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CHARACTERS AND ORTHOGONALITY IN FROBENIUS ALGEBRAS

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Matrix-theoretical proofs of orthogonality relations for the coefficients of representations of Frobenius algebras have been extensively developed in the literature. This paper has grown out of a desire to prove some of these orthogonality relations in a matrix-free, module-theoretic manner. Most of the attention is focused on characters, even though some of the results diverge from this restriction.

Sections 1 and 2 set the stage for the development of the principal theorems in § 3. There we derive the orthogonality relations and demonstrate a relation between characters and certain homogeneous modules over Frobenius algebras. In § 4 we apply these results to obtain some information about the character of the left regular module.

All rings are assumed to have an identity, and all modules are assumed to be unital. The Jacobson radical of a ring A will be written $J(A)$, or simply J.

Let A be a ring, M a simple left A-module, and let $A_M = \{a \in A : aM = 0\}$. Then A_M is a two-sided ideal of A, the *annihilator* of M. If L is a left A-module, $\mathrm{Soc}_M(L)$ will denote the *M-socle of L*, *i.e.*, the sum of all submodules of L isomorphic to M. We say L is *M-homogeneous* if $\mathrm{Soc}_M(L) = L$. In particular if A is left artinian, then A_M is a maximal two-sided ideal of A, and L is M-homogeneous if and only if $A_M L = 0$. We let S_M denote the M-socle $\mathrm{Soc}_M(A)$ of A. A *block* of A is an indecomposable ring direct summand of A [2, § 55]. We say *M belongs to* the block B of A if M is a composition factor of B regarded as a left A-module.

Let K be a field. If V is a vector space over K, let $(V : K)$ denote the K-dimension of V. Now assume A is a finite dimensional K-algebra. Then $A^* = \mathrm{Hom}_K(A, K)$ is an (A, A)-bimodule, where for $a \in A$ and $\lambda \in A^*$ we define

$$\begin{cases} (a\lambda)(x) = \lambda(xa) \\ (\lambda a)(x) = \lambda(ax) . \end{cases} \qquad (x \in A)$$

If $\lambda \in A^*$, we say λ is a *class function* if $\lambda(ab) = \lambda(ba)$ for all $a, b \in A$. Let $\mathrm{cf}(A)$ denote the set of class functions in A^*. Observe that $\lambda \in \mathrm{cf}(A)$ if and only if $a\lambda = \lambda a$ for all $a \in A$. We say $\chi \in A^*$ is an *A-character* if χ is the character of a (finite dimensional) left A-module. Clearly all A-characters belong to $\mathrm{cf}(A)$. The A-character χ is said to be *irreducible* if it is the character of a simple left A-module.

DEFINITION. Let A be a finite-dimensional K-algebra, K a field, and let M be a simple left A-module. We say an element $\lambda \in A^*$ *belongs to* M if $\lambda(A_M) = 0$. One checks that if λ belongs to M, then $a\lambda$ and λa belong to M for any $a \in A$. If χ is the character of M, then χ clearly belongs to M.

All other notation may be found in [2] or [3].

1. Socles of QF rings. A ring A is said to be a *QF* (quasi-Frobenius) ring if A is (left and right) artinian and ${}_AA$ is injective [2, (58.6)].

LEMMA 1.1. *Let A be a QF ring, M a simple left A-module, and $S_M = \mathrm{Soc}_M(A)$. Then S_M is a nonzero simple two-sided ideal of A.*

Proof. By [2, (58.13)], M is isomorphic to a minimal left ideal I in A, so $S_M \neq 0$. Moreover $S_M = \sum\{f(I): f \in \mathrm{Hom}_A(I, A)\}$ since $I \cong M$. By assumption ${}_AA$ is injective and from the "injective test lemma" [2, (57.14)] it follows that each $f \in \mathrm{Hom}_A(I, A)$ is right multiplication by some $a \in A$; that is, $f(I) = Ia \subseteqq IA$. Therefore $S_M \subseteqq IA$. But plainly $IA \subseteqq S_M$, and the lemma follows.

LEMMA 1.2. *Let A be a QF ring, M a simple left A-module, and $S_M = \mathrm{Soc}_M(A)$. Then the following are equivalent.*

(a) *S_M is a simple block of A to which M belongs*

(b) $S_M^2 \neq 0$

(c) *M is projective.*

Proof. That (a) implies (b) is obvious. For (b) implies (c), suppose M is not projective. By [2, (58.12)] M is isomorphic to the "bottom" constituent I of a principal indecomposable left ideal U, and as such, $I \subseteqq JU$ where J is the raidcal of A. Hence $I \subseteqq J$, and since $I \subseteqq S_M$, $S_M \subseteqq J$. But S_M is M-homogeneous and $J \subseteqq A_M$ so therefore

$$S_M^2 \subseteqq A_M S_M = 0 .$$

To show that (c) implies (a), notice that if M is projective, then M is injective [2, (58.14)] and therefore M is a direct summand of every module with M as a composition factor. Therefore every principal indecomposable module linked to M is isomorphic to M, so S_M is a block of A [2, § 55]. From (1.1) it follows that S_M is simple, completing the proof.

2. Frobenius algebras and class functions. Let A be a finite-dimensional K-algebra, K a field. We say that A is a *Frobenius algebra* if $A \cong A^* = \mathrm{Hom}_K(A, K)$ as left A-modules. To make this connection more explicit, we say that the pair (A, φ) is a *Frobenius algebra* if $\varphi: A \rightarrow A^*$ is a left A-isomorphism (see [4] and [2, (61.1)].) By [2, (61.1), (61.2)] a Frobenius algebra is necessarily a QF ring; hence the results of § 1 apply.

Let (A, φ) be a Frobenius algebra. If $(a_i), (b_i)$ is an ordered pair of bases for A such that $\varphi(b_i)(a_j) = \delta_{ij}$ for all i, j, we say the bases $(a_i), (b_i)$ are *φ-dual.* If (a_i) is any basis for (A, φ), there exists a second basis (b_i) such that $(a_i), (b_i)$ are φ-dual. (In most cases we are given an explicit isomorphism φ and certain φ-dual bases, although for an arbitrary Frobenius algebra A there may be many different choices for φ or φ-dual bases.)

LEMMA 2.1. *Let $(a_i), (b_i)$ be φ-dual bases for the Frobenius algebra (A, φ), and assume $\lambda \in A^*$. Then $\varphi^{-1}(\lambda) = \sum_i \lambda(a_i)b_i$.*

Proof. The proof is left to the reader.

LEMMA 2.2. *Let (A, φ) be a Frobenius algebra. Then the mapping $\tau = \tau_\varphi: A \rightarrow A$ given by $\tau(a) = \varphi^{-1}(\varphi(1)a)$ is a K-algebra automorphism of A.*

Proof. Observe that $\varphi(\tau(a)) = \varphi(1)a$ for each $a \in A$. Because φ is a left A-homomorphism and A^* is an (A, A)-bimodule, it follows easily that

$$\varphi(x\tau(a)) = \varphi(x)a\,, \qquad (x, a \in A)\,. \tag{2.3}$$

If $x = \varphi^{-1}(\lambda)$ for $\lambda \in A^*$, taking φ^{-1} on both sides of (2.3) gives

$$\varphi^{-1}(\lambda)\tau(a) = \varphi^{-1}(\lambda a)\,, \qquad (a \in A)\,. \tag{2.4}$$

Now for $a, b \in A$, $\varphi(\tau(ab)) = \varphi(1)ab = \varphi(\tau(a))b = \varphi(\tau(a)\tau(b))$ by (2.3), and therefore since φ is one-to-one, $\tau(ab) = \tau(a)\tau(b)$. If $\tau(a) = 0$, then $0 = \varphi(\tau(a)) = \varphi(1)a$, so $\varphi(b)a = b(\varphi(1)a) = 0$ for all $b \in A$. But then $0 = (\varphi(b)a)(1) = \varphi(b)(a)$ for all $b \in A$, and it follows that $a = 0$ since $\varphi(A) = A^*$. Therefore τ is one-to-one. Verification of the remaining properties of τ is left to the reader.

The reader may observe that $\tau = \tau_\varphi$ is the inverse of "Nakayama's automorphism" [5, Th. 1].

DEFINITION. Let (A, φ) be a Frobenius algebra with automorphism $\tau = \tau_\varphi$ as in (2.2). Define $Z_\tau(A) = \{a \in A: ba = a\tau(b) \text{ for all } b \in A\}$.

LEMMA 2.5. *Let (A, φ) be a Frobenius algebra with automorphism $\tau = \tau_\varphi$ as in (2.2). Then $\varphi^{-1}(\mathrm{cf}\,(A)) = Z_\tau(A)$.*

Proof. If $a \in Z_\tau(A)$, then $ba = a\tau(b)$ for all $b \in A$. Then by (2.3), $b\varphi(a) = \varphi(ba) = \varphi(a\tau(b)) = \varphi(a)b$ for all $b \in B$, so $\varphi(a) \in \mathrm{cf}\,(A)$. The converse is similar.

COROLLARY 2.6. *Let (A, φ), τ be as in* (2.5). *If $\lambda \in \mathrm{cf}\,(A)$ and $\lambda^* = \varphi^{-1}(\lambda)$, then $A\lambda^* = \lambda^* A$.*

Proof. By (2.5), $\lambda^* \in Z_\tau(A)$. Since τ is an automorphism of A, $A\lambda^* = \lambda^*\tau(A) = \lambda^* A$.

3. **Orthogonality relations, socles and characters.** *Throughout this section let (A, φ) be a Frobenius algebra.* Recall that if M is a simple left A-module, then $A_M = \{a \in A : aM = 0\}$, and for $\lambda \in A^*$, λ belongs to M if $\lambda(A_M) = 0$.

LEMMA 3.1. *Let M be a simple left A-module, and assume λ belongs to M, $\lambda \in A^*$. Set $\lambda^* = \varphi^{-1}(\lambda)$. Then $A_M\lambda^* = 0$. In particular if N is any left A-module, then $A\lambda^* N$ is M-homogeneous.*

Proof. If $b \in A_M$, then $b\lambda = 0$ because $(b\lambda)(A) = \lambda(Ab) \subseteq \lambda(A_M) = 0$. Therefore $b\lambda^* = \varphi^{-1}(b\lambda) = 0$, $b \in A_M$, and so $A_M\lambda^* = 0$. For N a left A-module, $A_M(A\lambda^* N) = A_M\lambda^* N = 0$, so $A\lambda^* N$ is M-homogeneous.

THEOREM 3.2. (*Orthogonality relations*). *Let M and N be simple left A-modules, and assume λ, μ belong to M, N, respectively, where $\lambda, \mu \in A^*$. Set $\lambda^* = \varphi^{-1}(\lambda)$. If $\lambda^* N \neq 0$, then $M \cong N$. In particular if $\mu(\lambda^*) \neq 0$, then $M \cong N$.*

Proof. Suppose $\lambda^* N \neq 0$. Since N is simple, $A\lambda^* N = N$. By (3.1), N must then be M-homogeneous, i.e., $M \cong N$. In particular if $\mu(\lambda^*) \neq 0$, then plainly $\lambda^* N \neq 0$, so by above $M \cong N$.

COROLLARY 3.3. *Let M, N, λ, μ be as in* (3.2), *and set*

$$\lambda^* = \varphi^{-1}(\lambda),\ \mu^* = \varphi^{-1}(\mu)\ .$$

If $\lambda^*\mu^* \neq 0$, then $M \simeq N$.

Proof. Assume $\lambda^*\mu^* \neq 0$. Then $\lambda^*\mu \neq 0$, and therefore for some $a \in A$, $0 \neq (\lambda^*\mu)(a) = \mu(a\lambda^*) = (\mu a)(\lambda^*)$. But μa belongs to N, so by (3.2), $M \cong N$.

COROLLARY 3.4. (*Orthogonality relations for characters*). *Let M, N be simple left A-modules with characters χ and ζ, respectively. Set $\chi^* = \varphi^{-1}(\chi)$. If $\zeta(\chi^*) \neq 0$, then $M \cong N$ and $\chi = \zeta$. In particular if (a_i), (b_i) are φ-dual bases for A, and if $\sum_i \chi(a_i)\zeta(b_i) \neq 0$, then $M \cong N$ and $\chi = \zeta$.*

Proof. The first part follows directly from (3.2) since the character of a simple module belongs to that module. Moreover modules uniquely determine their characters. For the second part we need only apply this and (2.1).

THEOREM 3.5. *Let χ be the character of the simple left A-module M, and assume $\chi \neq 0$. Set $\chi^* = \varphi^{-1}(\chi)$. Then $A\chi^* = \mathrm{Soc}_M(A) = S_M$. In particular either $(A\chi^*)^2 = 0$, or $A\chi^*$ is a simple block of A.*

Proof. Since $\chi \in \mathrm{cf}(A)$, (2.6) implies that $A\chi^* = \chi^*A = A\chi^*A$. By (3.1), $A\chi^*$ is M-homogeneous, so $0 \neq A\chi^* \subseteqq S_M$. Since S_M is a simple two-sided ideal (1.1), $A\chi^* = S_M$. The second assertion follows from this and (1.2).

COROLLARY 3.6. *Let χ^* be as in (3.5). If $(\chi^*)^2 = 0$, then $S_M^2 = 0$. If $(\chi^*)^2 \neq 0$, then χ^* is a unit in the simple block $A\chi^* = S_M$.*

Proof. By (3.5), $A\chi^* = S_M$. If $(\chi^*)^2 = 0$, then by (2.6),

$$S_M^2 = A\chi^*A\chi^* = A(\chi^*)^2 = 0 .$$

On the other hand if $(\chi^*)^2 \neq 0$, then $S_M^2 \neq 0$, so by (1.2) $S_M = A\chi^*$ is a simple block of A in which χ^* is clearly a unit.

COROLLARY 3.7. *Let χ, χ^* be as in (3.5). If $\chi(\chi^*) \neq 0$, then χ^* is a unit in the block $A\chi^* = S_M$.*

Proof. If $\chi(\chi^*) \neq 0$, then $\chi^*\chi \neq 0$ (in A^*), so by applying φ^{-1} we have $(\chi^*)^2 = \chi^*\varphi^{-1}(\chi) = \varphi^{-1}(\chi^*\chi) \neq 0$. By (3.6), χ^* is a unit in S_M.

REMARK. The converse to (3.7) is false. For example let F be a field of characteristic $p \neq 0$, and let F_p be the complete ring of p-by-p matrices over F. Then F_p is a Frobenius algebra with isomorphism φ given by $\varphi(X)(Y) = \mathrm{trace}\,(YX)$, $X, Y \in F_p$. One checks that if χ is the character of the unique simple left F_p-module M, then $\chi^* = \varphi^{-1}(\chi) = I_p$ where I_p is the p-by-p identity matrix. Then $\chi(\chi^*) = p = 0$ since F has characteristic p, but χ^* is clearly a unit in F_p.

The procedure for finding the socle of the left regular module ${}_AA$ of a Frobenius algebra (3.5) may also be applied to arbitrary left A-modules and in particular to principal indecomposable modules [2, (54.3)].

PROPOSITION 3.8. *Let M, χ^* be as in* (3.5), *and assume N is any left A-module. If $(\chi^*)^2 \neq 0$, then $\chi^* N = \mathrm{Soc}_M(N)$. If $(\chi^*)^2 = 0$, then $\chi^* N \subseteqq \mathrm{Soc}_M(JN)$ where J is the radical of A.*

Proof. Assume $(\chi^*)^2 \neq 0$. Then $S_M = \chi^* A$ is the simple block of A to which M belongs (3.6), and $S_M + A_M = A$. If L is any M-homogeneous left A-module, then $L = AL = S_M L = \chi^* L$. By (3.1), $A\chi^* N = \chi^* N$ is M-homogeneous, so $\chi^* N \subseteqq \mathrm{Soc}_M(N)$. But by above $\mathrm{Soc}_M(N) = \chi^* \mathrm{Soc}_M(N) \subseteqq \chi^* N$. Therefore $\chi^* N = \mathrm{Soc}_M(N)$. Now if $(\chi^*)^2 = 0$, then $\chi^* \in J = J(A)$ (since $\chi^* \in S_M$ and $S_M^2 = 0$ by (3.6)) so that $\chi^* N \subseteqq JN$. But $\chi^* N$ is M-homogeneous by (3.1), and therefore $\chi^* N \subseteqq \mathrm{Soc}_M(JN)$.

EXAMPLE. Equality does not necessarily occur in the relation $\chi^* N \subseteqq \mathrm{Soc}_M(JN)$ when $(\chi^*)^2 = 0$. For example let G be an abelian group of order a prime $p > 2$, and let F be a field of characteristic p. Then the group-algebra FG is a commutative local ring which is clearly a Frobenius algebra [2, Remark 2, p. 440] with isomorphism φ, say. Moreover if J denotes the radical of FG, then $J^p = 0$ and $J^{p-1} = \mathrm{Soc}(FG)$.

Let M be a simple left FG-module with character χ. Then all simple left FG-modules are isomorphic to M, and in particular

$$J^{p-1} = \mathrm{Soc}_M(FG) \cong M .$$

If $\chi^* = \varphi^{-1}(\chi)$, then $\chi^* \in \mathrm{Soc}_M(FG) = J^{p-1}$, and $(\chi^*)^2 = 0$. Let $N = J$, viewed as a left FG-module. Then $\chi^* N \subseteqq J^{p-1} J = J^p = 0$, so $\chi^* N = 0$. But $\mathrm{Soc}_M(JN) = \mathrm{Soc}_M(J^2) = J^{p-1} \neq 0$. Hence $\chi^* N \neq \mathrm{Soc}_M(JN)$.

THEOREM 3.9. *Let f be a primitive idempotent in A, and let M be a simple left A-module with character $\chi \neq 0$. Set $\chi^* = \varphi^{-1}(\chi)$. Then $\chi^* f \neq 0$ if and only if $\chi^* Af = \mathrm{Soc}(Af) \cong M$.*

Proof. By [2, (58.12)], $\mathrm{Soc}(Af)$ is simple. If $\chi^* f \neq 0$, then $0 \neq A\chi^* Af = \chi^* Af \subseteqq \mathrm{Soc}_M(Af) \subseteqq \mathrm{Soc}(Af)$ by (2.6) and (3.1), so

$$\chi^* Af = \mathrm{Soc}(Af) \cong M .$$

Conversely if $\chi^* Af = \mathrm{Soc}(Af)$, then $A\chi^* f = \chi^* Af \neq 0$ by (2.6), and therefore $\chi^* f \neq 0$.

4. **Applications.** *In this section (A, φ) is a Frobenius algebra, and ρ is the character of the left regular module ${}_A A$.*

LEMMA 4.1. *Let (a_i), (b_i) be φ-dual bases for (A, φ). Then*

$$\varphi^{-1}(\rho) = \sum_i a_i b_i .$$

Proof. This proof is identical to that of [3, (3.3)].

Let $M_1, \cdots, M_s$ be a complete set of (nonisomorphic) simple left A-modules with characters $\chi_1, \cdots, \chi_s$, respectively. Let m_i denote the multiplicity of M_i as a composition factor of ${}_AA$. It follows that $\rho = \sum_{i=1}^s m_i\chi_i$. If we set $\rho^* = \varphi^{-1}(\rho)$ and $\chi_i^* = \varphi^{-1}(\chi_i)$ for $1 \leqq i \leqq s$, then

$$\rho^* = \sum_{i=1}^{s} m_i\chi_i^* \; . \tag{4.2}$$

Now assume χ_k is any irreducible A-character, $1 \leqq k \leqq s$. For any $c \in A$, $c\chi_k$ belongs to M_k, and by (4.2),

$$\chi_k(\rho^*c) = (c\chi_k)(\rho^*) = \sum_{i=1}^{s} m_i(c\chi_k)(\chi_i^*) \; . \tag{4.3}$$

But by (3.2), $(c\chi_k)(\chi_i^*) = 0$ unless $M_i \cong M_k$, i.e., unless $i = k$. Therefore (4.3) implies that $\chi_k(\rho^*c) = m_k(c\chi_k)(\chi_k^*) = m_k\chi_k(\chi_k^*c)$. Together with (2.1) and (4.1) this gives the following (compare [3, (3.12)]).

THEOREM 4.4. *Let ρ be the character of the left regular module ${}_AA$, and let M be a simple left A-module with character χ. Set $\chi^* = \varphi^{-1}(\chi)$. Assume M appears as a composition factor of ${}_AA$ with multiplicity m. Then for any $c \in A$,*

$$\chi(\rho^*c) = m\chi(\chi^*c) \; .$$

In particular if (a_i), (b_i) are φ-dual bases for (A, φ), then for any $c \in A$,

$$\chi\Big(\sum_i a_ib_ic\Big) = m \sum_i \chi(a_i)\chi(b_ic) \; .$$

COROLLARY 4.5. *Let G be a finite group of order $|G|$, and let K be a field of characteristic p such that p does not divide $|G|$. Assume that K is a splitting field for KG. Then for any irreducible KG-character χ and for any $h \in G$,*

$$|G|\chi(h) = \chi(1)\sum_{g \in G} \chi(g)\chi(g^{-1}h) \; .$$

Proof. (Compare [2, (31.6)].) Let M be the simple left KG-module with character χ. Since KG is semi-simple (Maschke) and K-split, M appears exactly $(M: K)$ times as a composition factor of KG, and $\chi(1) = (M: K)$ in K. Finally KG is a Frobenius algebra with respect to the particular isomorphism φ for which (g), (g^{-1}) are φ-dual bases. By (4.4), $|G|\chi(h) = \chi(\sum_{g \in G} gg^{-1}h) = \chi(1)\sum_{g \in G}\chi(g)\chi(g^{-1}h)$ as desired.

Assume that B is a simple, finite-dimensional K-algebra, K a field, and let M be a simple left B-module. Set $D = \mathrm{Hom}_B(M, M)$.

Then D is a division ring, finite-dimensional over K, and B is isomorphic to D_m, the full ring of m-by-m matrices over D, for some positive integer m. It follows that M appears m times as a composition factor of ${}_BB$, and that $(M: K)m = (D: K)m^2$. Therefore $(M: K) = (D: K)m$. We combine these remarks with (4.4) to obtain the following.

COROLLARY 4.6. *Assume (A, φ) is a Frobenius algebra over a field K of characteristic zero, and let M be a simple projective left A-module with character χ. Set $\chi^* = \varphi^{-1}(\chi)$, $\rho^* = \varphi^{-1}(\rho)$, and $D = \mathrm{Hom}_A(M, M)$. Then for some $c \in A$, $\chi(\rho^* c) \neq 0$, and*

$$(D: K) = \chi(\rho^* c)^{-1}\chi(1)\chi(\chi^* c) .$$

In particular if $(a_i), (b_i)$ are φ-dual bases for (A, φ), then for some $c \in A$, $\chi(\sum_i a_i b_i c) \neq 0$ and for any such c,

$$(D: K) = \chi\Bigl(\sum_i a_i b_i c\Bigr)^{-1}\chi(1)\sum_i \chi(a_i)\chi(b_i c) .$$

Proof. Because M is projective and A is a QF ring, the M-socle S_M of A is a simple block of A to which M belongs by (1.2). Moreover $\chi(1) = (M: K)$ in K, and since K has characteristic zero, $\chi \neq 0$. By (3.6), $S_M = \chi^* A$ and $A = S_M + A_M$, so $0 \neq \chi(A) = \chi(S_M) = \chi(\chi^* A)$. Therefore for some $c \in A$, $\chi(\chi^* c) \neq 0$. But (4.4) shows that $\chi(\rho^* c) = m\chi(\chi^* c)$ where m is the multiplicity of M as a composition factor of ${}_AA$, and therefore $\chi(\rho^* c) \neq 0$. The remarks above the statement of this corollary show that $(M: K) = (D: K)m$. Combining these results, we have that

$$(D: K) = \chi(\rho^* c)^{-1}\chi(1)\chi(\chi^* c) .$$

The last statement follows from this, (2.1) and (4.1).

REMARK. The hypothesis that M be projective is essential in the above corollary, for if M is not projective (equivalently, if $(\chi^*)^2 = 0$) one can show from (1.2) and (3.6) that $\chi(\chi^* c) = 0$ for all $c \in A$.

COROLLARY 4.7. *Hypotheses as in (4.6). If $(a_i), (b_i)$ are φ-dual bases for A such that $\sum_i a_i b_i = (A: K)$, then*

$$(D: K) = (A: K)^{-1}\textstyle\sum_i \chi(a_i)\chi(b_i) .$$

Proof. Certainly $\chi(\rho^*) = \chi(\sum_i a_i b_i) = (A: K)\chi(1) \neq 0$. Therefore we may take $c = 1$ in the proof of (4.6), and the conclusion follows.

REMARKS. One might be surprised to observe that the hypotheses of (4.7) forces (A, φ) to be a semi-simple symmetric algebra [3]. For we know that $\rho \in \mathrm{cf}(A)$, so by (2.5) and (4.1), $(A: K) = \sum_i a_i b_i = \varphi^{-1}(\rho) \in Z_\tau(A)$. But then $1 \in Z_\tau(A)$ and therefore for any $b \in A$, $b = b \cdot 1 = 1 \cdot \tau(b) = \tau(b)$. We conclude that τ is the identity automomorphism which forces φ to be an (A, A)-bimodule isomorphism (2.3). Therefore (A, φ) is a symmetric algebra. Semi-simplicity follows from [3, (1.7)].

If G is a finite group, K a field of characteristic zero which is a splitting field for KG (e.g., K algebraically closed), then the sequence (2.1), (3.1), (3.2), (3.4), (4.1), (4.4) and (4.5) gives an elementary matrix-free proof of the orthogonally relations for irreducible characters of G.

The references below include a partial bibliography of papers relating to orthogonality relations for Frobenius algebras.

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