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A GENERAL PHILLIPS THEOREM FOR C^*-ALGEBRAS AND SOME APPLICATIONS

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In this paper Phillips's theorem is extended to a C^*-algebra setting and, by virtue of this extension, several results on interpolation are generalized and improved.

1. Introduction. Let N be the set of positive integers with the discrete topology and let $m(N)$ denote the bounded complex functions on N. We may identify $m(N)$ with $C(\beta N)$, where βN denotes the Stone-Cech compactification of N. A well known and useful result due to Phillips is the following.

THEOREM. *Let $\{f_n\}$ be a sequence in the dual of $C(\beta N)$ that converges weak* to zero. Then*

$$\lim_{m\to\infty} \sum_{p=m}^{\infty} |f_n(\delta_p)| = 0$$

uniformly in n, where δ_p is the characteristic function of the set $\{p\}$.

In §3 we extend this result to a C^*-algebra setting and we give several applications of this result. For example, we extend and improve several results on interpolation due to Bade [3] and Akemann [2]. A commutative version of our result was proved by Conway [7].

2. Preliminaries. Let A be a C^*-algebra. By a double centralizer on A, we mean a pair (R, S) of functions from A to A such that $aR(b) = S(a)b$ for a, b in A, and we denote the set of all double centralizers on A by $M(A)$. If $(R, S) \in M(A)$, then R and S are continuous linear operators on A and $||R|| = ||S||$. So $M(A)$ under the usual operations of addition, multiplication, and involution is a C^*-algebra, where $||(R, S)|| = ||R||$. If we define the map $\mu_0: A \to M(A)$ by the formula $\mu_0(a) = (L_a, R_a)$, where $L_a(b) = ab$ and $R_a(b) = ba$ for all $b \in A$, then μ_0 is an isometric $*$-isomorphism from A into $M(A)$ and $\mu_0(A)$ is a closed two-sided ideal of $M(A)$. Hence throughout this paper we will view A as a closed two-sided ideal of $M(A)$. For a more detailed account of the theory of double centralizers on a C^*-algebra, we refer the reader to [4] and [13].

Let B be a C^*-algebra and let A be a closed two-sided ideal of B. We define the strict topology β_A for B to be that locally convex topology generated by the seminorms $(\lambda_a)_{a\in A}$ and $(\rho_a)_{a\in A}$, where $\lambda_a(x) = ||ax||$

and $\rho_a(x) = ||xa||$, and we let B_{β_A} denote B under the strict topology generated by A. When A and B are understood (specifically, when $B = M(A)$) we let β denote the strict topology for B generated by A. The topological algebra $M(A)_\beta$ is complete and the unit ball of A is β dense in the unit ball of $M(A)$.

We will now state a result due to Busby that is very useful in computing the double centralizer algebra of a C^*-algebra.

THEOREM 2.1. *Let B be a C^*-algebra, let A be a closed two-sided ideal of B, and let $A^0 = \{x \in B \mid xA = 0\}$. Let the map $\mu: B \rightarrow M(A)$ be defined by $\mu(x) = (L_x, R_x)$, where $L_x(a) = xa$ and $R_x(a) = ax$ for each a in A. Then the following statements are true:*

(1) *The map μ is a $*$-homomorphism of B into $M(A)$; consequently, μ is an isometry if and only if $A^0 = 0$.*

(2) *If $A^0 = 0$ and every β_A-Cauchy net in the unit ball of A converges in the β_A topology to some element of the unit ball of B, then μ is an isometric $*$-isomorphism of B onto $M(A)$.*

Proof. For a proof, see [4, Proposition 3.7, p. 83].

COROLLARY 2.2. *If B is a W^*-algebra and $A^0 = 0$, then μ is an isometric $*$-isomorphism of B onto $M(A)$.*

Proof. Let $\{a_\alpha\}$ be a β_A-Cauchy net in the unit ball of A. Since the unit ball of B is compact in the weak operator topology, we can assume that $\{a_\alpha\}$ converges in the weak operator topology to some element x in the unit ball of B. Since $\{a_\alpha\}$ is β_A-Cauchy, it is straightforward by [4, Th. 3.9(i), p. 84] to show that $\{a_\alpha\}$ converges to x in the β_A-topology. The conclusion now follows from Theorem 2.1.

If B is a W^*-algebra, then it is straightforward to show that A^0 is a two-sided ideal of B that is closed in the weak operator topology. Hence A^0 has an identity q that commutes with each element of B. If follows that the quotient algebra B/A^0 is isometrically $*$-isomorphic to the W^*-algebra $(1-q)B(1-q)$. Now define the map $\mu': B/A^0 \rightarrow M(A)$ by the formula $\mu'(x + A^0) = \mu(x)$ for each x in B. Since $\ker \mu = A^0$, we see that μ' is well defined. Due to the fact that $\{x \in B/A^0 \mid x(A/A^0) = 0\} = \{0\}$, we get

COROLLARY 2.3. *If B is a W^*-algebra, then $M(A)$ is a W^*-algebra and the map μ' is an isometric $*$-isomorphism of B/A^0 onto $M(A)$; that is, $M(A) \cong M(A/A^0)$.*

EXAMPLE. Let H be a Hilbert space, let $B(H)$ be the bounded linear operators on H, and let $B_0(H)$ be the compact linear operators on H. It is well known that $B_0(H)$ is a closed two-sided ideal of $B(H)$. Since $B(H)$ is a W^*-algebra and $\{x \in B(H) \mid xB_0(H) = 0\} = \{0\}$, we have that $B(H)$ is the double centralizer algebra of $B_0(H)$.

EXAMPLE. Let B be a finite dimensional C^*-algebra, let S be a locally compact paracompact Hausdorff space, and let $\beta(S)$ denote the Stone-Cech compactification of S. Let $C(\beta(S), B)$ denote the space of all B-valued continuous functions on $\beta(S)$ and let $C_0(S, B) = \{x \in C(\beta(S), B) \mid x(t) = 0, t \in \beta(S) - S\}$. It is clear that under the usual pointwise operations and sup-norm that $C(\beta(S), B)$ is a C^*-algebra and $C_0(S, B)$ is a closed two-sided ideal of $C(\beta(S), B)$. Now it is straightforward to show that a β-Cauchy net in the unit ball of $C_0(S, B)$ converges to a B-valued continuous function on S that is uniformly bounded. Since a bounded B-valued continuous function on S can be uniquely extended to B-valued continuous functions on $\beta(S)$, Theorem 2.1 gives us that $C(\beta(S), B)$ is the double centralizer algebra of $C_0(S, B)$.

PROPOSITION 2.4. *Let B be a C^*-algebra and A a closed two-sided ideal of B. Then $B^*_{\beta_A}$, the dual of B_{β_A}, can be identified under the natural mapping as a closed subspace of B^*.*

Proof. The proof will follow from a variation of the argument given for [13, Corollary 2.3, p. 635].

PROPOSITION 2.5. *Let B be a C^*-algebra and let A be a closed two-sided ideal of B. If f is a bounded linear functional on B, then there exists a unique decomposition $f = f^0 + f^1$ such that $f^0 \in B^*_{\beta_A}$ and $f^1 \in A^\perp$. Consequently, $B^* = B^*_{\beta_A} \oplus A^\perp$.*

Proof. For a proof, see [14, Corollary 2.7].

REMARK. For each $f \in B^*$ we will always let f^0 and f^1 denote those unique linear functionals in $B^*_{\beta_A}$ and $A^\perp$ respectively that satisfy $f = f^0 + f^1$.

DEFINITION. Let A be a C^*-algebra. A subset K of $M(A)^*_\beta$ is said to be *tight* if K is uniformly bounded and if for some, or for each, approximate identity $\{e_\lambda\}$ for A we have

$$\|(1 - e_\lambda)f(1 - e_\lambda)\| \to 0$$

uniformly on K. Here $(1 - e_\lambda)f(1 - e_\lambda)(x) = f((1 - e_\lambda)x(1 - e_\lambda))$ for each $x \in M(A)$.

THEOREM 2.6. *Let A be a C^*-algebra. Then a subset K of $M(A)^*_\beta$ is β-equicontinuous if and only if K is tight.*

Proof. For a proof, see [13, Theorem 2.6, p. 636].

3. A general Phillips theorem for C^*-algebras.

In this section we will study sequential convergence in the dual of a double centralizer algebra. In particular, we prove a general Phillips theorem for C^*-algebras and we give some applications of it.

DEFINITION. An approximate identity $\{e_\lambda \mid \lambda \in \Lambda\}$ for the C^*-algebra A is said to be *well behaved* if and only if the following properties are satisfied.

(1) $e_\lambda \geqq 0$ for each $\lambda \in \Lambda$.

(2) If $\lambda_2 > \lambda_1$, then $e_{\lambda_2} e_{\lambda_1} = e_{\lambda_1}$.

(3) If $\lambda_1, \lambda_2, \cdots$ is a strictly increasing sequence in Λ and $\lambda \in \Lambda$, then there exists a positive integer N such that for all $n, m > N$ we have $e_\lambda(e_{\lambda_n} - e_{\lambda_m}) = 0$.

REMARK. If S is a locally compact paracompact Hausdorff space, then S can be expressed as the union of a collection $\{S_\alpha \mid \alpha \in I\}$ of pairwise disjoint open and closed σ-compact subsets of S. Since each C^*-algebra $C_0(S_\alpha)$ has a countable approximate identity and $C_0(S) \cong (\sum C_0(S_\alpha))_0$, it follows by Proposition 3.1 and Proposition 3.2 that $C_0(S)$ has a well behaved approximate identity. Now let H be a Hilbert space and $\{p_\alpha\}_{\alpha \in I}$ be a maximal family of orthogonal projections on H. It is straightforward to show that $\{p_\alpha\}_{\in I\alpha}$ is a series approximate identity for $B_0(H)$, the space of all compact operators on H, consequently, by Proposition 3.1, $B_0(H)$ has a well behaved approximate identity. Finally, suppose A is a C^*-algebra such that $M(A)$ is isometrically isomorphic to A^{**}, the bidual of A. By some recent results of E. McCharen or by [15, Theorem 5.1, p. 533] A is dual, consequently, $A \cong (\sum B_0(H_\alpha))_0$, where $\{H_\alpha\}$ is a family of Hilbert spaces (see [11]). Hence by Proposition 3.2 A has a well behaved approximate identity.

PROPOSITION 3.1. *Let A be a C^*-algebra and suppose one of the following conditions holds:*

(1) *A has a countable approximate identity;*

(2) *A has a series approximate identity* (see [2, p. 527]).

Then A has a well behaved approximate identity.

Proof. It is straightforward to verify that A has a well behaved approximate identity when (2) holds. Therefore assume A has a countable approximate identity $\{c_n\}$. We can also assume $c_n \geqq 0$, since $c_n^* c_n$ is an approximate identity for A. Let $b = \sum_{n=1}^{\infty} c_n/2^n$. Then b is a strictly positive element of A in the sense of [1, p. 749]. Hence A contains a countable increasing abelian approximate identity $\{d_n\}$ [1, Theorem 1, p. 749]. Let A_0 denote the maximal commutative subalgebra of A that contains $\{d_n\}$. Then we can view A_0 as $C_0(\mathscr{M})$, the complex-valued continuous functions that vanish at ∞ on the maximal ideal space $\mathscr{M}$ of A_0. Since A_0 has a countable approximate identity $\{d_n\}$, it follows by [5, Theorem 4.1, p. 160] that $\mathscr{M}$ is σ-compact. It is straightforward to show that A_0 has a well behaved countable approximate identity $\{e_n\}$. We now wish to show that $\{e_n\}$ is an approximate identity for A. Let $a \in A$ and $\varepsilon > 0$. Choose a positive integer m so that $||a - d_m a|| < \varepsilon|2$ and then choose a positive integer N so that $||(d_m - e_n d_m)|| < \varepsilon/2||a||$ for integers $n \geqq N$. It follows that $||a - e_n a|| \leqq ||(1 - e_n)(a - d_m a)|| + ||(d_m - e_n d_m)a|| < \varepsilon$ for $n \geqq N$. Hence $\{e_n\}$ is a well behaved approximate identity for A and the proof is complete.

PROPOSITION 3.2. *Let $\{A_\delta | \delta \in \varDelta\}$ be a family of C^*-algebras. If each A_δ has a well behaved approximate identity, then the sub-direct sum $(\sum_{\delta \in \varDelta} A_\delta)_0$ has a well behaved approximate identity* (see [12, p. 106] for definition of $(\sum_{\delta \in \varDelta} A_\delta)_0$).

Proof. For each $\delta \in \varDelta$ let $\{e_{\delta\,\lambda} | \lambda \in \Lambda_\delta\}$ be a well behaved approximate identity for A_δ, and let $\mathscr{F}$ denote the family of all finite subsets of $\varDelta$. Let Σ denote the set of all functions σ whose domain $D_\sigma \in \mathscr{F}$ and has the property that $\sigma(\delta) \in \Lambda_\delta$ for each $\delta \in D_\sigma$. We define the binary relation $\geqq$ in Σ by the following formula: $\sigma_2 \geqq \sigma_1$ if and only if $D_{\sigma_2} \geqq D_{\sigma_1}$ and $\sigma_2(\delta) \geqq \delta_1(\delta)$ for each $\delta \in D_{\sigma_1}$. It is straightforward to verify that Σ under $\geqq$ is a directed set. Now for each $\sigma \in \Sigma$ define d_σ in $(\sum_{\delta \in \varDelta} A_\delta)_0$ by the following formula $d_\sigma(\delta) = e_{\delta,\sigma(\delta)}$ for each $\delta \in D_\sigma$ and $d_\sigma(\delta) = 0$ otherwise. It is straightward to verify that $\{d_\sigma | \sigma \in \Sigma\}$ is a well behaved approximate identity for $(\sum_{\delta \in \varDelta} A_\delta)_0$.

The next result extends Phillips' theorem to a C^*-algebra setting. A commutative version of this result was proved by Conway [7, Theorem 2.2, p. 55].

THEOREM 3.3. *Suppose A is a C^*-algebra with a well behaved approximate identity. If $\{f_n\}$ is a sequence in $M(A)^*$ that converges weak* to zero, then $\{f_n^0\}$ is tight and converges weak* to zero.*

Proof. It is clear that $\{f_n\}$ is uniformly bounded, so without loss of generality we can assume $\{f_n\}$ is uniformly bounded by 1. Since $\|f_n\| \geqq \|f_n|A\| = \|f_n^0|A\| = \|f_n^0\|$, we have that $\{f_n^0\}$ is also uniformly bounded by 1. Let $\{e_\lambda|\lambda \in \Lambda\}$ be a well behaved approximate identity for A and suppose $\{f_n^0\}$ is not tight. Then there exists an $\varepsilon > 0$ such that $\{\lambda \in \Lambda: \sup_n \|(1 - e_\lambda)f_n^0(1 - e_\lambda)\| \geqq 4\varepsilon\}$ is cofinal in Λ and since a cofinal subnet of a well behaved approximate identity is also one, we may assume

$$\sup_n \|(1 - e_\lambda)f_n^0(1 - e_\lambda)\| \geqq 4\varepsilon \tag{3.1}$$

for all $\lambda \in \Lambda$. We may then define inductively sequences $n_1 < n_2 < \cdots$ and $\lambda_1 < \lambda_2 < \cdots$ such that $\|(1-e_{\lambda_k})f_{n_k}^0(1-e_{\lambda_k})\| \geqq 4\varepsilon$ and $\|e_{\lambda_{k+1}}f_{n_k}^0 e_{\lambda_{k+1}} - f_{n_k}^0\| < \varepsilon$ by using the following: (3.1); $\lim_\lambda \|(1 - e_\lambda)g(1 - e_\lambda)\| = 0$, $g \in M(A)_\beta^*$; $\lim_\lambda \|e_\lambda g e_\lambda - g\| = 0$, $g \in M(A)_\beta^*$. It then follows that

$$\begin{aligned}\|(1 - e_{\lambda_k})e_{\lambda_{k+1}}f_{n_k}^0 e_{\lambda_{k+1}}(1 - e_{\lambda_k})\| &= \|(e_{\lambda_{k+1}} - e_{\lambda_k})f_{n_k}(e_{\lambda_{k1+1}} - e_{\lambda_k})\| \\ &\geqq 3\varepsilon\ .\end{aligned}$$

We then, for each k, choose $b_k = b_k^*$ in ball A such that $|f_{n_k}((e_{\lambda_{k+1}} - e_{\lambda_k})b_k(e_{\lambda_{k+1}} - e_{\lambda_k}))| \geqq \varepsilon$. Define $a_k = (e_{\lambda_{2k+1}} - e_{\lambda_{2k}})b_{2k}(e_{\lambda_{2k+1}} - e_{\lambda_{2k}})$ and let $g_k = f_{n_{2k}}$. Then we have:
(i) $|g_k(a_k)| \geqq \varepsilon$; (ii) $a_j a_k = 0$ for $j \neq k$; (iii) for each $\lambda \in \Lambda$, there exists a positive integer N such that $a_k e_\lambda = 0$ for $k \geqq N$.

Now let $\alpha = \{\alpha_a\}_{k=1}^\infty$ be an element of l^∞. By virtue of (ii) and (iii) the sequence of partial sums $\{\sum_{k=1}^n \alpha_k a_k\}$ is uniformly bounded by $\|\alpha\|_\infty$ and is β-Cauchy. Since $M(A)_\beta$ is complete [4, Proposition 3.6, p. 83], $\{\sum_{k=1}^n \alpha_k a_k\}$ has a β-limit $\sum_{k=1}^\infty \alpha_k a_k$ that is also bounded by $\|\alpha\|_\infty$. Next, define the bounded linear map $T: l^\infty \to M(A)$ by the formula

$$T(\alpha) = \sum_{k=1}^\infty \alpha_k a_k$$

for each $\alpha \in l^\infty$. Let T^* denote the adjoint of T. Since T is continuous, T^* is a weak* continuous mapping of $M(A)^*$ into $(l^\infty)^*$. From our hypothesis on $\{f_n\}$ it follows that $\{T^*(g_k)\}$ converges to 0 weak*. Hence, by Phillips theorem [8, p. 32],

$$\lim_{m\to\infty} \sum_{q=m}^\infty |T^* g_k(\delta_q)| = \lim_{m\to\infty} \sum_{q=m}^\infty |g_k(a_q)| \to 0$$

uniformly in k, where δ_k is the Kronecker delta function. Therefore there exists a positive integer m such that $|g_m(a_m)| \leqq \sum_{q=m}^\infty |g_m(a_q)| < \varepsilon$. This contradicts (i), so $\{f_n^0\}$ is tight.

Note that $\{f_n^0\}$ is now equicontinuous on $M(A)_\beta$ and converges pointwise on a dense subset and hence (by a well known result) converges weak*. The proof is now complete.

By virtue of Proposition 3.1 and the previous remark, the following result is an improvement of [13, Theorem II, p. 634].

COROLLARY 3.4. *Suppose A has a well behaved approximate identity. If K is a relatively weak* countably compact subset of $M(A)_\beta^*$, then K is tight. Consequently, $M(A)_\beta$ is a strong Mackey space* (hence, in particular, is a Mackey space).

Proof. The proof that K is tight is similar to the one given for Theorem 3.3. Since $M(A)_\beta$ is a strong Mackey space if and only if each weak* compact subset of $M(A)_\beta^*$ is β-equicontinuous, it follows from Theorem 2.6 that $M(A)_\beta$ is a strong Mackey space.

REMARK. In [6, p. 481] Conway showed that if S is the ordinals less than the first uncountable ordinal and $A = C_0(S)$, then $M(A)_\beta$ is not even a Mackey space. Therefore it follows that $C_0(S)$ does not have a well behaved approximate identity.

The next result extends [5, Theorem 5.1, p. 161].

COROLLARY 3.5. *If A has a well behaved approximate identity, then $(MA)_\beta^*$ is weakly sequentially complete.*

Proof. If $\{f_n\}$ is a weak* Cauchy sequence in $M(A)_\beta^*$, then there exists a unique linear functional f in $M(A)^*$ with $f_n \to f$ weak*. It follows that $f_n - f \to 0$ weak*. Thus, by Theorem 3.3, $(f_n - f)^0 \to 0$ weak*. But by virtue of Proposition 2.5 $(f_n - f)^0 = f_n^0 - f^0 = f_n - f^0$. This implies that $f_n \to f^0$ weak*. Hence $f = f^0$ and the proof is complete.

The next result generalizes and improves results due to Bade [3, Theorem 1.1, p. 149] and Akemann [2, Theorem 2.3, p. 527] (see our Corollaries 3.9 and 3.8).

THEOREM 3.6. *Suppose A is a C^*-algebra with a well behaved approximate identity $\{e_\lambda \mid \lambda \in \Lambda\}$. If X is a Banach space and $T\colon X \to M(A)$ is a bounded linear map with $T(X) + A = M(A)$, then there exists a $\lambda \in \Lambda$ such that $(1 - e_\lambda)M(A)(1 - e_\lambda) = (1 - e_\lambda)T(X)(1 - e_\lambda)$.*

Proof. For each $\lambda \in \Lambda$ let E_λ denote the uniform closure of the linear space $\{e_\lambda a + a e_\lambda - e_\lambda a e_\lambda \mid a \in M(A)\}$ and let $T_\lambda\colon X \to M(A)/E_\lambda$ be the bounded linear map defined by $T_\lambda(x) = T(x) + E_\lambda$. We will now show that there exists a λ in Λ so that T_λ maps X onto $M(A)/E_\lambda$. Suppose no such λ exists. Let $\lambda_1 \in \Lambda$. By virtue of [10, 487–8] and the fact that $(M(A)/E_\lambda)^*$ is isometrically isomorphic to $E_\lambda^\perp$, we can choose f_1 in $E_{\lambda_1}^\perp$ so that $||f_1|| = 1$ and $||T^*(f_1)|| < 1$, where T^* denotes the adjoint of T. Having defined $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $f_1, f_2, \cdots, f_n$ we can choose, by virtue of [13, Corollary 2.2, p. 635], $\lambda_{n+1} > \lambda_n$ so that

$$||e_{\lambda_{n+1}} f_n^0 e_{\lambda_{n+1}} - f_n^0|| < \frac{1}{n} . \tag{3.2}$$

Now as before choose f_{n+1} in $E_{\lambda_{n+1}}^\perp$ so that

$$||f_{n+1}|| = 1 \quad \text{and} \quad ||T^*(f_{n+1})|| < \frac{1}{n+1} . \tag{3.3}$$

We will now show that the sequence $\{f_n\}$ converges weak* to 0. Let $a \in M(A)$ and let $\varepsilon > 0$. By our hypothesis there exists an $x \in X$ and a $c \in A$ such that $a = T(x) + c$. Now choose $\lambda \in \Lambda$ so that $||c - e_\lambda c|| < \varepsilon/3$. Next choose a positive integer N such that for each integer $n \geqq N$ we have $(e_{\lambda_{n+1}} - e_{\lambda_n})e_\lambda = 0$, $||x||/n < \varepsilon/3$, and $||c||/n < \varepsilon/3$. It follows from (3.2), (3.3), and the fact $f_n \in E_{\lambda_n}^\perp$ that for each integer $n \geqq N$

$$\begin{aligned}
|f_n(a)| &\leqq |f_n(T(x))| + |f_n^0(e_\lambda c)| + |f_n^0(c - e_\lambda c)| \\
&\leqq ||T^* f_n|| \, ||x|| + ||c - e_\lambda c|| + |(1 - e_{\lambda_n}) f_n^0 (1 - e_{\lambda_n}) e_\lambda c)| \\
&\leqq \varepsilon/3 + \varepsilon/3 + ||f_n^0 - e_{\lambda_{n+1}} f_n^0 e_{\lambda_{n+1}}|| \, ||c|| \\
&\quad + |(e_{\lambda_{n+1}} - e_{\lambda_n}) f_n^0 (e_{\lambda_{n+1}} - e_{\lambda_n})(e_\lambda c)| \\
&< \varepsilon .
\end{aligned}$$

Hence $f_n \to 0$ weak*.

Since $f_n \to 0$ weak*, we have by Theorem 3.3 that $\{f_n^0\}$ is tight and converges weak* to zero. Moreover, we will show that $||f_n^0|| \to 0$. Let $\varepsilon > 0$. Choose $\lambda \in \Lambda$ so that $||(1 - e_\lambda) f_n^0 (1 - e_\lambda)|| < \varepsilon/2$ for each positive integer n. Next choose a positive integer N so that for each integer $n \geqq N$, $e_\lambda(e_{\lambda_{n+1}} - e_{\lambda_n}) = 0$ and $3/n < \varepsilon/2$. Since $f_n \in E_{\lambda_n}^\perp$, it is straightforward to verify that $f_n^0 = (1 - e_{\lambda_n}) f_n^0 (1 - e_{\lambda_n})$. It follows that for $n \geqq N$

$$||f_n^0|| \leqq ||(1 - e_\lambda) f_n^0 (1 - e_\lambda)|| + ||e_\lambda f_n^0 + f_n^0 e_\lambda - e_\lambda f_n^0 e_\lambda|| .$$

Replacing f_n^0 in the second term by $e_{\lambda_{n+1}} f_n^0 e_{\lambda_{n+1}} - g_n$, $g_n = -f_n^0 + e_{\lambda_{n+1}} f_n^0 e_{\lambda_{n+1}}$, we get

$$\begin{aligned}
||f_n^0|| &< \varepsilon/2 + ||e_\lambda e_{\lambda_{n+1}} f_n^0 e_{\lambda_{n+1}} + e_{\lambda_{n+1}} f_n^0 e_{\lambda_{n+1}} e_\lambda - e_\lambda e_{\lambda_{n+1}} f_n^0 e_{\lambda_{n+1}} e_\lambda|| \\
&\qquad + 3||f_n^0 - e_{\lambda_{n+1}} f_n^0 e_{\lambda_{n+1}}|| \\
&< \varepsilon/2 + 0 + \varepsilon/2 \\
&\leqq \varepsilon
\end{aligned}$$

for $n \geqq N$. Hence $\|f_n^0\| \to 0$.

Since the map $(x, c) \to T(x) + c$ is a bounded linear map from $x \oplus A$ onto $M(A)$ by hypothesis, the open mapping theorem gives a constant k such that if $a \in M(A)$ and $\|a\| \leqq 1$, then there exists an $x \in X$ and $c \in A$ with $\|x\| + \|c\| \leqq k$ and $T(x) + c = a$. Then we have

$$\begin{aligned}|f_n(a)| &\leqq |f_n(T(x))| + |f_n(c)| \\ &\leqq \|T^*f_n\|\,\|x\| + \|f_n^0\|\,\|c\| \\ &\leqq k\left(\frac{1}{n} + \|f_n^0\|\right).\end{aligned}$$

This implies that $\|f_n\| \leqq k(1/n + \|f_n^0\|)$. It follows that $\|f_n\| \to 0$, which contradicts the fact that $\|f_n\| = 1$. Hence there exists a λ_0 in Λ so that T_{λ_0} maps X onto $M(A)/E_{\lambda_0}$.

Finally choose $\lambda > \lambda_0$. Let $a \in M(A)$. Since T_{λ_0} maps X onto $M(A)/E_{\lambda_0}$, there exists an $x \in X$ and $b \in E_{\lambda_0}$ such that $T(x) = a + b$. Due to the fact that $(1 - e_\lambda)b(1 - e_\lambda) = 0$, we have $(1 - e_\lambda)T(x)(1 - e_\lambda) = (1 - e_\lambda)a(1 - e_\lambda)$. Hence $(1 - e_\lambda)T(X)(1 - e_\lambda) = (1 - e_\lambda)M(A)(1 - e_\lambda)$ and our proof is complete. The idea of this proof comes from [2, Theorem 2.3, p. 527].

The next result is a generalization of Phillips theorem that c_0 is not complemented in l^∞. It also shows (i) (using Conway's result that $C_0(S)$ is complemented in $C(S)$ implies S is pseudo-compact) that $A = C_0(S)$ is never complemented in $C(S)$ when S is paracompact and noncompact, (ii) the compacts are uncomplemented in $B(H)$ unless H is finite dimensional.

COROLLARY 3.7. *Let A be a C^*-algebra with well behaved approximate identity. If A is without an identity, then A is not complemented in $M(A)$.*

Proof. Suppose A is complemented in $M(A)$; that is, suppose there exists, a closed subspace X of $M(A)$ such that $X \oplus A = M(A)$. Then by Theorem 3.6 there exists a $\lambda \in \Lambda$ such that $(1 - e_\lambda)X(1 - e_\lambda) = (1 - e_\lambda)M(A)(1 - e_\lambda)$. Since e_λ is not an identity for A, there exists an $a \in A$ such that $(1 - e_\lambda)a(1 - e_\lambda) \neq 0$. It follows that there exists an x in X such that $(1 - e_\lambda)x(1 - e_\lambda) = (1 - e_\lambda)a(1 - e_\lambda)$, or equivalently, $x = (1 - e_\lambda)a(1 - e_\lambda) + e_\lambda x e_\lambda - e_\lambda x - x e_\lambda$. But this implies that $x = 0$, since $x \in A \cap X$. This contradicts the fact that $(1 - e_\lambda)a(1 - e_\lambda) \neq 0$. Hence A is not complemented in $M(A)$ and the proof is complete.

COROLLARY 3.8. *Let B be a W^*-algebra and let A be a closed two-sided ideal of B with a well behaved approximate identity $\{e_\lambda \mid \lambda \in \Lambda\}$. If X is a Banach space and $T\colon X \to B$ is a bounded linear map such* *that $T(X) + A = B$, then there exists a λ in Λ such that*

$$(1 - e_\lambda)T(X)(1 - e_\lambda) = (1 - e_\lambda)B(1 - e_\lambda) .$$

Proof. Let $A^0 = \{x \in B \mid xA = 0\}$. Since A^0 is a two-sided ideal of B that is closed in the weak operator topology, A^0 has an identity q that commutes with each element of B. Let $X_0 = \{x \in X \mid qT(x) = 0\}$. Then define the bounded linear map $T_0: X_0 \to B/A^0$ by the formula $T_0(x) = T(x) + A^0$ for each x in X_0. We now wish to show that $T_0(X_0) + A/A^0 = B/A^0$. Let $a \in B$. It is clear that $a + A^0 = a - qa + A^0$. By hypothesis, there exists an $x \in X$ and a $c \in A$ such that $T(x) + c = (1 - q)a$. This means $qT(x) = q(1 - q)a - qc = 0$, so $x \in X_0$. Hence $T_0(X_0) + A/A^0 = B/A^0$. By Corollary 2.3 $M(A) = B/A^0$. Therefore, by Theorem 3.6, there exists λ in Λ such that

$$(1 - e_\lambda)B(1 - e_\lambda)/A^0 = (1 - e_\lambda)T(X_0)(1 - e_\lambda)/A^0 . \tag{3.4}$$

We will now show that $(1 - e_\lambda)B(1 - e_\lambda) = (1 - e_\lambda)T(X)(1 - e_\lambda)$. Let $a \in B$. Then by virtue of (3.4) there exists an $x \in X_0$ and $c \in A^0$ such that $(1 - e_\lambda)a(1 - e_\lambda) = (1 - e_\lambda)T(x)(1 - e_\lambda) + c$. This implies $(1 - e_\lambda)(1 - q)a(1 - e_\lambda) = (1 - e_\lambda)T(x)(1 - e_\lambda)$. Hence

$$(1 - e_\lambda)(1 - q)B(1 - e_\lambda) = (1 - e_\lambda)T(X_0)(1 - e_\lambda) . \tag{3.5}$$

Now let $b \in B$. By hypothesis there exists a $y \in X$ such that $qT(y) = qb$. Set $a = b - T(y)$. By (3.5) there exists an $x \in X_0$ such that

$$(1 - e_\lambda)T(x)(1 - e_\lambda) = (1 - e_\lambda)(1 - q)a(1 - e_\lambda) .$$

It follows that

$$\begin{aligned}(1 - e_\lambda)b(1 - e_\lambda) &= (1 - e_\lambda)((1 - q)b + qb)(1 - e_\lambda)\\ &= (1 - e_\lambda)((1 - q)b + qT(y))(1 - e_\lambda)\\ &= (1 - e_\lambda)((1 - q)b - (1 - q)T(y) + T(y))(1 - e_\lambda)\\ &= (1 - e_\lambda)((1 - q)(b - T(y)))(1 - e_\lambda)\\ &\quad + (1 - e_\lambda)T(y)(1 - e_\lambda)\\ &= (1 - e_\lambda)T(x)(1 - e_\lambda) + (1 - e_\lambda)T(y)(1 - e_\lambda)\\ &= (1 - e_\lambda)T(x + y)(1 - e_\lambda) .\end{aligned}$$

Hence $(1 - e_\lambda)B(1 - e_\lambda) = (1 - e_\lambda)T(X)(1 - e_\lambda)$ and our proof is complete.

Let B be a C^*-algebra, let Ω be a compact Hausdorff space, and let $C(\Omega, B)$ denote the space of all B-valued continuous functions on Ω. Let Q be a closed subset of Ω. A linear subspace X of $C(\Omega, B)$ is said to interpolate $C(Q, B)$ if $X \mid Q = C(Q, B)$. More briefly, we call Q an interpolation set for X. In [3] Bade investigated a class of theorems which state for appropriate B, Ω, Q, and X that if X interpolates $C(Q, B)$, then X interpolates $C(V, B)$ for some closed neighborhood V of Q. In paticular, Bade showed (see [3, Theorem 1.1, Theorem 2.1, pp. 149, 157]) that this happens whenever the following hold: B is the complex numbers; $\Omega = \beta(S)$, where S is a locally compact, σ-compact or discrete, Hausdorff space; $Q = \beta S - S$; X is a closed linear subspace of $C(\Omega, B)$. We will now give a natural specialization of Theorem 3.6 that extends Bade's results to a noncommutative setting.

COROLLARY 3.9. *Let B be a finite dimensional C^*-algebra and let S be a locally compact paracompact Hausdorff space. Let X be a closed linear subspace of $C(\beta(S), B)$ such that $X \mid \beta(S) - S = C(\beta(S) - S, B)$. Then there exists a closed neighborhood V of $\beta(S) - S$ in $\beta(S)$ such that $X \mid V = C(V, B)$.*

Proof. It is straightforward to show that $C_0(S, B)$ has a well behaved approximate identity $\{e_\lambda \mid \lambda \in \Lambda\}$ such that each e_λ has compact support. Since the double centralizer algebra of $C_0(S, B)$ is $C(\beta(S), B)$, the conclusion follows from Theorem 3.6.

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