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ANALYTIC SHEAVES ON KLEIN SURFACES

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Morphisms of Klein surfaces are discussed from the sheaf-theoretic standpoint, and the cohomology of an analytic sheaf on a Klein surface is computed.

0. Let $\mathfrak{X}$ be a Klein surface [1], [2]; that is, $\mathfrak{X}$ consists of an underlying space X, which is a surface with boundary, and a family of equivalent dianalytic atlases on X. If (U_α, z_α) is such an atlas, then $z_\alpha: U_\alpha \to \boldsymbol{C}^+$ is a homeomorphism of the open set U_α in X onto an open subset of $\boldsymbol{C}^+ = \{z \in \boldsymbol{C} \mid \operatorname{Im}(z) \geqq 0\}$. The functions z_α must thus be real on $U_\alpha \cap \partial X$, and it is required that $z_\alpha \circ z_\beta^{-1}$ be dianalytic, that is, either analytic or antianalytic on each component of $z_\beta(U_\alpha \cap U_\beta)$.

In this paper we define the structure sheaf of $\mathfrak{X}$, show that the concept of morphism given in [1], [2] coincides with the concept of a morphism of ringed spaces, and compute the cohomology of analytic sheaves on $\mathfrak{X}$. If $\mathscr{F}$ is an analytic sheaf on X, and $\widetilde{\mathscr{F}}$ is the lift of $\mathscr{F}$ to the complex double $\widetilde{\mathfrak{X}}$ of $\mathfrak{X}$, then there is a natural isomorphism

$$H^q(\widetilde{\mathfrak{X}}, \widetilde{\mathscr{F}}) \cong \boldsymbol{C} \otimes_{\boldsymbol{R}} H^q(\mathfrak{X}, \mathscr{F}) .$$

1. **The structure sheaf $\mathscr{O}_{\mathfrak{X}}$.** We define the structure sheaf $\mathscr{O}_{\mathfrak{X}} = \mathscr{O}$ on $\mathfrak{X}$ as follows. If U is open in X, let $\mathscr{O}(U)$ be the ring of holomorphic functions on U (in the sense of [1], [2]). If $U \supset U'$, then the inclusion map is a morphism of Klein surfaces and we have a natural map $\rho^U_{U'}: \mathscr{O}(U) \to \mathscr{O}(U')$ (this is not quite an ordinary restriction map since the elements of $\mathscr{O}(U)$ are not quite functions). In particular, if (U_α, z_α) and (U_β, z_β) are dianalytic charts on $\mathfrak{X}$, $U_\alpha \supset U_\beta$, then

$$\mathscr{O}(U_\alpha) \cong \left\{ \begin{array}{l} f: U_\alpha \to C \mid f(U_\alpha \cap \partial X) \subset \boldsymbol{R}\} , \\ \qquad\qquad\qquad\qquad \text{and } f \circ z_\alpha^{-1} \text{ analytic} \end{array} \right\}$$

and

$$\rho^{U_\alpha}_{U_\beta}(f) = \begin{cases} f \mid U_\beta \text{ where } z_\alpha \circ z_\beta^{-1} \text{ is analytic} \\ \bar{f} \mid U_\beta \text{ where } z_\alpha \circ z_\beta^{-1} \text{ is antianalytic} . \end{cases}$$

It is easily checked that this defines a sheaf of local R-algebras on $\mathfrak{X}$.

Let $\mathfrak{X}, \mathfrak{Y}$ be Klein surfaces, $f: \mathfrak{Y} \to \mathfrak{X}$ a continuous map. Then f is a morphism [1] if $f(\partial Y) \subset \partial X$ and if for every point $p \in Y$ there are dianalytic charts (V, w) and (U, z) at p and $f(p)$, and an analytic function h on $w(V)$, such that

$$\begin{tikzcd} V \arrow[rr, dashed, "f|V"] \arrow[d, "w"'] & & U \arrow[d, "z"] \\ \boldsymbol{C}^+ \arrow[r, "h"'] & \boldsymbol{C} \arrow[r, "\phi"'] & \boldsymbol{C}^+ \end{tikzcd}$$

commutes (ϕ is the folding map, $\phi(a + bi) = a + |b|i$).

Recall that a ringed space morphism $\mathfrak{Y} \to \mathfrak{X}$ is a pair (f, θ) where $f: Y \to X$ is continuous and $\theta: \mathcal{O}_{\mathfrak{X}} \to f_* \mathcal{O}_{\mathfrak{Y}}$ is a morphism of sheaves of rings [4, p. 36]. Here $f_*\mathcal{O}_{\mathfrak{Y}}$ is the direct image sheaf: $f_*\mathcal{O}_{\mathfrak{Y}}(U) = \mathcal{O}_{\mathfrak{Y}}(f^{-1}(U))$.

THEOREM 1. *Let $\mathfrak{X}$, $\mathfrak{Y}$ be Klein surfaces, and let $f: Y \to X$ be a nonconstant continuous map. Then the following are equivalent*:

(i) *f is a morphism*;

(ii) *there exists a morphism $\theta: \mathcal{O}_{\mathfrak{X}} \to f_*\mathcal{O}_{\mathfrak{Y}}$ of sheaves of $\boldsymbol{R}$-algebras.*

Under these conditions the morphism θ is unique, so f can be made in a unique way into a morphism of ringed spaces.

Proof. (i) $\Rightarrow$ (ii). Let $U \supset U'$ be open in X. From the commutative diagram:

$$\begin{tikzcd} f^{-1}(U) \arrow[d, "f"'] & f^{-1}(U') \arrow[l] \arrow[d, "f"] \\ U & U' \arrow[l] \end{tikzcd}$$

of morphisms of Klein surfaces we deduce a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{\mathfrak{X}}(U) \arrow[r] \arrow[d] & \mathcal{O}_{\mathfrak{X}}(U') \arrow[d] \\ \mathcal{O}_{\mathfrak{Y}}(f^{-1}(U)) \arrow[r] & \mathcal{O}_{\mathfrak{Y}}(f^{-1}(U')) \end{tikzcd}$$

of morphisms of $\boldsymbol{R}$-algebras, and this defines an $\boldsymbol{R}$-algebra morphism $\theta: \mathcal{O}_{\mathfrak{X}} \to f_* \mathcal{O}_{\mathfrak{Y}}$.

(ii) $\Rightarrow$ (i). Let $p \in Y$, and let (V, w), (U, z) be dianalytic charts at p, $f(p)$, with $f(V) \subset U$. Let z^* be the image of z in $\mathcal{O}_{\mathfrak{Y}}(V)$ under

$$(*) \qquad \mathcal{O}_{\mathfrak{X}}(U) \to \mathcal{O}_{\mathfrak{Y}}(f^{-1}(U)) \to \mathcal{O}_{\mathfrak{Y}}(V) .$$

Set $h = z^* \circ w^{-1}$. We claim $f | V = z^{-1} \circ \phi \circ h \circ w$, i.e. that $z \circ (f | V) = \phi \circ z^*$. It clearly suffices to show that $z(f(p)) = \phi(z^*(p))$. If this does not hold, then

$$g = \frac{1}{[z - z^*(p)][z - \overline{z^*(p)}]}$$

is holomorphic at $f(p)$, and shrinking U, V if necessary, we may assume $g \in \mathscr{O}_{\mathfrak{X}}(U)$. We let g^* denote its image under (*) in $\mathscr{O}_{\mathfrak{Y}}(V)$. But $g^* = 1/[z^* - z^*(p)][z^* - \overline{z^*(p)}]$ which is not defined at p.

We still need to show that $f(\partial Y) \subset \partial X$. Let $q \in X$. Then $\mathscr{O}_{\mathfrak{X},q}$ is an $\boldsymbol{R}$-algebra which contains a copy of $\boldsymbol{C}$ if and only if $q \notin \partial X$. The $\mathscr{O}_{\mathfrak{X},q}$ algebra $(f_* \mathscr{O}_{\mathfrak{X}})_q$ is isomorphic to

$$\prod_{f(p)=q} \mathscr{O}_{\mathfrak{Y},p} ,$$

so $q \notin \partial X$, $f(p) = q$ implies $p \notin \partial Y$.

We now check that θ is unique. Let U be open in X, $g \in \mathscr{O}_{\mathfrak{X}}(U)$, $p \in f^{-1}(U)$. Let (V, w) be a dianalytic chart at p with $V \subset f^{-1}(U)$. Let g^* be the image of g in $\mathscr{O}_{\mathfrak{Y}}(V)$ under (*). Then using the above arguments, either $g^*(p) = gf(p)$ or $g^*(p) = \overline{gf(p)}$. If g is nonconstant, only one of these can yield an analytic function. If g is constant it can be expressed as a sum of nonconstant functions. Hence g^*, and thus θ, are uniquely determined. The theorem is proved.

By an analytic sheaf of $\mathfrak{X}$ we mean an $\mathscr{O}_{\mathfrak{X}}$-module. If $\mathscr{F}$ is an analytic sheaf on $\mathfrak{X}$ and $f\colon \mathfrak{Y} \to \mathfrak{X}$ is a morphism then $f^*\mathscr{F}$ is the sheaf associated to the presheaf $V \to \mathscr{O}_{\mathfrak{Y}}(V) \otimes_{\mathscr{O}_{\mathfrak{X}}(fV)} \mathscr{F}(fV)$.

PROPOSITION 2. *If $\mathscr{F}$ is a coherent analytic sheaf on $\mathfrak{X}$, then $f^*\mathscr{F}$ is a coherent analytic sheaf on $\mathfrak{Y}$.*

Proof. The proof given in [5, p. 47] for Riemann surfaces carries over to the Klein surface case.

2. The complex double. Let $\mathfrak{X}$ be a Klein surface, $\pi\colon \tilde{\mathfrak{X}} \to \mathfrak{X}$ its complex double. Recall that if (U_α, z_α) is a dianalytic atlas on $\mathfrak{X}$, then $(\tilde{U}_\alpha, \tilde{z}_\alpha)$ is a dianalytic atlas on $\tilde{\mathfrak{X}}$, where $\tilde{U}_\alpha = \pi^{-1}(U_\alpha) = U'_\alpha \cup U''_\alpha$, $U'_\alpha \cap U''_\alpha = \pi^{-1}(U_\alpha \cap \partial X)$, and π maps U'_α and U''_α each homeomorphically onto U_α. The function $\tilde{z}_\alpha$ is defined by

$$\tilde{z}_\alpha(p) = \begin{cases} z_\alpha(p) & p \in U'_\alpha \\ \overline{z_\alpha(p)} & p \in U''_\alpha \end{cases} .$$

U'_α is identified with U'_β where $z_\alpha \circ z_\beta^{-1}$ is analytic, and with U''_β where $z_\alpha \circ z_\beta^{-1}$ is anti-analytic. This construction yields the Riemann surface (without boundary) $\tilde{\mathfrak{X}}$ as a double cover of $\mathfrak{X}$, folded along ∂X.

If U is open in X, let $\tilde{U} = \pi^{-1}(U)$. We denote the structure sheaf of $\tilde{\mathfrak{X}}$ by $\tilde{\mathscr{O}}$.

PROPOSITION 3. *There is a canonical isomorphism*

$$(\dagger) \qquad \boldsymbol{C} \otimes_{\boldsymbol{R}} \mathscr{O}(U) \overset{\cong}{\to} \widetilde{\mathscr{O}}(\tilde{U})$$

for every open set $U \subset X$.

Proof. We may cover U by dianalytic charts (U_α, z_α). It then suffices to verify (†) for U_α, since $\mathscr{O}(U)$ is the difference kernel of $\prod_\alpha \widetilde{\mathscr{O}}(\tilde{U}_\alpha) \rightrightarrows \prod_{\alpha,\beta} \widetilde{\mathscr{O}}(\tilde{U}_\alpha \cap \tilde{U}_\beta)$ and $\boldsymbol{C} \otimes_{\boldsymbol{R}}$ is exact.

Let σ be the canonical anti-involution of $\tilde{\mathfrak{X}}$ which commutes with π, and let κ denote complex conjugation. If we identify $\mathscr{O}(U_\alpha)$ with its image in $\widetilde{\mathscr{O}}(\tilde{U}_\alpha)$ then we see

$$\mathscr{O}(U_\alpha) = \{g \in \widetilde{\mathscr{O}}(\tilde{U}_\alpha) \mid g = \kappa g \sigma\} .$$

But any $g \in \mathscr{O}(U_\alpha)$ can be written as

$$g = \tfrac{1}{2}(g + \kappa g \sigma) + \tfrac{1}{2}(g - \kappa g \sigma)$$

and hence the canonical map

$$\boldsymbol{C} \otimes_{\boldsymbol{R}} \mathscr{O}(U_\alpha) \to \widetilde{\mathscr{O}}(\tilde{U}_\alpha)$$

is surjective. This map is easily seen to be injective, completing the proof.

If $\mathscr{F}$ is an analytic sheaf on $\mathfrak{X}$, let $\tilde{\mathscr{F}} = \pi^* \mathscr{F}$.

THEOREM 4. *There is a canonical isomorphism*

$$\boldsymbol{C} \otimes_{\boldsymbol{R}} \mathscr{F}(\mathfrak{X}) \cong \tilde{\mathscr{F}}(\tilde{\mathfrak{X}}) .$$

Proof. We may choose a base for the topology of X consisting of sets of the form U_α, where (U_α, z_α) is a dianalytic atlas on X. Then sets of the form U'_α, U''_α(where $U_\alpha \cap \partial X = \varnothing$) and of the form $\tilde{U}_\alpha$(where $U_\alpha \cap \partial X \neq \varnothing$) form a base B for the topology of $\tilde{\mathfrak{X}}$. Since $\widetilde{\mathscr{O}}(\tilde{U}) \otimes_{\mathscr{O}(U)} \mathscr{F}(U) \cong \boldsymbol{C} \otimes_{\boldsymbol{R}} \mathscr{F}(U)$, it suffices to show that the sequence

$$(\dagger\dagger) \qquad \begin{aligned} 0 \to \widetilde{\mathscr{O}}(\tilde{\mathfrak{X}}) \otimes_{\mathscr{O}(\mathfrak{X})} \mathscr{F}(\mathfrak{X}) &\to \prod_{V \in B} \widetilde{\mathscr{O}}(V) \otimes_{\mathscr{O}(\pi V)} \mathscr{F}(\pi V) \\ &\rightrightarrows \prod_{V, W \in B} \widetilde{\mathscr{O}}(V \cap W) \otimes_{\mathscr{O}} (\pi(V \cap W)) \mathscr{F}(\pi(V \cap W)) . \end{aligned}$$

is exact. When U'_α and U''_α are disjoint then $\widetilde{\mathscr{O}}(\tilde{U}_\alpha) = \widetilde{\mathscr{O}}(U'_\alpha) \times \widetilde{\mathscr{O}}(U''_\alpha)$ so (††) may be replaced by

$$0 \to \widetilde{\mathscr{O}}(\mathfrak{X}) \otimes_{\mathscr{O}(X)} \mathscr{F}(\mathfrak{X}) \to \prod_{\alpha} \widetilde{\mathscr{O}}(\tilde{U}_\alpha) \otimes_{\mathscr{O}(U_\alpha)} \mathscr{F}(U_\alpha)$$
$$\rightrightarrows \prod_{\alpha,\beta} \widetilde{\mathscr{O}}(\tilde{U}_{\alpha\beta}) \otimes_{\mathscr{O}(U_{\alpha\beta})} \mathscr{F}(U_{\alpha\beta})$$

and this last is exact because of Proposition 3 and the fact that $\mathscr{F}$ is a sheaf.

Since the functors $\mathscr{F} \to \boldsymbol{C} \otimes_R \mathscr{F}(\mathfrak{X})$ and $\mathscr{F} \to \widetilde{\mathscr{F}}(\tilde{\mathfrak{X}})$ are canonically isomorphic, so are their derived functors [3], and we have

THEOREM 5. *Let $\mathscr{F}$ be an analytic sheaf on the Klein surface $\mathfrak{X}$. Then there is a canonical isomorphism*

$$H^q(\tilde{\mathfrak{X}}, \widetilde{\mathscr{F}}) \cong \boldsymbol{C} \otimes_R H^q(\mathfrak{X}, \mathscr{F})$$

for all $q \geqq 0$.

COROLLARY. (Cartan Theorem B) *Let $\mathfrak{X}$ be a non-compact Klein surface, $\mathscr{F}$ a coherent analytic sheaf on $\mathfrak{X}$. Then $H^q(\mathfrak{X}, \mathscr{F}) = 0$ for all $q \geqq 1$*

Proof. Use Theorem 5 and Proposition 2 to reduce to the case of a non-compact Riemann surface [6, p. 270].

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