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STRICTLY CYCLIC OPERATOR ALGEBRAS

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This paper is concerned with the structure of abelian algebras $\mathscr{A}$ of operators on Hilbert space $\mathscr{H}$ such that $\mathscr{A}x = \mathscr{H}$ for some vector x in H. It is shown that if a transitive algebra $\mathscr{T}$ contains such an algebra then $\mathscr{T}$ is dense in the weak topology on $\mathscr{L}(\mathscr{H})$. It is also shown that when an algebra of this type is semi-simple then it is a reflexive operator algebra. The algebras investigated have the property that every densely defined linear trans-formation commuting with the algebra is bounded.

Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. The study of subalgebras of $\mathscr{L}(\mathscr{H})$ has primarily dealt with self-adjoint algebras. The literature on non-self-adjoint subalgebras of $\mathscr{L}(\mathscr{H})$ is far less complete. This paper is concerned with a class of non-self-adjoint subalgebras, the strictly cyclic abelian subalgebras. The first application of these algebras will be to the theory of transitive algebras. A subalgebra $\mathscr{T}$ of $\mathscr{L}(\mathscr{H})$ is *transitive* if the only closed subspace of $\mathscr{H}$ invariant for every operator in $\mathscr{T}$ are $\mathscr{H}$ and $\{0\}$. W. B. Arveson showed that a knowledge of the (possibly) unbounded linear transformations commuting with a transitive algebra $\mathscr{T}$ can be used to decide if $\mathscr{T}$ is dense in the weak operator topology on $\mathscr{L}(\mathscr{H})$ (it is not kown if every transitive algebra of operators on an infinite dimensional Hilbert space must be weakly dense in $\mathscr{L}(\mathscr{H})$).

Arveson also proved that every transitive algebra containing a maximal abelian self-adjoint algebra is weakly dense in $\mathscr{L}(\mathscr{H})$. E. Nordgren, H. Radjavi, and P. Rosenthal used Arveson's techniques to show that if $\mathscr{H}$ is separable, then every transitive algebra of operators containing a certain type of weighted shift must be dense in $\mathscr{L}(\mathscr{H})$. It is shown that every transitive algebra containing a strictly cyclic abelian algebra is weakly dense in $\mathscr{L}(\mathscr{H})$. It has been shown that the weakly closed algebras generated by certain weighted shifts are strictly cyclic. This class of shifts properly contains the class of shifts mentioned above. In particular, several examples of shifts generating strictly cyclic algebras are neither compact nor quasi-nilpotent.

In § 3 we develop some tests for strict cyclicity of abelian algebras. In § 5 we show that certain stictly cyclic abelian algebras are unitarily equivalent to multiplication operator algebras on functional Hilbert spaces (Theorem 5.1), and are examples of reflexive operator algebras. We then give examples of strictly cyclic abelian algebras on spaces of arbitrary dimension and show that there exist non-singly generated strictly cyclic abelian algebras.

2. Preliminaries. *A subalgebra* $\mathscr{A}$ *of* $\mathscr{L}(\mathscr{H})$ *is* cyclic *if*

$$\mathscr{A}x_0 = \{Ax_0 : A \text{ in } \mathscr{A}\}$$

is strongly dense in $\mathscr{H}$ *for some vector* x_0 *in* $\mathscr{H}$. $\mathscr{A}$ *is* strictly cyclic *if* $\mathscr{A}x_0 = \mathscr{H}$. *The vector* x_0 *is called cyclic for* $\mathscr{A}$ *in the former case and strictly cyclic in the latter.*

If $\mathscr{A}$ is abelian and x_0 is cyclic for $\mathscr{A}$, then x_0 is also separating for $\mathscr{A}$, i.e., if A is in $\mathscr{A}$ and $Ax_0 = 0$, then $A = 0$. It follows that for each x in $\mathscr{A}x_0$ there is a unique operator A_x in $\mathscr{A}$ such that $A_x x_0 = x$. Let ρ be the mapping $x \to A_x$ of $\mathscr{A}x_0$ onto $\mathscr{A}$. It is clear that ρ is a bijective linear transformation.

If $\mathscr{A}$ is a subalgebra of $\mathscr{L}(\mathscr{H})$ and T is a possibly unbounded linear transformation with domain $D(T)$, then by "T commutes with $\mathscr{A}$" we mean for every A in $\mathscr{A}$, $A(D(T))$ is contained in $D(T)$ and $AT = TA$ on $D(T)$. T is *closed* if graph $(T) = \{\langle x, Tx\rangle : x \text{ in } D(T)\}$ is closed in $\mathscr{H} \oplus \mathscr{H}$. T_1 is an extension of T_2 if $D(T_1)$ contains $D(T_2)$ and $T_1 = T_2$ on $D(T_2)$. A linear transformation is *closable* if it has a closed extension. It is easy to see T is closable if and only if whenever $\{x_n\}$ is a sequence in $D(T)$ converging strongly to 0, then either Tx_n diverges or Tx_n converges strongly to 0.

In the remainder of this paper $\mathscr{A}$ is assumed to be an abelian subalgebra of $\mathscr{L}(\mathscr{H})$ with cyclic vector x_0. We note that for any x and y in $\mathscr{A}x_0$, $A_y A_x x_0 = A_y x = A_x y$. Also, $A_x y$ is in $\mathscr{A}x_0$. We will assume $\mathscr{H}$ is infinite dimensional, $\mathscr{A}$ is weakly closed, and $\|x_0\| = 1$.

3. Conditions equivalent to strict cyclicity. We showed in [6] that $\mathscr{A}$ is strictly cyclic if and only if ρ is continuous with respect to the strong topology on $\mathscr{A}x_0$ and the uniform topology on $\mathscr{A}$. Also, ρ^{-1} is a contraction since $\|A_x\| \geqq \|A_x x_0\| = \|x\|$.

For each x in $\mathscr{H}$ define the linear transformation U_x by

$$D(U_x) = \mathscr{A}x_0 \quad \text{and} \quad U_x y = A_y x\ .$$

LEMMA 3.1. *Each* U_x *commutes with* $\mathscr{A}$, *and if* $\mathscr{A}$ *is maximal abelian, then* U_x *is bounded if and only if* x *is in* $\mathscr{A}x_0$.

Proof. Let y and z be in $\mathscr{A}x_0$ and let $w = A_y z$. Then

$$\begin{aligned} A_y U_x z &= A_y A_z x = A_w x \\ &= U_x w = U_x A_y z , \end{aligned}$$

showing U_x commutes with $\mathscr{A}$.

Now suppose $\mathscr{A}$ is maximal abelian. If U_x is bounded, let A be the bounded operator extending U_x. Then A commutes with $\mathscr{A}$. Thus $x = U_x x_0 = A x_0$ is in $\mathscr{A} x_0$. The converse is trivial.

COROLLARY 3.2. *$\mathscr{A}$ is strictly cyclic if and only if $\mathscr{A}$ is maximal abelian and each U_x is bounded.*

Proof. By Lemma 3.1 it suffices to show every strictly cyclic abelian algebra is maximal abelian. Let $\mathscr{A}$ be strictly cyclic and suppose B is a bounded operator commuting with $\mathscr{A}$. Then for every y in $\mathscr{H}$, $By = B A_y x_0 = A_y B x_0 = A_{Bx_0}(y)$, showing $B = A_{Bx_0}$.

LEMMA 3.3. *$\mathscr{A}$ is strictly cyclic if and only if $\mathscr{A}$ is maximal abelian and the dual space of $\mathscr{A}$ consists entirely of the maps $A_x \rightarrow (x, y)$, y in $\mathscr{H}$.*

Proof. Suppose first $\mathscr{A}$ is strictly cyclic. Then $\mathscr{A}$ is maximal abelian and if f is a continuous linear functional on $\mathscr{A}$, then the composition $f \circ \rho$ is a continuous linear functional on $\mathscr{H}$. Thus there is a unique y in $\mathscr{H}$ such that $f(A_x) = f(\rho(x)) = (x, y)$ for every x in $\mathscr{H}$. Conversely, suppose these are the only continuous linear functionals on $\mathscr{A}$. Then for each pair x, y in $\mathscr{H}$ there is a vector $K(x, y)$ in $\mathscr{H}$ such that for every A in $\mathscr{A}$, $(Ax, y) = (Ax_0, K(x, y))$. Since $\mathscr{A} x_0$ is dense, $K(x, y)$ is uniquely defined. Also, it is easy to see for fixed x the map $K_x \colon y \rightarrow K(x, y)$ is an everywhere defined linear transformation. Fix x in $\mathscr{H}$ and let z be in $\mathscr{A} x_0$. Then for every y in $\mathscr{H}$, $(A_z x, y) = (z, K(x, y))$. But $A_z x = U_x z$ so that for all y in $\mathscr{H}$ and z in $\mathscr{A} x_0$, $(U_x z, y) = (z, K(x, y))$. Thus U_x^* is everywhere defined (in fact, $U_x^* y = K(x, y)$). Since the adjoint of every linear transformation is closed, U_x^* is closed and everywhere defined. Thus U_x^* is bounded and U_x^{**} is then a bounded extension of U_x. By Corollary 3.2, $\mathscr{A}$ is strictly cyclic.

The next lemma yields information about the spectra of operators in a strictly cyclic abelian algebra and will be used in § 4 and § 5.

LEMMA 3.4. *If $\mathscr{A}$ is strictly cyclic, then there is a nonzero y in $\mathscr{H}$ such that $A_x^* y = (y, x) y$ for every x in $\mathscr{H}$.*

Proof. Since $\mathscr{A}$ is a commutative Banach algebra with identity, there is a nonzero multiplicative linear functional f on $\mathscr{A}$. By Lemma 3.3 there is a y in $\mathscr{H}$ such that $f(A_x) = (x, y)$ for every x in $\mathscr{H}$. Let x and z be in $\mathscr{H}$, and let $w = A_x z$. Then $A_x A_z = A_w$ and so $f(A_w) = f(A_x)(A_z)$, i.e.,

$$(A_x z, y) = (x, y)(z, y) = (z, (y, x)y) .$$

Thus $A_x^* y = (y, x)y$.

4. Transitivity and strict cyclicity. We begin this section with a brief summary of Arveson's analysis of transitive algebras. This material is found in [1].

Let $\mathscr{T}$ be a subalgebra of $\mathscr{L}(\mathscr{H})$. For N a positive integer, $\mathscr{T}$ is *N-fold transitive* if for every linearly independent set $\{x_1, x_2, \cdots, x_N\}$ in $\mathscr{H}$, and for every set $\{y_1, y_2, \cdots, y_N\}$ in $\mathscr{H}$. There is a sequence $\{T_k\}$ in $\mathscr{T}$ such that $\text{limit}_{k\to\infty} T_k x_i = y_i$, $i = 1, 2, \cdots, N$. Note that 1-fold transitivity is transivity.

LEMMA (Arveson). *A subalgebra $\mathscr{T}$ of $\mathscr{L}(\mathscr{H})$ is weakly dense if and only if $\mathscr{T}$ is N-fold transitive for every positive integer N.*

THEOREM 4.1 (Arveson). *Let $\mathscr{T}$ be a transitive subalgebra of $\mathscr{L}(\mathscr{H})$. Then*

(a) *$\mathscr{T}$ is not 2-fold transitive if and only if there exists a non-scalar closed linear transformation commuting with $\mathscr{T}$; and*

(b) *if $N \geqq 2$ and $\mathscr{T}$ is N-fold but not $(N+1)$-fold transitive, then there exist linear transformations $T_1, T_2, \cdots, T_N$ with common dense domain D such that each T_i commutes with $\mathscr{T}$, no T_i is closable, and $\{\langle x, T_1x, T_2x, \cdots, T_Nx\rangle : x \text{ in } D\}$ is closed in $\mathscr{H} \oplus \cdots \oplus \mathscr{H}$* ($N+1$ copies).

We now examine the linear transformations commuting with a strictly cyclic abelian algebra $\mathscr{A}$.

LEMMA 4.2. *Let T be a linear transformation commuting with $\mathscr{A}$. Then either T is closable or there is a nonzero A in $\mathscr{A}$ such that*

$$A(D(T)) = 0 .$$

Proof. Suppose T is not closable. Then there is a sequence $\{x_n\}$ of vectors in $D(T)$ such that x_n converges to 0 but Tx_n converges to a non-zero vector y. Let z be in $D(T)$. Then

$$\begin{aligned} A_y z = A_z y &= A_z(\operatorname*{limit}_n \; Tx_n) \\ &= \operatorname*{limit}_n \; (A_z T x_n) = \operatorname*{limit}_n \, (TA_{x_n} z) \\ &= \operatorname*{limit}_n \; (A_{x_n} Tz) = 0 \; . \end{aligned}$$

LEMMA 4.3. *Let $\mathscr{M}$ be a linear submanifold of $\mathscr{H}$* (not necessarily closed) *with x_0 in the closure of $\mathscr{M}$. If $\mathscr{M}$ is invariant for $\mathscr{A}$, then $\mathscr{M} = \mathscr{H}$.*

Proof. Since ρ is continuous and $A_{x_0} = I$, there is a vector x in $\mathscr{M}$ with $\| I - A_x \| < 1$. In particular, A_x is invertible. Since $\mathscr{A}$ is maximal abelian, A_x^{-1} is in $\mathscr{A}$. But then $x_0 = A_x^{-1} A_x x_0 = A_x^{-1} x$ is in $\mathscr{M}$, and so for any y in $\mathscr{H}$, $y = A_y x_0$ is in $\mathscr{M}$.

COROLLARY 4.4. *Every densely defined linear transformation commuting with $\mathscr{A}$ is everywhere defined and bounded.*

Proof. Let T be a densely defined linear transformation commuting with $\mathscr{A}$. By Lemma 4.2 T is closable, and by Lemma 4.3 $D(T) = \mathscr{H}$. By the closed graph theorem T is bounded.

We are now ready to prove the main result of this section.

THEOREM 4.5. *Let $\mathscr{T}$ be a transitive algebra containing a strictly cyclic abelian algebra $\mathscr{A}$. Then $\mathscr{T}$ is weakly dense in $\mathscr{L}(\mathscr{H})$.*

Proof. By Corollary 4.5 every densely defined linear transformation commuting with $\mathscr{T}$ is bounded. Thus by 4.1 it suffices to show that every bounded operator commuting with $\mathscr{T}$ is a scalar multiple of I. Let A be a bounded operator commuting with $\mathscr{T}$ (and consequently with $\mathscr{A}$). Then A is in $\mathscr{A}$ and so by Lemma 3.4 there is a nonzero vector y and a scalar $\bar{a}$ such that $A^* y = \bar{a} y$. It follows that Range$(A - aI)$ is not dense in $\mathscr{H}$. But $A - aI$ commutes with $\mathscr{T}$ and so Range$(A\text{-}aI)$ is invariant for $\mathscr{T}$. Since $\mathscr{T}$ is transitive, Range$(A - aI)$ is either dense or $\{0\}$. Thus Range$(A - aI) = \{0\}$, i.e., $A = aI$.

5. Semisimplicity and strict cyclicity. A commutative Banach algebra $\mathscr{B}$ is *semisimple* if for every x in $\mathscr{B}$, there is a multiplicative linear functional f on $\mathscr{B}$ such that $f(x) \neq 0$. Some of the examples we gave in [6] of strictly cyclic abelian algebras are semisimple (e.g., the weakly closed algebra generated by the weighted shift with weights $\{(n+1)/n\}$). The collection of all multiplicative linear functionals on $\mathscr{B}$ will be denoted $\mathscr{M}_a$ and is called the maximal ideal space of $\mathscr{B}$.

Let $\mathscr{A}$ be a strictly cyclic abelian subalgebra of $\mathscr{L}(\mathscr{H})$, with the notation of § 2. For each y in $\mathscr{H}$, let y^* be the linear functional $y^*(A_x) = (x, y)$, and let $\mathscr{N}(\mathscr{A})$ be the collection of all y in $\mathscr{H}$ such that y^* is multiplicative. If $\mathscr{N}(\mathscr{A})$ is given the relative weak Hilbert space topology and $\mathscr{M}_{\mathscr{A}}$ is given the maximal ideal space topology [8; p. 110], the map $y \to y^*$ is a homeomorphism between $\mathscr{N}(\mathscr{A})$ and $\mathscr{M}_{\mathscr{A}}$(this is just the identification of $\mathscr{H}$ with its dual space restricted to $\mathscr{N}(\mathscr{A})$). In particular, $\mathscr{N}(\mathscr{A})$ is compact in the weak Hilbert space topology. A short calculation shows a vector y is in $\mathscr{N}(\mathscr{A})$ if and only if $(x_0, y) = 1$ and y is an eigenvector for the adjoint of every operator in $\mathscr{A}$.

We see that $\mathscr{A}$ is semisimple if and only if for every x in $\mathscr{H}$ there is a y in $\mathscr{N}(\mathscr{A})$ such that $(x, y) \neq 0$. This is equivalent to saying $\mathscr{N}(\mathscr{A})$ spans $\mathscr{H}$ (i.e., the smallest closed subspace of $\mathscr{H}$ containing $\mathscr{N}(\mathscr{A})$ is $\mathscr{H}$). Before continuing the discussion of semisimple strictly cyclic algebras, it is necessary to discuss fuctional Hilbert spaces. A Hilbert space $\mathscr{F}$ is a *functional Hilbert space* if there is a set X such that

(i) the elements of $\mathscr{F}$ are complex valued functions on X;

(ii) each point evaluation is a continuous linear functional on $\mathscr{F}$; and

(iii) for each x in X there is an f in $\mathscr{F}$ such that $f(x) \neq 0$.

We will denote such a functional Hilbert space by $(\mathscr{F}, X)$.

If $(\mathscr{F}, X)$ is a functional Hilbert space and g is a complex valued function on X such that gf is in $\mathscr{F}$ for every f in $\mathscr{F}$, then the linear transformation M_g: $f \to gf$ is called a multiplication operator. An easy application of (ii) and the closed graph theorem shows every multiplication operator on a functional Hilbert space is bounded.

In [5; p. 32] it is shown that a bounded operator A on an abstract Hilbert space $\mathscr{H}$ is unitarily equivalent to a multiplication operator on a functional Hilbert space if and only if the eigenvectors of A^* span $\mathscr{H}$. This easily generalizes to the following: If $\mathscr{A}$ is a subalgebra of $\mathscr{L}(\mathscr{H})$ and

$$X = \{x \text{ in } \mathscr{H}: x \text{ is an eigenvector for } A^* \text{ for all } A \text{ in } \mathscr{A}\}$$

spans $\mathscr{H}$, then $\mathscr{A}$ is unitarily equivalent to an algebra of multiplication operators on a functional Hilbert space. The idea is if u is a vector in $\mathscr{H}$, let u' be defined on X by $u'(x) = (u, x)$. Then define $\| u' \| = \| u \|$ and let U be the unitary transformation $Uu = u'$. If A is in $\mathscr{A}$, then $UAU^{-1} = M_f$ where $A^*x =$ (complex conjugate of $f(x))x$ for every x in X.

We now return to the case of $\mathscr{A}$ a semisimple, strictly cyclic abelian algebra. Then $\mathscr{N}(\mathscr{A})$ spans $\mathscr{H}$, and by the preceding remarks $\mathscr{H}$ is unitarily equivalent to a functional Hilbert space $(\mathscr{F}, \mathscr{N}(\mathscr{A}))$.

THEOREM 5.1. *Let $\mathscr{A}$ be a semisimple, strictly cyclic abelian subalgebra of $\mathscr{L}(\mathscr{H})$. Then $\mathscr{A}$ is unitarily equivalent to the algebra of all multiplication operators on a functional Hilbert space $(\mathscr{F}, \mathscr{N}(\mathscr{A}))$. Moreover, each f in $\mathscr{F}$ is continuous and there is a constant M such that for every f in $\mathscr{F}$,*

$$\| f \|_\infty = \max \{ | f(x) |: x \text{ in } \mathscr{N}(\mathscr{A}) \leqq M \| f \| .$$

Proof. We have only to show each f in $\mathscr{F}$ is continuous and satisfies the norm inequality. Let f be in $\mathscr{F}$ and let z be in $\mathscr{H}$ such that $Uz = f$. Then for every x in $\mathscr{N}(\mathscr{A})$, $f(x) = (z, x)$, showing f is continuous. Since $\mathscr{N}(\mathscr{A})$ is weakly compact, it is bounded, say, by M. Thus, for every x in $\mathscr{N}(\mathscr{A})$,

$$| f(x) | = | (z, x) | \leqq \| z \| \, \| x \| \leqq M \, \| z \| = M \| f \| .$$

REMARKS. 1. The continuity and norm inequality in Theorem 5.1 could have been ascertained by considering $\mathscr{H}$ as a Banach algebra with $\| z \|_1 = \| A_z \|$ and using the theory of the Gelfand transform.

2. The bound M on $\mathscr{N}(\mathscr{A})$ can be chosen to be the norm of ρ, i.e, $\sup \{ \| A_z \| : \| z \| = 1 \}$, since for each x in $\mathscr{N}(\mathscr{A})$,

$$\| x \|^4 = (x, x)\,(x, x) = (A_x x, x) \leqq \| A_x \| \, \| x \|^2 \leqq \| x \|^3 \, \| \rho \| .$$

Finally, we show that semisimple, strictly cyclic abelian algebras are examples of reflexive operator algebras. A subalgebra $\mathscr{B}$ of $\mathscr{L}(\mathscr{H})$ is *reflexive* if for every B in $\mathscr{L}(\mathscr{H})$, if B leaves invariant all the closed invariant subspaces of $\mathscr{B}$, then B is in $\mathscr{B}$. Reflexive algebras are studied in [2] and [9].

THEOREM 5.2. *If $\mathscr{A}$ is a semisimple, strictly cyclic abelian algebra, then $\mathscr{A}$ is reflexive.*

Proof. It is easy to see that an algebra $\mathscr{B}$ is reflexive if and only if $\mathscr{B}^* = \{B^*: B \text{ in } \mathscr{B}\}$ is reflexive. We show that $\mathscr{A}^*$ is reflexive. Suppose B is a bounded operator leaving invariant all the closed invariant subspaces of $\mathscr{A}^*$. For each x in $\mathscr{N}(\mathscr{A})$, the one-dimensional space spanned by x is invariant for $\mathscr{A}^*$ and hence for B. Since $\mathscr{N}(\mathscr{A})$ spans $\mathscr{H}$ it follows that B commutes with $\mathscr{A}^*$.

Since $\mathscr{A}^*$ is maximal abelian, B is in $\mathscr{A}^*$.

REMARK. It is not true that every strictly cyclic abelian algebra is reflexive. Let $\mathscr{H}$ be separable, with orthonormal basis $\{e_n\}_{n=0}^{\infty}$ and let S be the weighted shift operator $Se_n = (1/2^n)e_{n+1}$. R. Gellar [4] showed that the weakly closed algebra $\mathscr{A}$ generated by S is strictly cyclic, and W. Donoghue [3] proved that the only closed subspaces invariant for $\mathscr{A}$ are $\{0\}$ and $V_{k=n}^{\infty} e_k$, $n = 0, 1, \cdots$. These subspaces are invariant for any operator whose matrix relative to $\{e_i\}$ is lower triangular.

We now show that there exist strictly cyclic abelian algebras on Hilbert spaces of any dimension. We then conclude this paper by showing that for any Hilbert space $\mathscr{H}$ of dimension greater than 2, $\mathscr{L}(\mathscr{H})$ contains a non-singly generated strictly cyclic abelian algebra.

Let $\mathscr{H}$ be an arbitrary complex Hilbert space. For vectors u and v in $\mathscr{H}$, $u \otimes v$ is the operator on $\mathscr{H}$ defined by $(u \otimes v)(x) = (x, u)v$. Let x_0 be a fixed unit vector in $\mathscr{H}$, and for each x in $\mathscr{H}$ let

$$A_x = (x, x_0)\, P + x_0 \otimes x\ ,$$

where P is the orthogonal projection of $\mathscr{H}$ onto $\{x_0\}^{\perp}$. Let

$$\mathscr{A} = \{A_x\colon x \text{ in } \mathscr{H}\}\ .$$

LEMMA 5.3. *$\mathscr{A}$ is an abelian subalgebra of $\mathscr{L}(\mathscr{H})$ and x_0 is strictly cyclic for $\mathscr{A}$.*

Proof. Clearly $\mathscr{A}$ is a linear subspace of $\mathscr{L}(\mathscr{H})$ with

$$\lambda A_x + A_y = A_{\lambda x + y}\ .$$

Also, for every x in $\mathscr{H}$,

$$\begin{aligned} A_x x_0 &= (x, x_0)\ Px_0 + (x_0, x_0)x \\ &= \|x_0\|^2 x = x\ , \end{aligned}$$

so x_0 is strictly cyclic for $\mathscr{A}$. It remains to show that $\mathscr{A}$ is an abelian algebra. Let x and y be in $\mathscr{H}$. Then

$$\begin{aligned} A_x A_y = {} & (x, x_0)(y, x_0)P \\ & + (x, x_0)P(x_0 \otimes y) \\ & + (x_0 \otimes x)\,(y, x_0)P + (x_0 \otimes x)\,(x_0 \otimes y)\ . \end{aligned}$$

We note that for any vectors u and v, and for any bounded operator T,

$$T(u \otimes v) = u \otimes (Tv)$$

and

$$(u \otimes v)T = (T^*u) \otimes v .$$

Thus

$$\begin{aligned} A_x A_y &= (x, x_0)(y, x_0)P + (x, x_0)x_0 \otimes (Py) \\ &\quad + \{(y, x_0)Px_0 \otimes x\} + \{x_0 \otimes [(x_0 \otimes x)y]\} \\ &= (x, x_0)(y, x_0)P + (x, x_0)x_0 \otimes (Py) \\ &\quad + x_0 \otimes [(x_0 \otimes x)y] \\ &= (x, x_0)(y, x_0)P + x_0 \otimes [(x, x_0)Py + (x_0 \otimes x)y] . \end{aligned}$$

Let $z = (x, x_0)Py + (x_0 \otimes x)y$. Then $A_x y = z$ and so

$$\begin{aligned} (z, x_0) &= ((x, x_0)Py, x_0) + ((x_0 \otimes x)y, x_0) \\ &= ((x_0 \otimes x)y, x_0) = (y, x_0)(x, x_0) . \end{aligned}$$

Thus

$$\begin{aligned} A_x A_y &= (z, x_0)P + (x_0 \otimes z) \\ &= A_z , \end{aligned}$$

showing that $\mathscr{A}$ is an algebra.

To show that $\mathscr{A}$ is abelian it suffices to show that $A_x y = A_y x$ for every pair x, y of vectors in $\mathscr{H}$. We have

$$\begin{aligned} A_x y &= (x, x_0)Py + (y, x_0)x \\ &= (x, x_0)[y - (y, x_0)x_0] + (y, x_0)x \\ &= (x, x_0)y - (x, x_0)(y, x_0)x_0 + (y, x_0)x \\ &= (y, x_0)[x - (x, x_0)x_0] + (x, x_0)y \\ &= A_y x . \end{aligned}$$

Assume now that the dimension of $\mathscr{H}$ is at least 3. We show that $\mathscr{A}$ is not the commutant of any operator. This will show that $\mathscr{A}$ is not singly generated. For if $\mathscr{A}$ is generated by an operator A, then since $\mathscr{A}$ is maximal abelian $\mathscr{A}$ is the algebra of all operators commuting with A, i.e., $\mathscr{A}$ is the commutant of A.

To show $\mathscr{A}$ is not the commutant of an operator it suffices to show that for every A in $\mathscr{A}$ there is an operator T such that $AT = TA$ but T is not in $\mathscr{A}$. Let A_x be in $\mathscr{A}$. We may assume that $(x, x_0) = 0$ since an operator commutes with A_x if and only if it commutes with $A_x - (x, x_0)I = A_{x-(x,x_0)x_0}$. Choose y in $\mathscr{H}$, $y \neq 0$, such that y is orthogonal to both x_0 and x. Finally, let $T = y \otimes x$. Then

$$Tx_0 = (x_0, y)x = 0 .$$

Since x_0 is separating for $\mathscr{A}$ and $T \neq 0$, T is not in $\mathscr{A}$. However,

$$\begin{aligned} TA_x &= (y \otimes x)(x_0 \otimes x) \\ &= x_0 \otimes [(y \otimes x)x] \\ &= x_0 \otimes [(x, y)x] \\ &= x_0 \otimes 0 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} A_xT &= (x_0 \otimes x)(y \otimes x) \\ &= y \otimes [(x_0 \otimes x)x] \\ &= y \otimes [(x, x_0)x] \\ &= y \otimes 0 \\ &= 0 . \end{aligned}$$

In particular T commutes with A_x.

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