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REAL-VALUED CHARACTERS OF METACYCLIC GROUPS

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The nonlinear real-valued irreducible characters of metacyclic groups are determined, and the defining relations are given for the metacyclic groups with every nonlinear irreducible character real-valued.

Consider the metacyclic group

$$G = \langle a, b \mid a^n = b^m = 1, a^k = b^t, b^{-1}ab = a^r \rangle$$

where $r^t - 1 \equiv kr - k \equiv 0 \pmod n$ and $t \mid m$. Let s be a positive divisor of n and let t_s be the smallest positive integer such that $r^{t_s} \equiv 1 \pmod s$. Let χ_s be a linear character of $\langle a \rangle$ with kernel $\langle a^s \rangle$ and $\bar{\chi}_s$ be an extension of χ_s to $K_s = \langle a, b^{t_s} \rangle$, see [1]. From [1] the induced character $\bar{\chi}_s^G$ is irreducible of degree t_s and every irreducible character of G is some $\bar{\chi}_s^G$.

Assume $\bar{\chi}_s^G$ is nonlinear. Then $K_s \subset G \neq K_s$. From Lemma 1 of [2], $\bar{\chi}_s^G$ is real-valued if and only if there is $y \in G$ such that $\langle K_s, y \rangle / D_s$ is dihedral or quaternion, where D_s is the kernel of $\bar{\chi}_s$. Assume such a y exists. Since G/K_s is cyclic, t_s is even and we may let $y = b^{t_s/2}$. Hence $r^{t_s/2} \equiv -1 \pmod s$ and $\bar{\chi}_s(b^{t_s}) = \pm 1$. Since $\bar{\chi}_s(b^{t_s})^{t/t_s} = \chi_s(a^k)$, $\bar{\chi}_s(b^{t_s}) = \pm 1$ implies either (i) $s \mid (k, n)$, or (ii) $s \mid 2(k, n)$, $s \nmid (k, n)$, and $t = t_s$. When (i) occurs, $\bar{\chi}_s(b^{t_s}) = 1$ if t/t_s is odd and $\bar{\chi}_s(b^{t_s}) = \pm 1$ if t/t_s is even. When (ii) occurs $\bar{\chi}_s(b^{t_s}) = -1$. Note that if $\bar{\chi}_s(b^{t_s}) = -1$ then $\bar{\chi}_s^G$ is not realizable in the real field. Using [1] the number of the nonlinear irreducible real-valued characters not realizable in the real field is $\Sigma'' \phi(s)/t_s$ where Σ'' is over all positive divisors s of n such that t_s is even, $r^{t_s/2} \equiv -1 \pmod s$ and either (i) $s \mid (k, n)$ and t/t_s even or (ii) $s \mid 2(k, n)$, $s \nmid (k, n)$, and $t = t_s$. The number of the nonlinear irreducible real-valued characters is $\Sigma' \phi(s) | t_s + \Sigma'' \phi(s)/t_s$ where Σ' is defined in [2, § 2].

Let $\pi = \{p \mid p \text{ an odd prime dividing } n\}$.

THEOREM. *Assume G, given as above, is non-abelian. Then every nonlinear irreducible character of G is real-valued if and only if n, m, k, t, t_p and r satisfy one of the conditions below.*

(a) *$m = t$ with either* (i) *$4 \nmid n$, $2 \mid t$, and $t_p = t$ for all $p \in \pi$,* (ii) *$4 \nmid n$, $4 \mid t$, and $t_p = t/2$ for all $p \in \pi$, or* (iii) *$4 \mid n$, $r = -1$, $t = 2$ or $t = 4$.*

(b) *$m = 2t$ with either* (i) *$2 \| n$, $2 \mid t$, and $t_p = t$ for all $p \in \pi$, or*

(ii) $4 \mid n$, $r = -1$, *and* $t = 2$.

REMARK. Since (b)—(i) is the semi-direct product $\langle a^2\rangle \circ \langle b\rangle$, and thus a special case of (a)—(ii), we have (b)—(ii) (where G is quaternion) the only nonsplitting case.

Proof. Consider χ_n with kernel $\langle 1\rangle$. Assume $m/t > 2$. Since $b^t \in \langle a\rangle \subseteqq K_n = \langle a, b^{t_n}\rangle$ we have $\chi_n(b^t)$, and hence $\bar{\chi}_n^G(b^t) = t_n\chi_n(b^t)$, complex. Thus $m/t \leqq 2$, i.e., $m = t$ or $m = 2t$. Assume $t_s = 1$ for some $s \mid n$, $s > 2$. Since $b^{-1}a^{n/s}b = a^{n/s}$ we have $\bar{\chi}_n^G(a^{n/s}) = t_n\chi_n(a^{n/s})$ complex. Thus $t_s > 1$ for all $s \mid n$, $s > 2$.

Consider χ_s with kernel $\langle a^s\rangle$, $s > 2$. Assume $t/t_s > 2$. Then the extension $\bar{\chi}_s$ to $K_s = \langle a, b^{t_s}\rangle$ can be chosen such that $\bar{\chi}_s(b^{t_s})$ is complex. Thus $\bar{\chi}_s^G(b^{t_s}) = t_s\bar{\chi}_s(b^{t_s})$ is complex and hence $t = t_s$ or $t = 2t_s$. Also, since G/K_s is cyclic of even order, $2 \mid t_s$.

Let p and q be in π, $p \neq q$ and assume $t_p = t$ and $t_q = t/2$. Then $t_{pq} = t$. Thus $K_{pq} = \langle a\rangle$, and since $b^{-t/2}ab^{t/2} \notin a^{-1}\langle a^{pq}\rangle$, it follows that $\langle a, b^{t/2}\rangle/\langle a^{pq}\rangle$ is neither dihedral nor quaternion, a contradiction. Thus either $t_p = t$ for all $p \in \pi$ or $t_p = t/2$ for all $p \in \pi$.

Now assume $\lambda \mid n$, $\lambda = 2^e$, $e > 1$. Then t_λ is a power of 2. Consider χ_λ. Then since $\langle a, b^{t_\lambda/2}\rangle/\langle a^\lambda\rangle$ is dihedral or quaternion, it follows that $r^{t_\lambda/2} \equiv -1 \pmod{\lambda}$. The only solution is $r \equiv -1 \pmod{\lambda}$. Thus $t_\lambda = 2$. As above, if $e > 1$ then $2 = t_\lambda = t_p$ for all $p \in \pi$, so $r \equiv -1 \pmod{n}$.

Assume $t_p = t$, $2 \mid t$, for all $p \in \pi$. If n is odd then $t = m$, and if $n = 2v$, v odd, then $t = m$ or $2t = m$. If $4 \mid n$ then $r = -1$ and $2 = t_4 = t_p = t = m$ or $2t = m = 4$.

Assume $t_p = t/2$, $4 \mid t$, for all $p \in \pi$. If $4 \nmid n$ then $t = m$. [The case $m = 2t$, $n = 2v$, v odd can not occur, since then we have $b^t = a^v \neq 1$, $K_n = \langle a, b^{t/2}\rangle$, which implies $\bar{\chi}_n(b^{t/2}) = \pm\sqrt{-1}$ and thus $\bar{\chi}_n^G$ is complex.] If $4 \mid n$ then $r = -1$ and $2 = t_4 = t_p = t/2$. Thus $4 = t = m$. [The case $m = 2t = 8$ cannot occur for a similar reason as above.]

The above give all the cases in the Theorem. Conversely if G is as in the Theorem and $s \mid n$, $s > 2$, it is easy to show that $\langle a, b^{t_s/2}\rangle/D_s$ is dihedral or quaternion, where D_s is the kernel of $\bar{\chi}_s$, and thus $\bar{\chi}_s^G$ is real-valued. This completes the proof.

We remark that a similar result to Theorem 1 of [2], with a parallel proof, could be given for the real-valued characters of metabelian groups.

REFERENCES

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