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NOTE ON THE OPEN MAPPING THEOREM

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The open mapping and closed graph theorems are usually stated in terms of metrizable topological groups which are complete in a one-sided uniformity. It would be desirable to use only the weaker hypothesis of completeness in the two-sided uniformity. Important results of this sort are already known, and it is our purpose to strengthen those results by removing the separability hypotheses.

There are three uniformities normally used in connection with a topological group: the left, right, and two-sided uniformities. Completeness in the left uniformity is equivalent to completeness in the right uniformity, and stronger than completeness in the two-sided uniformity. If the group is metrizable, then it always has at least one left invariant metric; completeness in the left uniformity is equivalent to completeness in this metric (it does not matter which left-invariant metric is chosen). Still in the case of a metrizable group, completeness in the two-sided uniformity is equivalent to topological completeness, i.e., the existence of at least one metric in which the group is complete as a metric space (see [3], Exercise $Q(d)$, p. 212 for proof). Every topological group has a completion in the two-sided sense, but not necessarily in the one-sided sense (more precisely, the completion in the left-uniformity, though it is a complete uniform space, is not in general a group). When the one-sided completion does exist, it coincides with the two-sided completion. Two important examples of metrizable topological groups which are complete in the two-sided but not the one-sided sense are the full permutation group of a countably infinite set and the unitary group of an infinite-dimensional, separable Hilbert space (with the strong operator topology). Both of these examples are separable. Nonseparable examples of less naturality can be obtained by taking the direct product of one of the above with a nonseparable Banach space.

Lemma 1 below, which is the separable case of our theorem, was essentially proved by Banach [1], though his statement of it (Satz 8) was weaker. Its present statement and brief proof are the same as in Corollary 3.2 of Pettis [5]. Corollary 1 of the theorem and the corollary to Lemma 1 have some independent interest. They state that if N is a closed normal subgroup of the group G, which is metrizable and complete in its two-sided uniformity, then G/N is also complete. The corresponding result for one-sided completeness is trivial and is true even if N is not normal. (This case where N is not normal is, so far as we know, the only case where the results for one-sided completeness remain significantly superior to those for two-sided completeness.) Lemma 2, which is corollary 2 to the theorem, may also have some independent interest. Our theorem itself, when strengthened by the addition of the accompanying remark, differs from the "standard"[1] version of the open mapping theorem ([2], Theorem 3, p. 90) only in the type of completeness used. Our version of the closed graph theorem, Corollary 3, is slightly weaker than the analog of the "standard" version in that both groups are assumed metrizable.

LEMMA 1. *Let G and H be separable metrizable topological groups which are complete in their two-sided uniformities and π: $G \to H$ a continuous homomorphism such that $\overline{\pi(U)}$ is a neighborhood of the identity for each neighborhood U of the identity in G. Then π is open.*

REMARK. It is easy to see that the hypothesis on $\overline{\pi(U)}$ will be satisfied if π is surjective. This special case is Satz 8 of [1].

Proof. Let U be an open nighborhood of the identity in G. Then $\pi(U)$ is analytic and hence ([4], p. 94-95) almost open. But $\pi(U)$ is second category in H. (If $\pi(U) \subseteqq \bigcup_n F_n$, F_n closed, then some $\pi^{-1}(F_n) \cap U$ has interior. Hence $\overline{\pi\,(\pi^{-1}(F_n)} \subseteqq F_n$ has interior.) Hence (see [5] or [3], Exercise $P(b)$, p. 211) $\pi(U) \cdot \pi(U)^{-1}$ is a neighborhood of the identity in H.

COROLLARY. *If G is a polonais group* (i.e., a separable, metrizable, and topologically complete topological group) *and N is a closed normal subgroup, then G/N is again complete and hence polonais.*

Proof. Let H be the completion of G/N.

LEMMA 2. *Let G and H be metrizable topological groups which are complete in their two-sided uniformities and π: $G \to H$ a continuous homomorphism such that $\overline{\pi(U)}$ is a neighborhood of the identity in H for each neighborhood U of the identity in G and such that $\overline{\pi(G)} = H$. Then for each closed separable subgroup H_0 of H, there is a closed separable subgroup G_0 of G such that π is a relatively open map of G_0 onto H_0.*

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[1] A perhaps better choice for the "standard" version appears in [**3**; p. 213]. It has a weaker hypothesis.

Proof. Let $U_1 \supseteqq U_2 \supseteqq U_3 \cdots$ be a fundamental system of neighborhoods of the identity in G. Let $V_j (j = 1, 2, \cdots)$ be open neighborhoods of the identity in H such that $\overline{\pi(U_j)} \supseteqq V_j$. Let G_1 be a closed separable subgroup of G such that $\overline{\pi(G_1)} \supseteqq H_0$ and

$$\overline{\pi(G_1 \cap U_j)} \supseteqq H_0 \cap V_j .$$

Let $H_1 = \overline{\pi(G_1)}$. Similarly, define G_n $(n = 2, 3 \cdots)$ and $H_n (n = 2, 3 \cdots)$ so that for $n \geqq 1$:

(i) $G_n \subseteqq G_{n+1}$, $H_n \subseteqq H_{n+1}$,

(ii) $\overline{\pi(G_n)} = H_n$,

(iii) $\overline{\pi(G_n \cup U_j)} \supseteqq H_{n-1} \cap V_j$,

(iv) G_n, H_n are closed and separable.

Let $G_\infty = \overline{\bigcup_{n=1}^{\infty} G_n}$ and $H_\infty = \overline{\bigcup_{n=1}^{\infty} H_n}$. Then by Lemma 1, π is a relatively open map of G_∞ onto H_∞. Let $G_0 = \pi^{-1}(H_0) \cap G_\infty$.

THEOREM. *Let G and H be metrizable topological groups which are complete in their two-sided uniformities and π: $G \to H$ a continuous homomorphism such that $\overline{\pi(U)}$ is a neighborhood of the identity in H for each neighborhood U of the identity in G. Then π is open.*

REMARK. It is easy to remove all hypotheses on H (except that H be a Hausdorff topological group) from the statement of the theorem. (It is easy to construct a countable base at e and hence prove H metrizable. H can then be completed.)

Proof. Since $\overline{\pi(G)}$ is an open subgroup of H, we may assume $\pi(G)$ dence. Let y_n be a sequence in H so that $\lim y_n = e$. Then $\{y_n\}$ is imbedded in a closed separable subgroup H_0 of H. Choose G_0 as in Lemma 2. Then there are x_n in G_0 such that $\pi(x_n) = y_n$ and $\lim x_n = e$.

COROLLARY 1. *If G is a metrizable topological group which is complete in its two-sided uniformity and N a closed normal subgroup, then G/N is complete in its two-sided uniformity.*

COROLLARY 2. *With G, N as in Corollary* 1, *any closed separable subgroup of G/N is the image of a closed separable subgroup of G.*

COROLLARY 3. *If G and H are as in the thorem, and f: $G \to H$ is a homomorphism such that f has a closed graph, and $f^{-1}(U)$ is a neighborhood of the identity in G for each neighborhood of the identity U in H, then f is continuous.*

Proof. Apply the theorem to the projection of K onto G, where K is the graph of f.

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