes whose outer normal directions fall in ω. In particular the area function of the vector sum $K + tE$, where t is nonnegative and E is the unit ball, is a polynomial of degree $n - 1$ in t whose coefficients are also measures over $\mathscr{B}$. To within a binomial coefficient, the coefficient of t^{n-p-1} in this polynomial is called the area function of order p. For $p = 1$ and $p = n - 1$ necessary and sufficient conditions for a measure over $\mathscr{B}$ to be an area function of order p are known, but for intermediate values of p only certain necessary conditions are known. Here a new necessary condition is established. It is a bound on those functional values of an area function of order p which correspond to special sets of $\mathscr{B}$. These special sets are closed, small circles of geodesic radius α less than $\pi/2$; the bound depends on α, p and the diameter of K. This necessary condition amplifies an old observation: area functions of order less than $n - 1$ vanish at Borel sets consisting of single points.

To examine area functions in detail, we write $\Pi(u)$ for the support plane to K whose outer normal direction corresponds to the point u on the unit spherical surface Ω. For ω in $\mathscr{B}$ set

$$B(\omega) = \bigcup_{u \in \omega} (\Pi(u) \cap K) .$$

The area function of K at ω is the $(n - 1)$-dimensional measure of $B(\omega)$; we denote this by $S(K, \omega)$. $S(K + tE, \omega)$ is a polynomial of degree $n - 1$ in t; the coefficient of

$$\binom{n-1}{p} t^{n-p-1}, \text{ where } \binom{n-1}{p} = \frac{(n-1)!}{p!(n-p-1)!} ,$$

is the area function of order p at ω and is written $S_p(K, \omega)$. In particular

$$S_{n-1}(K, \omega) = S(K, \omega), S_0(K, \omega) = S(E, \omega) .$$

If at each boundary point of K there is a unique outer normal

u and principal radii of curvature $R_1(u), \cdots, R_{n-1}(u)$ and if $\{R_1, \cdots, R_p\}$ signifies the p^{th} elementary symmetric function of these radii, then

$$S_p(K, \omega) = \int_\omega \{R_1, \cdots, R_p\} d\omega \Big/ \binom{n-1}{p} .$$

For general convex bodies the total area of order p is a special mixed volume; in detail

$$S_p(K, \Omega) = nV(\underbrace{K, \cdots, K}_{p}, \underbrace{E, \cdots, E}_{n-p}) .$$

Let v be any fixed point on Ω and let ω_α be the set of u on Ω for which

$$(u, v) \geqq \cos \alpha, 0 < \alpha < \pi/2 ,$$

where (u, v) denotes the inner product of u and v. We shall prove that

$$(1) \qquad S_p(K, \omega_\alpha) \leqq AD^p \sin^{n-p-1} \alpha \sec \alpha = AD^p f_p(\alpha) ,$$

for $p = 1, 2, \cdots, n-1$, where D is the diameter of K and A depends neither on α nor on K.

A. D. Aleksandrov [1] and W. Fenchel and B. Jessen [3] introduced such area functions. They showed that for a measure Φ over $\mathscr{B}$ to be an area function of order $n-1$, it is necessary and sufficient that, for any u'

$$(2) \qquad \int_\Omega (u', u)\Phi(d\omega(u)) = 0, \int_\Omega |(u', u)|\Phi(d\omega(u)) > 0 ,$$

where these are Radon integrals. Aleksandrov showed also that (2), while necessary for Φ to be a p^{th} order area function when $p < n-1$, are not sufficient. In part this depended on the observation that

$$(3) \qquad S_p(K, \{v\}) = 0$$

for each v on Ω and $p < n-1$. By letting α tend to zero, we see that (3) is a consequence of (1).

Necessary and sufficient conditions for Φ to be an area function of order one are given in [4] and [5]. Inequality (1) for $p = 1$ was proved in the latter paper and plays a significant part. Items of background are in these papers and [2] and [3].

1. We first show that if (1) holds for convex polyhedra, then it is true for all convex bodies.

Given any convex body K we can find convex polyhedra K_m, $m =$

$1, 2, \cdots$, which approximate K to within $1/m$ in the sense of the metric

$$\delta(K, K_m) = \max_{u \in \Omega} |H(u) - H_m(u)| ,$$

where H and H_m are the support functions of K and K_m. For the diameters D and D_m of these bodies we have

$$\lim_{m \to \infty} D_m = D .$$

Let $\varepsilon > 0$ be such that $\alpha + \varepsilon < \pi/2$; denote by η_ε the open set of u on Ω for which

$$(u, v) > \cos(\alpha + \varepsilon) .$$

Clearly

$$\omega_\alpha \subset \eta_\varepsilon \subset \omega_{\alpha+\varepsilon} . \tag{4}$$

By Theorem IX of [3], $S_p(K_m, \omega)$ converges weakly to $S_p(K, \omega)$ as m tends to infinity. This implies [3, p. 8] that

$$\liminf_{m \to \infty} S_p(K_m, \eta_\varepsilon) \geqq S_p(K, \eta_\varepsilon) \geqq S_p(K, \omega_\alpha) \tag{5}$$

since η_ε is open. We have used (4) and the monotonicity of $S_p(K, \omega)$ in ω for the final inequality.

Also from (4), the monotonicity of S_p, and the assumption of (1) for polyhedra, we get

$$S_p(K_m, \eta_\varepsilon) \leqq AD_m^p f_p(\alpha + \varepsilon) . \tag{6}$$

Hence, because D_m tends to D, (5) and (6) yield

$$S_p(K, \omega_\alpha) \leqq AD^p f_p(\alpha + \varepsilon) .$$

The left side does not depend on ε and so inequality (1) holds for K.

2. To prove (1) for convex polyhedra K we form, from a given K, four convex bodies K_1, K_2, K_3, K_4 for which

$$S_p(K_j, \omega_\alpha) \leqq S_p(K_{j+1}, \omega_\alpha), j = 1, 2, 3 , \tag{7}$$

and

$$S_p(K_1, \omega_\alpha) = S_p(K, \omega_\alpha) , \tag{8}$$

$$S_p(K_4, \omega_\alpha) = AD^p f_p(\alpha) . \tag{9}$$

As a matter of notation $\Pi_j(u)$ signifies the support plane to K_j with outer unit normal u. We write ∂P for the boundary of any set P.

The body K_1 is to be the convex closure of $B(\omega_\alpha)$. Since

$$\bigcup_{u \in \omega_\alpha} (K_1 \cap \Pi_1(u)) = B(\omega_\alpha)$$

(8) holds. Also K_1 is polyhedral.

Let $\mathfrak{H}_1(u)$ signify the half-space with outer normal u which is bounded by $\Pi_1(u)$. Of course, for u in ω_α, $\mathfrak{H}_1(u)$ is the half-space with outer normal u bounded by $\Pi(u)$. Since $\alpha < \pi/2$, the intersection of those $\mathfrak{H}_1(u)$ for which

$$(u, v) \leqq \cos \alpha$$

is a convex polyhedron $K_2 \supseteqq K_1$. Here v, as before, is the centre of ω_α; we write ω'_α for those u on Ω which satisfy the last inequality. Clearly

$$\bigcup_{u \in \omega'_\alpha} (K_1 \cap \Pi_1(u)) = \bigcup_{u \in \omega'_\alpha} (K_2 \cap \Pi_2(u))$$

and so

$$S_p(K_1, \omega'_\alpha) = S_p(K_2, \omega'_\alpha) \ . \tag{10}$$

On the other hand $K_1 \subseteqq K_2$ implies that

$$S_p(K_1, \Omega) \leqq S_p(K_2, \Omega) \ .$$

This is a consequence of the representation of these total area functions as mixed volumes and the known monotonicity of mixed volumes $V(K, \cdots, K, E, \cdots, E)$ in K, cf. [2]. The additinity of area functions, our last inequality and (10) yield (7) for $j = 1$.

The rest of the proof is treated in separate sections. In §3 we describe a plane Π_0 normal to v, which cuts K so that $B(\omega_\alpha)$, and hence K_2, lies in one of the half-spaces determined by Π_0. Call this half-space $\mathfrak{H}_0$. We take K_3 to be the intersection of $\mathfrak{H}_0$ with

$$\cap \mathfrak{H}(u) = \cap \mathfrak{H}_1(u)$$

where these intersections are taken over those u in the common boundary of ω_α and ω'_α, i.e., those u for which

$$(u, v) = \cos \alpha \ .$$

The body K_3 contains K_2. To determine Π_0 it is necessary to consider circular cones of the form

$$(v, x - x_0) + ||x - x_0|| \sin \alpha \leqq 0 \ . \tag{11}$$

The norm is Euclidean. The vertex of such a cone is x_0; the axial ray within the cone has the direction $-v$; these cones are translates of one another. We choose x_0 so that the resulting cone contains K and the distance from K to the plane

$$(v, x - x_0) = 0$$

is as small as possible. We call this tangent cone C.

In §4 (7) is proved for $j = 2$.

K_4 is $C \cap \mathfrak{H}_0$. This intersection is clearly a convex body which contains K_3. In §5 we prove (7) for $j = 3$. Finally (9) follows from a direct calculation sketched in §6.

3. Let us introduce a Cartesian coordinate system with origin at the vertex x_0 of C and such that $v = (-1, 0, \cdots, 0)$. The description of C takes the form

$$x_1 \geqq \tan \alpha (x_2^2 + \cdots + x_n^2)^{1/2}$$

and the distance from K, which is in C, to the plane $x_1 = 0$ is minimal.

This means that each half-space

$$u_2 x_2 + \cdots + u_n x_n \geqq 0 \tag{12}$$

must contain a point of $B(\omega_\alpha) \cap \partial C$ for the following reason. If $\partial K \cap \partial C$ had no points in (12), a small translation of K in the direction u would cause $\partial K \cap \partial C$ to be empty; a subsequent small translation in the direction v would reduce the distance from K to $x_1 = 0$. Hence (12) contains a point x of $\partial C \cap \partial K$. The tangent plane to ∂C at x is a support plane of ∂K and the outer normal to this support plane makes an angle of measure α with v, i.e., falls in ω_α. Thus x is also in $B(\omega_\alpha)$ as asserted.

We define conical bodies C_1 and C_2 to be the intersection of C with the half-spaces

$$x_1 \leqq D \tan \alpha, \; x_1 \leqq 2D \tan \alpha$$

respectively.

We first prove that

$$B(\omega_\alpha) \cap \partial C \subseteqq C_1 \, . \tag{13}$$

Suppose to the contrary that there is a y in $B(\omega_\alpha) \cap \partial C$ for which $y_1 > D \tan \alpha$. Since the radius of the intersection of C with

$$x_1 = D \tan \alpha$$

is D, a ball of radius D, centred at y, lies in a half-space of the form

$$u_2 x_2 + \cdots + u_n x_n < 0 \tag{14}$$

for some u. As noted in the previous paragraph, there is a point x

in the complement of (14) which is in $B(\omega_\alpha) \cap \partial C$. This would give two points x and y in K separated by a distance greater than the diameter D of K. The contradiction establishes (13).

Next we demonstrate

$$B(\omega_\alpha) \subseteqq C_2 . \tag{15}$$

Again the proof is by contradiction. Imagine z to be a point in $B(\omega_\alpha)$ for which $z_1 > 2D \tan \alpha$. z cannot be on the x_1-axis for the following reason. Let Π be a support plane to K which contains z. There must be a half-space of the form (12) in which the points of $\Pi \cap \partial C$ lie in the half-space

$$x_1 > 2D \tan \alpha .$$

This implies that the points of $\partial K \cap \partial C$ which lie in (12) are at a distance exceeding $2D$ from z which, again, contradicts the fact that D is the diameter of K.

Let z' be the point nearest to z on the x_1-axis. Set

$$u = (z - z')/||z - z'|| ;$$

u is orthogonal to v and z' and so

$$0 < (u, z' - z) = -(u, z) .$$

Thus z satisfies

$$u_2 z_2 + \cdots + u_n z_n < 0 .$$

There is also a point x of

$$B(\omega_\alpha) \cap \partial C_1 = B(\omega_\alpha) \cap \partial C_2$$

in the complementary half-space. Therefore the distance $||z - x||$ must exceed the distance between $(2D \tan \alpha, 0, \cdots, 0)$ and the intersection of ∂C_1 with the plane

$$x_1 = D \tan \alpha .$$

That is to say

$$||z - x|| > (D^2 + D^2 \tan^2 \alpha)^{1/2} > D .$$

This is impossible for x and z in K which completes the proof of (15).

The plane

$$x_1 = 2D \tan \alpha$$

is the cutting plane Π_0 of §2; the conical convex body C_2 is K_4.

4. From the definitions of K_2 and K_3 we see that their support planes $\Pi_2(u)$ and $\Pi_3(u)$ coincide whenever their outer normal directions u are in ω_α. Hence for such u, since $K_2 \subseteqq K_3$,

$$K_2 \cap \Pi_2(u) \subseteqq K_3 \cap \Pi_3(u) \ ;$$

there is certainly equality when u is in the interior of ω_α. Inequality (7) for $j = 2$ follows from the next lemma, to the proof of which this section is devoted.

LEMMA. *Let K and K' be two convex polyhedral bodies whose support planes with outer normal direction u are denoted by $\Pi(u)$ and $\Pi'(u)$. If*

$$K \cap \Pi(u) \subseteqq K' \cap \Pi'(u) \tag{16}$$

for each u in some Borel set ω of Ω, then

$$S_p(K, \omega) \leqq S_p(K', \omega), \ \textit{for} \ \ p = 1, 2, \cdots, n-1 \ .$$

We first require a description of $S_p(K, \omega)$ where K is polyhedral. In this we follow work, as yet unpublished, of J. Zelver.

Consider a set of the form $K \cap \Pi(u)$; this is a p-face e_p when e_p lies in a p-dimensional flat but not in a $(p-1)$-dimensional flat. The outer unit normals to support planes of K which contain e_p sweep out a closed, geodesically convex set $\omega(e_p)$ on Ω which is in $\mathscr{B}$ and is $(n-p-1)$-dimensional. Throughout $\omega(e_p)$ we distribute mass with constant density $\lambda_p(e_p)$ equal to the p-dimensional volume of e_p. Thus if ω is any subset of $\omega(e_p)$ which is in $\mathscr{B}$ and if $\mu_{n-p-1}(\omega)$ is its $(n-p-1)$-dimensional volume, then the mass falling in ω is $\lambda_p(e_p)\mu_{n-p-1}(\omega)$. The representation we seek is

$$S_p(K, \omega) = \sum_{*} \lambda_p(e_p)\mu_{n-p-1}(\omega \cap \omega(e_p)) \Big/ \binom{n-1}{p} \ , \tag{18}$$

where the starred summation is taken over all e_p in ∂K.

Consider the vector sum $K + tE$ and let $\Pi^*(u)$ signify its support plane with outer normal u. If x' is a point of

$$(K + tE) \cap \Pi^*(u) \ ,$$

then there is a unique point x in $K \cap \Pi(u)$ such that

$$x' - x = tu \ . \tag{19}$$

Suppose e_p to be the face of lowest dimension which contains x and let $\{\Pi(u')\}$ be the set of support planes of K which contain e_p where u' ranges over $\omega(e_p)$. We form

$$\bigcup_{*} \{(K + tE) \cap \Pi^*(u')\} , \tag{20}$$

where the starred union is taken over those u' in $\omega \cap \omega(e_p)$. If (20) is not empty, it is made up of points x' to each of which corresponds a unique x on

$$\bigcup_{*} (K \cap \Pi(u')) = e_p$$

for which (19) holds. Thus (20) is the Cartesian product of e_p with that part of the boundary of tE which is swept out by rays whose directions are in $\omega \cap \omega(e_p)$. Therefore, empty or not, the $(n-1)$-dimensional measure of (20) is

$$t^{n-p-1}\lambda_p(e_p)\mu_{n-p-1}(\omega \cap \omega(e_p)) .$$

We add up all such contributions to $S_{n-1}(K + tE, \omega)$ and obtain the sum

$$\sum_{p=1}^{n} t^{n-p-1} \sum_{*} \lambda_p(e_p)\mu_{n-p-1}(\omega \cap \omega(e_p)) .$$

On the other hand, from the generalized Steiner formula [3, p. 31], we have

$$S_{n-1}(K + tE, \omega) = \sum_{p=1}^{n} t^{n-p-1} \binom{n-1}{p} S_p(K, \omega) .$$

The comparison of coefficients of like powers of t in these two representations of $S_{n-1}(K + tE, \omega)$ yields (18).

Choose u in ω; neither set in (16) is empty and so $\Pi(u)$ and $\Pi'(u)$ share a common point, have the same normal direction and so coincide. We have

$$K' \cap \Pi(u) = e'_p$$

for some p. By (16) either $K \cap \Pi(u)$ is a face e_p of the same dimension p or this intersection is a face of lower dimension. In the latter case there is no contribution to the sum in (18), i.e., the left side of (17), whereas there would be a positive contribution to the right side of (17). In the former case, from (16) it follows that

$$\lambda_p(e'_p) \geqq \lambda_p(e_p) . \tag{21}$$

Also

$$\mu_{n-p-1}(\omega \cap \omega(e'_p)) = \mu_{n-p-1}(\omega \cap \omega(e_p)) . \tag{22}$$

To see this, we prove that the two argument sets in (22) coincide by showing that, for any u in Ω, we have $K \cap \Pi(u) \supseteqq e_p$ if and only if $K' \cap \Pi(u) \supseteqq e'_p$.

If $K' \cap \Pi(u) \supseteqq e'_p$, then $e_p \subseteqq e'_p \subseteqq \Pi(u)$ and e_p also lies in ∂K. Hence e_p lies in $K \cap \Pi(u)$. Suppose $e_p \subseteqq K \cap \Pi(u)$; then e_p lies in $\Pi(u)$. Since $e_p \subseteqq e'_p$ by (16) and these two sets have the same dimensionality, any point x in e'_p is a linear combination of $p+1$ suitable points in e_p. But, being such a combination of points in $\Pi(u)$, x must be in $\Pi(u)$. Thus e'_p is in both $\Pi(u)$ and K' and so in their intersection.

Substitution from (21) and (22) into the representation (18) as it applies to K and K' proves (17).

5. Our next step is to prove (7) for $j = 3$. We first settle the simplest case: $p = n - 1$. It is clear from the construction of K_3 and K_4 that, for $i = 3, 4$:

$$S_{n-1}(K_i, \Omega - \omega_\alpha) = S_{n-1}(K_i, \{-v\}) ,$$

$$S_{n-1}(K_i, \omega_\alpha) = S_{n-1}(K_i, \partial\omega_\alpha) ,$$

and

$$S_{n-1}(K_i, \partial\omega_\alpha) \cos \alpha = S_{n-1}(K_i, \{-v\}) .$$

Consequently

$$S_{n-1}(K_i, \Omega) = (1 + \cos \alpha) S_{n-1}(K_i, \omega_\alpha) .$$

Since $K_3 \subseteqq K_4$ and $S_{n-1}(K, \Omega)$ is increasing in K, it follows that (7) holds for $j = 3, p = n - 1$. For the cases $1 \leqq p < n - 1$ a more elaborate argument is needed.

We shall examine the behaviour of $S_p(K_i, \omega_\alpha)$ in K_i by studying that of

$$Q_i = \int_{\Omega - \omega_\alpha} (v, u) S_p(K_i, d\omega(u)), \; i = 3, 4 .$$

These integrals will be reduced to iterated integrals. For this purpose we let Ω_{n-1} denote the set of u on Ω which are orthogonal to v and we form, for each u in Ω_{n-1}, the vectors

$$u_\lambda = [(1 - \lambda)u + \lambda(-v)]/||(1 - \lambda)u + \lambda(-v)|| .$$

As before, v is the centre of ω_α. We have

$$(u_\lambda, v) = -\lambda/(\phi(\lambda))^{1/2} ,$$

where

$$\phi(\lambda) = 1 - 2\lambda + 2\lambda^2 .$$

Also, if s signifies arc length along the circle through v and u,

$$ds/d\lambda = 1/\phi(\lambda) .$$

Define $\lambda_0 < 0$ by

$$-\lambda_0 = \cos \alpha(\phi(\lambda_0))^{1/2} .$$

As u passes over Ω_{n-1} and λ over the interval $\lambda_0 < \lambda < 1$, u_λ sweeps out

$$\Omega - \omega_\alpha - \{-v\} .$$

For such u and λ:

$$\Pi_i(u_\lambda) \cap K_i = \Pi_i(u) \cap \Pi_0 \cap K_i = \Pi_i(u) \cap k_i ,$$

where we have set

$$k_i = K_i \cap \Pi_0 ,$$

and we recall that Π_0 is the support plane of K_i with outer normal $-v$. If we view each k_i as a nondegenerate convex body in the $(n-1)$-dimensional space Π_0, then the outer normals u to k_i fall in Ω_{n-1} and k_i has area functions

$$s_1(k_i, \eta), \cdots, s_{n-2}(k_i, \eta)$$

defined over the Borel sets η of Ω_{n-1}.

We write Q_i as an iterated integral

$$\int_{\lambda_0}^1 \frac{-\lambda}{(\phi(\lambda))^{1/2}}\left(\int_{\Omega_{n-1}} s_p(k_i, d\eta(u))\right)\frac{d\lambda}{\phi(\lambda)} = gS_p(k_i, \Omega_{n-1}) ,$$

where

$$g = \int_{\lambda_0}^1 \frac{-\lambda d\lambda}{(\phi(\lambda))^{3/2}} < 0 .$$

Here we have used the fact that the point $-v$ can be deleted from $\Omega - \omega_\alpha$ without affecting Q_i in virtue of (3) and the assumption that $p < n - 1$. Since $k_3 \subseteqq k_4$

$$s_p(k_3, \Omega_{n-1}) \leqq s_p(k_4, \Omega_{n-1})$$

and, from the negativity of g, it follows that

$$Q_3 \geqq Q_4 .$$

The first condition in (2), which is satisfied by any area function, shows that

$$Q_i + \int_{\omega_\alpha} (v, u_\lambda)S_p(K_i, d\omega(u_\lambda)) = 0 .$$

Hence, from our last inequality, we obtain

$$\int_{\omega_\alpha} (v, u_\lambda) S_p(K_3, d\omega(u_\lambda)) \leqq \int_{\omega_\alpha} (v, u_\lambda) S_p(K_4, d\omega(u_\lambda)) . \tag{23}$$

Let x_0 signify the vertex of the cone K_4 and denote by ω_α^0 the interior of ω_α. Then for all u in ω_α^0

$$K_4 \cap \Pi_4(u) = x_0$$

and, because $p \geqq 1$,

$$S_p(K, \omega_\alpha^0) = 0 .$$

Therefore on the right side of (23) the integration needs to be extended only over $\partial\omega_\alpha$ throughout which (v, u_λ) is $\cos\alpha$. This yields for the right side of (23)

$$\cos\alpha S_p(K_4, \omega_\alpha) .$$

Consider the left side of (23). For u_λ in ω_α we have

$$(v, u_\lambda) \geqq \cos\alpha$$

and so we may strengthen inequality (23) by replacing the left side by

$$\cos\alpha S_p(K_3, \omega_\alpha) .$$

After division by $\cos\alpha$ the strengthened inequality is just (7) for $j = 3, 1 \leqq p < n - 1$.

6. It remains to prove (9). In the Cartesian coordinate system of section three, K_4 is the set of points x for which

$$\tan\alpha(x_2^2 + \cdots + x_n^2)^{1/2} \leqq x_1 \leqq 2D\tan\alpha .$$

Let tE^* be the convex body formed by the intersection of the ball tE with the reflected polar cone to C, i.e.,

$$x_1 \leqq -ctn\alpha(x_2^2 + \cdots + x_n^2)^{1/2} .$$

The vector sum $K_4 + tE^*$ is a convex body of revolution whose radial distance $r(\xi)$ in the plane $x_1 = \xi$ has the representation

$$\begin{aligned} r(\xi) &= (t^2 - \xi^2)^{1/2}, \; -t \leqq \xi \leqq -t\cos\alpha ; \\ &= \xi ctn\alpha + tcsc\alpha, \; -t\cos\alpha \leqq \xi \leqq 2D\tan\alpha - t\cos\alpha ; \\ &= 2D\sec^2\alpha - \xi\tan\alpha, \; 2D\tan\alpha - t\cos\alpha \leqq \xi \leqq 2D\tan\alpha . \end{aligned} \tag{24}$$

The volume $V(K_4 + tE^*)$ is

$$\omega_{n-1}\int_{-t}^{2D\tan\alpha} r^{n-1}(\xi)d\xi/(n-1) . \tag{25}$$

Here ω_{n-1} is the area of the unit spherical surface in Euclidean $(n-1)$-dimensional space and is given by

$$\omega_{n-1} = 2\pi^{(n-1)/2}/\Gamma((n-1)/2) ,$$

where Γ is the usual gamma function.

We equate (25) with the Steiner polynomial

$$V(K_4 + tE^*) = \sum_{p=0}^{n} \binom{n}{p} t^{n-p} V_p(K_4, E^*) ,$$

where $V_p(K_4, E^*)$ is the mixed volume

$$V(\underbrace{K_4, \cdots, K_4}_{p}, \underbrace{E^*, \cdots, E^*}_{n-p}) .$$

Substitution from (24) into (25) and a comparison of coefficients of like powers of t yields

$$V_p(K_4, E^*) = \omega_{n-1}(2D)^p(\sin\alpha)^{n-p-1}\sec\alpha/n(n-1) . \tag{26}$$

We consider next the brush set (Bürstenmenge) $B_t(K_4, \omega_\alpha)$ which is formed from K_4 in the following manner. At each point x of

$$\bigcup_{u\in\omega_\alpha} (K_4 \cap \Pi_4(u))$$

we draw all segments $x + \theta u$, $0 < \theta \leqq t$, corresponding to u in ω_α. The union of these segments is $B_t(K_4, \omega_\alpha)$. Clearly this is

$$(K_4 + tE^*) - K_4$$

and so the volume $V_t(K_4, \omega_\alpha)$ of $B_t(K_4, \omega_\alpha)$ is

$$V(K_4 + tE^*) - V(K_4) = \sum_{p=0}^{n-1} \binom{n}{p} t^{n-p} V_p(K_4, E^*) .$$

On the other hand, cf. [3, p. 31],

$$V_t(K_4, \omega_\alpha) = \sum_{p=0}^{n-1} \binom{n}{p} t^{n-p} S_p(K_4, \omega_\alpha)/n .$$

A comparison of coefficients of like powers of t in these two representations of $V_t(K_4, \omega_\alpha)$ yields

$$S_p(K_4, \omega_\alpha) = n\, V_p(K_4, E^*)$$

and this, together with (26), gives (9) with

$$A = 2^p\omega_{n-1}/(n-1) .$$

This completes the proof of (1).

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Received June 6, 1968, and in revised form December 30, 1969.

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