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AMITSUR COHOMOLOGY OF ALGEBRAIC NUMBER RINGS

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A bound is given for the order of the Amitsur cohomology group $H^i(S/R, U)$ corresponding to an extension $R \subset S$ of rings of algebraic integers. The effect of inflation on the Chase-Rosenberg exact sequence involving Amitsur cohomology and split Brauer groups is also studied.

It is well known [9] that global class field theory, together with some results of Auslander-Goldman [2], leads to the determination of the split Brauer group $B(S/R)$ corresponding to an extension $R \subset S$ of rings of algebraic integers. Although the Amitsur cohomology group $H^2(S/R, U)$ is related to $B(S/R)$ by the Chase-Rosenberg exact sequence [5], $H^2(S/R, U)$ has only been computed in case $R = \boldsymbol{Z}$ and S is quadratic (see [10] and [8]). In this note we prove $H^i(S/R, U)$ is finite for all i (Corollary 2.2) and, as in [8], derive further information in case $i = 2$ by applying inflation to the Chase-Rosenberg sequence.

Throughout the paper, rings and algebras are commutative with unit elements and algebra homomorphisms are unitary. We assume familiarity with the Amitsur cohomology, Brauer group and Pic functors (see [4], [2] and [3] respectively) and with spectral sequences.

2. Finiteness of chomology. The aim of this section is to establish a bound for the order of Amitsur cohomology groups in the unit functor U for extensions of rings of algebraic integers.

PROPOSITION 2.1. *Let R be a Dedekind domain with quotient field K, S the integral closure of R in an n-dimensional separable field extension L of K, and T the integral closure of R in a normal closure F of L/K. Assume $U(T)$ can be generated by m elements. Then, for all $i \geqq 0$, the Amitsur cohomology group $H^i(S/R, U)$ is finite, with order at most $n^{(m(n!)^i)}$.*

Proof. We first observe that the cochain group $C^{i-1}(S/R, U) = U(S \otimes_R \cdots \otimes_R S) = U(S^i)$ is finitely generated for all $i \geqq 1$. By a standard argument [12, Chapter V, Theorem 7], S and T are module-finite faithful R-projectives such that $S \otimes_R K \cong L$; hence the R-rank of S is n. Let $G = \mathrm{gal}(F/K)$. Flatness provides injective R-algebra homomorphisms $S^i \to T^i \to F^i$; composition with the canonical isomorphism $F^i \to \prod_{G^{i-1}} F$(defined in [4, p. 18]) yields an injection $S^i \to \Pi T$ and, hence, a monomorphism of abelian groups $U(S^i) \to \Pi U(T)$. Since $\Pi U(T)$ is finitely generated, so is $U(S^i)$.

$H^i(S/R, U)$, being a quotient of a submodule of $U(S^{i+1})$, is therefore finitely generated. However, $H^i(S/R, U)$ is annihilated by n, the R-rank of S [1, Theorem 6], and is therefore finite.

As $|G| = [F: K] \leqq n!$, it follows that $\prod_{G^i} U(T)$ can be generated by $m(n!)^i$ elements. By the elementary theory of abelian groups, the same conclusion holds for $C^i(S/R, U)$ and, hence, for its quotient $H^i(S/R, U)$. Since $H^i(S/R, U)$ is n-torsion, the result is immediate.

COROLLARY 2.2. *If $R \subset S$ is an extension of rings of algebraic integers then, for all i, $H^i(S/R, U)$ is finite.*

Proof. If T is as above, a weak form of Dirichlet's unit theorem implies $U(T)$ is finitely generated, and the proposition applies.

3. Kernel of inflation. Let $f: S \to T$ be an R-algebra homomorphism and J an abelian group valued functor defined on a full subcategory of R-algebras containing all the tensor products S^n and T^n. The homomorphisms $J(f \otimes \cdots \otimes f): J(S^n) \to J(T^n)$ induce a map of Amitsur complexes $C(S/R, J) \to C(T/R, J)$ which yields *inflation* homomorphisms inf: $H^n(S/R, J) \to H^n(T/R, J)$.

In this section, we study the kernel of inf for the case $n = 2$ and $J = U$, the unit functor. The principal result, Remark 3.3, complements the torsion result in [8, Theorem 2.7], and may be used with Corollary 4.4 below to yield information about cohomology of rings of algebraic integers.

We begin by recalling the following result.

PROPOSITION 3.1. (Chase-Rosenberg [5, Theorem 7.2]). *Let $R \to S$ and $S \to T$ be R-algebra homomorphisms, and let J be a functor from R-algebras to abelian groups. For each $q \geqq 0$, let J_T^q be the functor given by $J_T^q(A) = H^q(A \otimes_R T/A, J)$. Then there exists a first quadrant spectral sequence $H^p(S/R, J_T^q) \Rightarrow H^{p+q}(T/R, J)$.*

The most important application of the proposition is to the case of an R-based topology (Definition [7, p. 86]) for which J is a sheaf and $\{R \to T\}$ a cover. Then the natural transformation $J \to J_T^0$ is an equivalence. In particular, the $E_2^{p,0}$ term of the above spectral sequence is just $H^p(S/R, J)$.

COROLLARY 3.2. *Let $R \to S$ and $S \to T$ be R-algebra homomorphisms such that T is faithfully flat over R. Then there is an exact sequence* $H^0(S/R, U_T^1) \to H^2(S/R, U) \xrightarrow{\inf} H^2(T/R, U)$.

Proof. A standard spectral sequence argument, applied to the proposition with $J = U$, provides an exact sequence

$$H^0(S/R, U_T^1) \to H^2(S/R, U_T^0) \xrightarrow{\text{edge}} H^2(T/R, U) .$$

However, U is a sheaf in the faithfully flat R-based topology [5, Proposition 3.9(a)], and so the preceding remark identifies U_T^0 with U. Then [11, Lemma 1.1 and Proposition 1.6] identifies the edge homomorphism with inf, to complete the proof.

REMARK 3.3. Let $R \subset S \subset T$ be a tower of rings such that T is faithfully flat over R and T is a rank n S-projective (rank defined in [3, p. 141]). Then ker [inf: $H^2(S/R, U) \to H^2(T/R, U)$] is n-torsion.

For the proof, it suffices to show $U_T^1(S)$ is n-torsion. However, [5, Corollary 4.6] yields natural isomorphisms

$$\ker [\operatorname{Pic}(S) \longrightarrow \operatorname{Pic}(S \otimes_R T] \cong H^1(S \otimes_R T/S, U)$$

and

$$\ker [\operatorname{Pic}(S) \longrightarrow \operatorname{Pic}(T)] \cong H^1(T/S, U) .$$

Hence, $U_T^1(S)$ embeds in $H^1(T/S, U)$ which is n-torsion [1, Theorem 6], to complete the proof.

4. Direct limit arguments.

We begin by recalling the Chase-Rosenberg exact sequence of low degree obtained from the direct limit of spectral sequences given by Proposition 3.1.

PROPOSITION 4.1. (Chase-Rosenberg [5, Theorem 7.6]). *Let S be a module-finite faithful and projective R-algebra. Then there exists an exact sequence natural in S:*

$$\begin{gathered} 0 \longrightarrow H^1(S/R, U) \longrightarrow \operatorname{Pic}(R) \longrightarrow H^0(S/R, \operatorname{Pic}) \longrightarrow H^2(S/R, U) \longrightarrow \\ B(S/R) \longrightarrow H^1(S/R, \operatorname{Pic}) \longrightarrow H^3(S/R, U) . \end{gathered}$$

The isomorphism of Amitsur cohomology with split Brauer group in the case of fields is contained in the next result, a slight improvement of [5, Corollary 7.7].

COROLLARY 4.2. *If S is a module-finite faithful and projective*

R-algebra with $\mathrm{Pic}(S^2) = 0$, *then the natural map* $H^2(S/R, U) \to B(S/R)$ *is an isomorphism.*

Proof. It suffices to prove $\mathrm{Pic}(S) = 0$. Since composition of a face map $S \to S^2$ with the contraction map $S^2 \to S$ is the identity on S, applying the functor Pic shows that the identity map on $\mathrm{Pic}(S)$ factors through $\mathrm{Pic}(S^2) = 0$.

REMARK 4.3. Let $R \subset S$ be rings of algebraic integers whose quotient fields form an n-dimensional extension, and let h be the class number of S. If $(h, n) = 1$, then the canonical map $H^2(S/R, U) \to B(S/R)$ is a monomorphism.

Proof. Since S is Dedekind, $\mathrm{Pic}(S)$ is isomorphic to the ideal class group of S[3, Exer. 21, p. 181], and so $H^0(S/R, \mathrm{Pic})$ is h-torsion. As noted in the proof of Proposition 2.1, S is a rank n R-projective and $H^2(S/R, U)$ is n-torsion. An application of Proposition 4.1 completes the proof.

For the remainder of this section, let R be a ring of algebraic integers with quotient field K, F an algebraic closure of K, and A the ring of all algebraic integers inside F. We proceed to study direct limits of groups of the form $H^n(N/R, U)$, where N ranges over the inclusion-directed collection of algebraic number overrings of R contained in A. By cofinality, this may be viewed as taking direct limits over the map-directed collection of module-finite R-faithfully flat domains [7, Ch. III, Remark 3.1(b)]. Interest in such direct limits is partially due to the Cech description of cohomological field dimensions [7, Ch. I, Theorem 3.13].

In the finite R-based topology [7, p. 105], the functor U is represented by $R[X, X^{-1}]$ which is an algebra-finite commutative cocommutative Hopf algebra over R. Consequently, [7, Ch. II, Remark 2.3(b)] shows that the canonical map $\varinjlim H^n(N/R, U) \to H^n(A/R, U)$ is an isomorphism for all $n \geqq 0$.

COROLLARY 4.4. *Applying* $\varinjlim_N$ *to the sequence of Proposition* 4.1 *yields an isomorphism* $H^1(A/R, U) \xrightarrow{\cong} \mathrm{Pic}(R)$ *and an exact sequence* $0 \to H^2(A/R, U) \to B(R) \to \varinjlim H^1(N/R, \mathrm{Pic}) \to H^3(A/R, U)$.

Proof. If P is a (not necessarily commutative) Azumaya R-algebra, let L be a finite dimensional subextension of F/K such that the class of $P \otimes_R K \otimes_K L \cong P \otimes_R L$ is trivial in $B(L)$. Let M be the integral closure of R in L. Since $B(M) \to B(L)$ is a monomorphism [2, Theorem 7.2], it follows that the class of $P \otimes_R M$ is trivial in $B(M)$. Thus $B(R) = \varinjlim B(N/R)$, and the above discussion shows that we need only prove $\varinjlim H^0(N/R, \mathrm{Pic}) = 0$. This, in turn, follows from the fact that $\varinjlim \overrightarrow{\mathrm{Pic}}(N) = 0$, which is an easy consequence of finiteness of class number (*cf.* [6, Theorem 20.14]).

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