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ON THE OPEN CONTINUOUS IMAGES OF PARACOMPACT ČECH COMPLETE SPACES

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This article characterizes the completely regular T_0 open continuous images of paracompact Čech complete spaces. The characterization involves three conditions equivalent to being such an image. The first is an intrinsic condition concerning the position of the space in any of its Hausdorff bicompactifications. This condition weakens the condition of Čech completeness by replacing the concept of G_δ-*set* by that of *set of interior condensation*. This replacement yields a notion of topological completeness which has certain advantages over Čech completeness and uniform completeness but which reduces to Čech completeness in the case of metrizable spaces. The second condition (Condition $\mathscr{K}$) is intrinsically defined with the use of a sequence of collections of open sets. It is an analogue of the notion of a regular T_0-space having a monotonically complete base of countable order. The third condition is that of being an open continuous image of a space which is the sum of open Čech complete subspaces. The main theorem thus displays four equivalent forms of a topological completeness property invariant under open continuous mappings between Tychonoff spaces.

The characterization mentioned (Theorem 4) complements the characterization [15] of the Hausdorff open continuous images of T_2 paracompact p-spaces as Hausdorff spaces of point-countable type in a way analogous to that in which the characterization [14] of regular T_0 open continuous images of complete metric spaces as regular T_0-spaces having monotonically complete bases of countable order complements Ponomarev's characterization [11] of the T_0 open continuous images of metrizable spaces as the T_0 first countable spaces. It is relevant to recall in this connection some results of Frolík [7] and Arhangel'skiĭ [4], respectively: The class of T_2 paracompact Čech complete spaces (T_2 paracompact p-spaces) is the class of T_2 perfect preimages of complete metric spaces (metric spaces).

In [13] it was shown that a Tychonoff open continuous image of a paracompact Čech complete space (in fact, of a metrically topologically complete space) is not necessarily Čech complete. This affords interesting contrast with the behavior guaranteed by Theorem 5: A Tychonoff open continuous image of a Tychonoff space complete in the sense of Condition $\mathscr{K}$ is also complete in the same sense.

Some results of the paper have antecedents in the classical theory of metrically topologically complete spaces (i.e., spaces which are homeomorphic to complete metric spaces). Particularly relevant is the theorem of Čech [6] that a metrizable space is topologically complete if and only if it is a G_δ-subset of its Stone-Čech bicompactification. Theorem 4 provides an analogue of this in which the concept of metric topological completeness is replaced by Condition $\mathscr{K}$ and the concept of G_δ-set is replaced by that of a set of interior condensation. Certain results on the invariance of G_δ-sets in Euclidean space under open continuous mappings also are relevant; they are referred to after Theorem 2.

The plan of exposition is to introduce appropriate concepts and prove three theorems concerning these concepts from which Theorem 4 readily follows. The first of these theorems gives a sufficient condition that a regular T_0-space be an open continuous image of a paracompact Čech complete space. The second theorem concerns the behavior of certain sets of interior condensation under open continuous mappings and the third one shows that the extrinsic condition mentioned implies the intrinsic condition.

2. **Terminology.** The terminology generally used here is much like that of [8] except that the null set convention is not used, i.e., all sets herein have elements. If A and B are sets, $A \cdot B$ denotes the intersection of A and B and $A + B$ denotes the sum or union of A and B. Spaces called *compact* in [8] are here called *bicompact* following the usage of [1]. The usage of *compact space* here is that of Fréchet, namely, that there exists no infinite subset of the space which does not have a limit point. If K is a collection of sets, K^* denotes the sum of the members of K. The term *inner limiting set* is synonymous with *G_δ-set*. A Tychonoff space S ($\equiv$ completely regular T_0-space) is said to be *Čech complete* [6] if and only if it is an inner limiting set in its Stone-Čech bicompactification. A space is said to be *metrically topologically complete* if and only if it has a topology-preserving metric in which it is complete. The letters i, j, k, and n are used to signify positive integers.

3. **Condition $\mathscr{K}$.** A subset M of a topological space (S, τ) is said to be of *countable character in* S [5] if and only if there exists a sequence $D_1, D_2, \cdots$ of elements of τ including M such that any member of τ which includes M also includes some D_n.

A space S is said to be of *point-countable type* [5] if and only if S is covered by a collection of bicompact sets of countable character in S.

In [15] it is shown that a Hausdorff space is of point-countable type if and only if it is an open continuous image of a Hausdorff paracompact p-space. All first-countable spaces and all Tychonoff p-spaces are of point-countable type and the property of being of point-countable type is preserved by open continuous mappings [5].

LEMMA 1 [15]. *In a Hausdorff space S the following properties are equivalent*:

(i) *S is of point-countable type.*

(ii) *If D is open in S and P is in D there exists a bicompact set β of countable character which contains P and is a subset of D.*

DEFINITION. A topological space (S, τ) is said to satisfy *Condition* $\mathscr{K}$ if and only if there exists a sequence $G_1, G_2, \cdots$ of subcollections of τ covering S such that: (1) For each n, if P belongs to an element g of G_n, there exists a member g' of G_{n+1} containing P such that g includes $\bar{g}'$. (2) If $g_1, g_2, \cdots$ is a sequence such that, for each n, g_n is a member of G_n including $\overline{g_{n+1}}$, then there exists a (nonempty) bicompact set β which is the common part of the terms of $g_1, g_2, \cdots$ such that any member of τ which includes β also includes some g_n.

Spaces satisfying Condition $\mathscr{K}$ are of point-countable type. Thus in the Hausdorff case Lemma 1 applies to such spaces. This remark is used in the proof of Theorem 1.

Theorem 1 is a major component of Theorem 4. It is related to the theorem of [15] stated above and also to other theorems regarding the existence of open mappings [2, 11, 14, 16]. Recall that the weight of a topological space is the smallest cardinal number m such that the space has a base of cardinal m [1].

THEOREM 1. *Suppose S is a regular T_0-space. If S satisfies Condition $\mathscr{K}$ then S is an open continuous image of a paracompact Čech complete space of the same weight as S.*

Proof. Let τ denote the topology of S. Since S satisfies Condition $\mathscr{K}$, it may be seen that there exists a sequence $H_1, H_2, \cdots$ of well-ordered subcollections of τ covering S such that these conditions are satisfied: (1) For each n, each h in H_n contains a point $X_{n,h}$ belonging to no preceding element of H_n. (2) If $n < k$ and P is a point of S, the closure of the first element of H_k containing P is a subset of the first element of H_n containing P. (3) If $h_1, h_2, \cdots$ is a sequence such that each h_n belongs to H_n and each h_n is the first element of H_n containing $X_{n+1,h_{n+1}}$, then there exists a bicompact point set β such that (a) β is the common part of the terms of $h_1, h_2, \cdots$ and (b) every element of τ which includes β also includes some h_n.

Let Γ denote the collection of all bicompact point sets γ of (S, τ) such that for some decreasingly monotonic sequence $D_{1,\gamma}, D_{2,\gamma}, \cdots$ of elements of τ including γ it is true that every element of τ in which γ lies includes some $D_{n,\gamma}$. Let L_S denote the weight of S. Let τ' denote a base for S such that $\bar{\bar{\tau}}' = L_S$.

There exists a meaning for the notation $U_{n,\omega}$ for positive integers n and certain ordered pairs ω, such that with respect to some sequences $\Omega_1, \Omega_2, \cdots$ and $Q_2, Q_3, \cdots$ of transformations these conditions are satisfied: (1) For each n, the range of Ω_n is a subcollection of τ covering S. (2) For each $n > 1$, Ω_n and Ω_{n-1} are, respectively, the domain and range of Q_n. (3) If for each n and ω in Ω_n, B_ω denotes the second term of ω, then for each $n > 1$ and ω in Ω_{n-1}, P belongs to B_ω if and only if P belongs to the second term of some member of $Q_n^{-1}(\omega)$. (4) For each n, $\overline{\overline{\Omega}}_n \leqq L_S + \aleph_0$. (5) For each n and ω in Ω_n, $U_{n,\omega}$ is a finite sub-collection of τ covering B_ω such that each u in $U_{n,\omega}$ contains a point $Y_{n,\omega,u}$ such that $\bar{u}$ is a subset of the first element of H_n that contains $Y_{n,\omega,u}$. (6) For each $n > 1$ and ω in Ω_{n-1}, if ω' belongs to $Q_n^{-1}(\omega)$ and u belongs to $U_{n,\omega'}$, then there exists an element u' of $U_{n-1,\omega}$ such that $B_\omega \cdot u'$ includes $\bar{u}$. (7) If γ belongs to Γ there exists some ω in Ω_1 such that B_ω includes γ and $D_{1,\gamma}$ includes B_ω. Moreover, if $n > 1$, and ω belongs to Ω_{n-1}, and γ is a subset of B_ω belonging to Γ there exists some ω' in $Q_n^{-1}(\omega)$ such that $B_{\omega'}$ includes γ and $D_{n,\gamma}$ includes $U_{n,\omega'}^*$. (This construction may be carried out using (7) as a starting point.)

Let E denote the set of all sequences $(\omega_1, P), (\omega_2, P), \cdots$ such that (1) each ω_n belongs to Ω_n, (2) each ω_n is $Q_{n+1}(\omega_{n+1})$, and (3) each B_{ω_n} contains P. Let ψ denote the collection to which an element belongs if and only if it is the sum of some sets D, such that for some n, some ω in Ω_n, and some subset U of B_ω belonging to τ, D is the set of all sequences $(\omega_1, P), (\omega_2, P), \cdots$ in E such that $\omega = \omega_n$ and U contains P. Let φ denote the transformation of E such that for each sequence $(\omega_1, P), (\omega_2, P), \cdots$ belonging to E, $\varphi[(\omega_1, P), (\omega_2, P), \cdots] = P$.

It may be shown that (E, ψ) is a regular T_0 topological space, that φ maps (E, ψ) onto (S, τ) continuously, and that $\varphi(D)$ belongs to τ for each D in ψ. If it can be shown that (E, ψ) has a metrically topologically complete upper semicontinuous decomposition into bicompact sets, it follows from Frolík's theorem [7] that (E, ψ) is a paracompact Čech complete space.

For each n and ω in Ω_n, let $t_{n,\omega}$ denote the set of all sequences $(\omega_1, P), (\omega_2, P), \cdots$ in E such that $\omega = \omega_n$ and B_ω contains P. Let V_n denote the collection of all these sets $t_{n,\omega}$. Let G denote the collection to which g belongs if and only if there exists a decreasingly monotonic sequence $v_1, v_2, \cdots$ of sets such that each V_n contains v_n and g is the common part of the sets v_n. It may be shown that there exists a transformation θ of G such that if (1) $\omega_1, \omega_2, \cdots$ is a sequence such that each ω_n belongs to Ω_n and each ω_n is $Q_{n+1}(\omega_{n+1})$ and (2) if g is the common part of the sets t_{n,ω_n}, then $\theta(g)$ is $B_{\omega_1} \cdot B_{\omega_2} \cdot \cdots$. Moreover, it may be seen that if g belongs to G, $\theta(g) = \varphi(g)$ and $\varphi | g$ is reversibly continuous.

Suppose that $\omega_1, \omega_2, \cdots$ is a sequence such that each ω_n belongs to Ω_n and each ω_n is $Q_{n+1}(\omega_{n+1})$. (A) If $u_1, u_2, \cdots$ is a sequence such that each u_n belongs to U_{n,ω_n} and each u_n includes $\overline{u_{n+1}}$ then for each n there exists a first element h_n of H_n that includes some term of $u_1, u_2, \cdots$. If $n < k$ there exists some $i > k$ such that u_i is a subset of h_n and of h_k. Moreover, u_i is a subset of the first element h of H_n containing Y_{i,ω_i,u_i}. So h does not precede h_n. Since h_n contains Y_{i,ω_i,u_i}, h does not follow h_n. Hence $h = h_n$. Similarly, h_k is the first element of H_k containing Y_{i,ω_i,u_i}. So h_n includes h_k. Thus h_n is the first element of H_n containing X_{k,h_k}. So each h_n includes $\overline{h_{n+1}}$ and there exists a bicompact point set β such that β is the common part of the sets h_n and every element of τ in which β lies includes some h_n. Hence each $\bar{u}_n$ intersects β. Moreover, $u_1 \cdot u_2 \cdot \cdots$ is a bicompact subset β' of β. If D is a member of τ in which β' lies but which includes none of the sets u_n, there exists a sequence $P_1, P_2, \cdots$ of distinct points of S such that each $[u_n - (D \cdot u_n)] \cdot h_n$ contains P_n. But a contradiction is involved, for $\{P_1\} + \{P_2\} + \cdots$ has a limit point X belonging to β, and each $\bar{u}_n$ contains X, and each u_n includes $\overline{u_{n+1}}$. (B) Since each $B_{\omega_{n-1}}$ includes U^*_{n,ω_n}, the terms of the sequence B_{ω_1}, $B_{\omega_2}, \cdots$ have a common part M. Since each B_{ω_n} includes $\overline{B_{\omega_{n+1}}}$, M is closed. If J is an infinite subset of M then, since each of the finite collections U_{n,ω_n} covers B_{ω_n}, each of the collections has an element having an infinite intersection with J. With application of Konig's lemma it may be seen that there exists a sequence $u_1, u_2, \cdots$ as in (A) such that each u_n contains infinitely many points of J. Since every member of τ in which the bicompact set $u_1 \cdot u_2 \cdot \cdots$ lies includes some u_n, J has a limit point belonging to $u_1 . u_2 . \cdots$. (C) Let B denote the collection of all sets β' as in (A). If W is a subcollection of ψ covering M then for each β in B there exists a finite subcollection T_β of W covering β. For each β there exists some positive integer n such that some element u of U_{n,ω_n} including β is a subset of T^*_β. Since $U_{1,\omega_1} + U_{2,\omega_2} + \cdots$ is countable, there exists a sequence $\beta_1, \beta_2, \cdots$ of elements of B such that $T_{\beta_1} + T_{\beta_2} + \cdots$ covers M. Thus M is Lindelofian. Since M is T_1 and compact, M is bicompact. For reasons similar to some involved in (A), every member of τ in which M lies includes some B_{ω_n}. (D) Let g denote $t_{1,\omega_1} . t_{2,\omega_2} . \cdots$. Since $\varphi | g$ is reversibly continuous, and $\varphi(g) = M$, g is a T_2 bicompact point set. With the use of (C) it may be shown that every element of ψ in which g lies includes some t_{n,ω_n}. Moreover, for each n every member of V_n includes every element of G that it intersects. Thus G is upper semicontinuous. (E) With straightforward application of the conditions on $\Omega_1, \Omega_2, \cdots$ and $Q_2, Q_3, \cdots$ it may be seen that if γ belongs to Γ there exists a sequence $\omega_1, \omega_2, \cdots$ as above such that γ is a point set M as in (B). Moreover, for each n and ω in Ω_n, if γ is a subset of B_ω belonging to Γ then there exists such a sequence ω_1, $\omega_2, \cdots$ such that $\omega_n = \omega$ and $\gamma = M$.

For each n, let Δ_n denote the collection to which δ belongs if and only if for some v in V_n, δ is the collection of all elements of G that intersect v. It may be seen that $\Delta_1 + \Delta_2 + \cdots$ is a σ-discrete base for G with respect to the (appropriate) quotient topology ξ. Thus, since (G, ξ) is a regular T_0-space, (G, ξ) is metrizable. With use of the above paragraphs it may be seen that if $D_1, D_2, \cdots$ is a decreasingly monotonic sequence such that each D_n belongs to $\Delta_n + \Delta_{n+1} + \cdots$, then $\{D_1\} + \{D_2\} + \cdots$ is a base for (G, ξ) at some point. It follows that (G, ξ) is metrically topologically complete. With the use of condition (4) on $\Omega_1, \Omega_2, \cdots$ and $Q_2, Q_3, \cdots$, it may be shown that if $\aleph$ is the weight of (E, ψ) then $\aleph + \aleph_0 = L_S + \aleph_0$.

4. Sets of interior condensation. In this section we introduce a concept which has been essential in the authors' discussions of topological completeness in structures more general than metrizable spaces [17, 18]. The concept plays a role in these more general settings comparable to that played by inner limiting sets in the theory of Moore spaces and metrizable spaces.

Suppose S is a subset of a set E. A sequence $G_1, G_2, \cdots$ of collections of subsets of E covering S is said to be a *monotonically contracting sequence of S in E* if and only if, for each n, if P is an element of S belonging to a member g of G_n, there exists a subset of g which contains P and belongs to G_{n+1}.

A subset S of a topological space E is said to be a *set of interior condensation in E* if and only if there exists a monotonically contracting sequence $G_1, G_2, \cdots$ of S in E such that (1) each element of each G_n is open in E, and (2) if P is a point belonging to each term of a sequence $g_1, g_2, \cdots$ such that, for each n, g_n belongs to G_n and includes g_{n+1}, then P belongs to S.

It is clear that inner limiting sets are sets of interior condensation. It may be shown that sets of interior condensation in Moore spaces are inner limiting sets in these spaces.

The proof of the next theorem demonstrates the invariance of the property of being an absolute set of interior condensation under open mappings between Tychonoff spaces. This theorem shows the advantage obtained by relaxing the condition of being an inner limiting set to that of being a set of interior condensation, for there exists an open mapping between Tychonoff spaces whose domain is an absolute inner limiting set (Čech complete) and whose range is not. Such an example may be obtained by applying Theorem 2 of [14] to any non-Čech-complete subspace of the space Ω of countable ordinals with the order topology [13]. (Every subspace of Ω is regular, T_0, and has a monotonically complete base of countable order and is therefore an open continuous image of a complete metric space.)

Recall first that a mapping φ of a space S onto a space R is said to be *inductively open* [3] if and only if there exists a subspace S' of S such that $\varphi|S'$ is open and $\varphi(S') = R$.

THEOREM 2. *Suppose S is a set of interior condensation in a T_2 bicompact space. Then any inductively open continuous image of S is a set of interior condensation in any T_2-space of which it is a dense subspace.*

Proof. Let (E, ψ) denote a T_2 bicompact space of which (S, τ) is a subspace. Let $G_1, G_2, \cdots$ denote a sequence of subcollections of ψ covering S which satisfies the conditions of the definition of set of interior condensation. Let φ denote a continuous mapping of S onto a space R and S' a subspace of S such that $\varphi|S'$ is open and $\varphi(S') = R$. Let (F, σ) denote a T_2-space of which R is a dense subspace. There exists a sequence $H_1, H_2, \cdots$ of well-ordered subcollections of ψ covering S such that these conditions are satisfied for each n: (S1) Each h in H_n contains a point of S not in any predecessor of h. (S2) If P is in S, $n < k$, and h and h' are the first elements of H_n and H_k that contain P, respectively, then h includes h'. (S3) If P is in E and $h_1, h_2, \cdots$ is a sequence such that each h_n is a member of H_n which includes h_{n+1} and contains P, then P belongs to S. By using a technique related to one used in the proof of the Lemma on page 261 of [16] it may be shown that there exists a sequence $W_1, W_2, \cdots$ of well-ordered subcollections of σ covering R such that these conditions are satisfied for each n: (W1) Each w in W_n contains a point of R not in any predecessor of w. (W2) If P is in R, $n < k$, and w and w' are the first elements of W_n and W_k, respectively, that contain P, then w includes w'. There also exists a function D such that for each n: (D1) If w is in W_n, $D_{n,w}$ is an element of ψ such that

$$\varphi(D_{n,w}\cdot S') = w\cdot R .$$

(D2) If P is in R, $n < k$, and w and w' are the first elements of W_n and W_k, respectively, that contain P, then $\bar{D}^E_{k,w'}$ is a subset of $D_{n,w}$. (D3) If w is in W_n, there exists a point X of $D_{n,w}\cdot S'$ such that the first element of H_n that contains X includes $D_{n,w}$.

Let V_1 denote W_1 and for each $n > 1$ let V_n denote the collection of all sets of the form $v\cdot w$ where w is in W_n and v belongs to V_{n-1} and contains a point of w not in any predecessor of w in W_n. Then $V_1, V_2, \cdots$ is a sequence of subcollections of σ covering R. If P is in the element v of V_n and w is the first element of W_{n+1} that contains P, then $v \cdot w$ is a subset of v which contains P and belongs to V_{n+1}. Suppose P is a point belonging to each term of a sequence $v_1, v_2, \cdots$ where each v_n belongs to V_n and includes v_{n+1}. For each n there exists a first w_n in W_n that includes a term of $v_1, v_2, \cdots$. For each n there exists $j > n+1$ such that v_j is a subset of w_n and w_{n+1}. The set v_j is of the form $v \cdot w$ where v is in V_{j-1}, w is in W_j, and v contains a point X of w not in any predecessor of w. Let h denote the first element of W_n that contains X. Then h includes $v \cdot w$. Hence w_n does not follow h. Since w_n contains X it follows that $w_n = h$. Similarly w_{n+1} is the first element of W_{n+1} that contains X. Hence w_n includes w_{n+1} and, letting D_n denote D_{n,w_n}, it follows that $\bar{D}^E_{n+1}$ is a subset of D_n. Since E is bicompact, it follows that the sets $D_1, D_2, \cdots$ have a (nonempty) closed and bicompact common part K. By condition (D3) for each n there exists a first h_n in H_n that includes a term of $D_1, D_2, \cdots$. By (D3) and an argument similar to one used just above it follows that, for each n, h_n includes h_{n+1}. Since K is a subset of each h_n it follows from (S3) that K is a subset of S. Suppose P does not belong to $\varphi(K)$. Since $\varphi(K)$ is bicompact and F is T_2, $\varphi(K)$ is closed and there exists an open set D which includes $\varphi(K)$ such that P is not in $\bar{D}$. Since R is dense in F, for each n, there exists Y_n in R such that Y_n belongs to $w_n - \bar{D}$ and there exists X_n in $D_n \cdot S'$ such that $\varphi(X_n) = Y_n$. There exists a point Z such that Z is either a limit point of the set of all X_n's or else $X_n = Z$ for infinitely many n. In either case Z belongs to K so that $\varphi(Z)$ is in $\varphi(K)$ and thus $\varphi(Z)$ is in D. Since φ is continuous this implies that infinitely many Y_n belong to D which involves a contradiction. Therefore P belongs to $\varphi(K)$ and hence to R. Thus R is a set of interior condensation in F.

COMMENT. A classical theorem of Sierpiński [12] on the invariance of G_δ-sets in Euclidean n-space under open continuous mappings may be derived from this theorem. An earlier result is by Mazurkiewicz [10]. See [9, pp. 430–431] for further discussion.

THEOREM 3. *Suppose S is a set of interior condensation in some T_2 bicompact space. Then S satisfies Condition $\mathscr{K}$.*

Proof. Suppose S is a set of interior condensation in a T_2 bicompact space (E, ψ). Let $G_1, G_2, \cdots$ denote a sequence of ψ which satisfies the conditions of the definition of set of interior condensation. As in the proof of Theorem 2 there exists a sequence $H_1, H_2, \cdots$ of well-ordered subcollections of ψ satisfying conditions (S1)–(S3) for each n. For each n, let V_n denote the collection of all sets of the form

$h \cdot S$ where h belongs to H_n. Suppose h and h' are in H_n and $h \cdot S = h' \cdot S$. Since h contains a point P of S not in any predecessor of h and h' contains P it follows that h does not follow h'. Similarly h' does not follow h, so that $h = h'$. Hence each V_n may be well ordered by the prescription that v precedes v' if and only if h precedes h', where h and h' are in H_n and $v = h \cdot S$, $v' = h' \cdot S$. Suppose P is in S. Let h denote the first element of H_n that contains P. Then $h \cdot S$ is the first element of V_n that contains P. Suppose P is in S, $n < k$, and v and v' are the first elements of V_n and V_k, respectively, that contain P. Suppose $v = h \cdot S$ and $v' = h' \cdot S$ where h is in H_n and h' is in H_k. Then h and h' are the first elements of H_n and H_k, respectively, that contain P so that h includes $\bar{h}'$. Hence v includes $\overline{v'}^S$ (the closure of v' with respect to S). Suppose that $v_1, v_2, \cdots$ is a sequence such that, for each n, v_n is an element of V_n that includes $\bar{v}_{n+1}^S$. For each n there exists a first h_n in H_n that includes a term of $v_1, v_2, \cdots$. For each n there exists $j > n+1$ such that v_j is a subset of h_n and h_{n+1}. There exists an h in H_j such that $v_j = h \cdot S$. Let X denote a point of $h \cdot S$ not in any predecessor of h. By an argument used in the proof of Theorem 2 it follows that h_n and h_{n+1} are the first elements of H_n and H_{n+1}, respectively, that contain X. It follows that h_n includes $\bar{h}_{n+1}$. Hence there exists a bicompact point set β which is the common part of the terms of $h_1, h_2, \cdots$ and which is a subset of S. Suppose for some n, v_n does not meet β. Then there exists a sequence $P_1, P_2, \cdots$ such that, for each k, P_k is in v_{n+k} and is not in β. For each k there exists $n_k > k$ such that P_{n_k} is in h_k. If A denotes $\{P_{n_1}\} + \{P_{n_2}\} + \cdots$, then $\bar{A}$ meets β. For if it does not, some h_k does not meet $\bar{A}$. Suppose P is in $\bar{A} \cdot \beta$. Then P is not in A so that every open set containing P contains infinitely many elements of A. But then P is in each $\bar{v}_n^S$ and, therefore, in each v_n which is a contradiction. It follows that the common part β' of the terms of $v_1, v_2, \cdots$ exists and is a bicompact subset of β. By an argument used in the proof of Theorem 1 it follows that any open set in E which includes β' also includes some v_n.

Let G_1' denote V_1. For each $n > 1$ let G_n' denote the collection of all sets of the form $g \cdot v$ where g is an open set of S, v is in V_n, g contains a point P of v not in any predecessor of v in V_n, and for some g' in G_{n-1}', $\bar{g}^S$ is a subset of g'. We shall show that $G_1', G_2', \cdots$ is a sequence enabling S to satisfy Condition $\mathscr{K}$. Suppose P is a point of g in G_n'. There exist a first v in V_{n+1} that contains P, and an open set g' of S containing P such that $\overline{g'}^S$ is a subset of g. Hence $g' \cdot v$ is a member of G_{n+1}' which contains P and $\overline{g' \cdot v}^S$ is a subset of g. Suppose $g_1, g_2, \cdots$ is a sequence such that, for each n, g_n belongs to G_n' and includes $\bar{g}_{n+1}^S$. For each n there exists a first v_n in

V_n that includes a term of $g_1, g_2, \cdots$. By an argument used in the proof of Theorem 2 it follows that, for each n, v_n includes $\bar{v}^S_{n+1}$. Thus a bicompact common part β of the terms of $v_1, v_2, \cdots$ exists. By an argument similar to one used in the preceding paragraph it follows that there exists a bicompact point set β' which is the common part of the terms of $g_1, g_2, \cdots$ such that any open set in which β' lies includes some g_n.

5. The characterization theorem.

THEOREM 4. *Suppose S is a Tychonoff space. Then the following conditions on S are equivalent*:

(a) *S is a set of interior condensation in one (equivalently, in each) of its Hausdorff bicompactifications.*

(b) *S satisfies Condition $\mathscr{K}$.*

(c) *S is an open continuous image of a paracompact Čech complete space.*

(d) *S is an open continuous image of a space which is a sum of open Čech complete subspaces.*

Proof. Condition (a) implies (b) by Theorem 3. Condition (b) implies (c) by Theorem 1. That (c) implies (d) is obvious. Any space as in (d) is an open continuous image of a Čech complete space. Since a Čech complete space is a set of interior condensation in its Stone-Čech bicompactification a Tychonoff open continuous image of it is a set of interior condensation in each of its T_2 bicompactifications by Theorem 2. Thus (d) implies (a).

COMMENT. The reader is referred to the introduction for a comparison of part of Theorem 4 with a theorem of Čech.

THEOREM 5. *Suppose S is a Tychonoff space which satisfies Condition $\mathscr{K}$. Then any Tychonoff inductively open continuous image of S satisfies Condition $\mathscr{K}$.*

Proof. If S satisfies $\mathscr{K}$ then S is a set of interior condensation in its Stone-Čech bicompactification by Theorem 4. By Theorem 2 any Tychonoff inductively open continuous image of S is a set of interior condensation in its Stone-Čech bicompactification. An application of Theorem 4 completes the proof.

REFERENCES

1. P. S. Aleksandrov ($\equiv$P. S. Alexandroff) and P. Uryson ($=$P. Urysohn), *Mémoire sur les espaces topologiques compacts*, Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. I, 14, no. **1** (1929), 1-96.
2. A. V. Arhangel'skiĭ, *On mappings of metric spaces*, Dokl. Akad. Nauk SSSR **145** (1962), 245-247 (Russian); Trans. in Soviet Math. Dokl., **3** (1962), 953-956.
3. ———, *On open and almost-open mappings of topological spaces*, Dokl. Akad. Nauk SSSR **147** (1962), 999-1002 (Russian).
4. ———, *On a class of spaces containing all metric and all locally bicompact spaces*, Mat. Sb., **67** (1965), 55-85 (Russian).
5. ———, *Bicompact sets and the topology of spaces*, Trudy Mosk. Mat. Obšč., **13** (1965), 3-55 (Russian).
6. E. Čech, *On bicompact spaces*, Ann. of Math., **38** (1937), 823-844.
7. Z. Frolík, *On the topological product of paracompact spaces*, Bull. Acad. Polon. Sci. Sér. Math., Astr., Phys., **8** (1960), 747-750.
8. J. L. Kelley, *General Topology*, Van Nostrand, Princeton, 1955.
9. C. Kuratowski, *Topology* Vol. I, New edition, Academic Press, New York, and Warsaw, 1966.
10. S. Mazurkiewicz, *Über Borelsche Mengen*, Bull. Acad. Cracovie, (1916), 490-494.
11. V. I. Ponomarev, *Axioms of countability and continuous mappings*, Bull. Acad. Polon. Sci., Sér. Sci. Math., Astr. Phys., **8** (1960), 127-133 (Russian).
12. W. Sierpiński, *Sur une propriété des ensembles* G_δ, Fund. Math., **16** (1930), 173-180.
13. H. H. Wicke, *The non-preservation of Čech completeness by an open continuous mapping between collectionwise normal* T_1*-spaces*, Notices of the Amer. Math. Soc., **13** (1966), 601.
14. ———, *The regular open continuous images of complete metric spaces*, Pacific J. Math., **23** (1967), 621-625.
15. ———, *On the Hausdorff open continuous images of Hausdorff paracompact p-spaces*, Proc. Amer. Math. Soc., **22** (1969), 136-140.
16. H. H. Wicke and J. M. Worrell, Jr., *Open continuous mappings of spaces having bases of countable order*, Duke Math. J., **34**, No. 2 (1967), 255-272.
17. ———, *On topological completeness of first-countable Hausdorff spaces*, submitted for publication.
18. J. M. Worrell, Jr. and H. H. Wicke, *Complete* μ*-spaces*, Notices of the Amer. Math. Soc., **14** (1967), 856.

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