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ON EXTREMAL FIGURES ADMISSIBLE RELATIVE TO RECTANGULAR LATTICES

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A theorem of Bender states that if a convex figure F contains no point of the two dimensional lattice G, where G is generated by the vectors $\bar{V}_1$ and $\bar{V}_2$ having enclosed angle θ, then $A(F) \leqq 1/2\, P(F)$ max $(|\bar{V}_1|, |\bar{V}_2| \sin\theta)$ where $|\bar{V}_1| \leqq |\bar{V}_2|$. In this paper, two questions are answered: (1) Among all convex figures of perimeter L which are admissible relative to a rectangular lattice G, which encloses the maximum area? (2) Can the constant 1/2 in Bender's theorem be improved? By using the result of (1), the "sharpest possible" inequality of the Bender type is found.

NOTATION.

$$w_1 = \min(|\bar{V}_1|, |\bar{V}_2| \sin\theta)$$
$$w_2 = \max(|\bar{V}_1|, |\bar{V}_2| \sin\theta)$$

$A(F)$ is the area of F, $P(F)$ is the perimeter of F. A figure F is *admissible* relative to the lattice G, if no points of G are in the interior of F.

THEOREM. *If F is an admissible convex figure relative to the lattice G, then*

(i) *for* $0 < P(F) \leqq \pi(w_1^2 + w_2^2)^{1/2}$, $A(F) \leqq (P^2(F))/(4\pi)$

(ii) *for* $\pi(w_1^2 + w_2^2)^{1/2} < P(F) < 4w_1 + \pi w_2$,

$$A(F) \leqq \frac{P^2(F)}{4\pi} - \frac{\left(P^2(F) - \pi\left(w_1 \sin q/2 + w_2 \cos\frac{q}{2}\right)\right)^2}{\pi(4 - \pi \sin q)}$$

where q is the root of equation (9).

(iii) *for* $4w_1 + \pi w_2 \leqq P(F)$, $A(F) \leqq 1/2\, w_2 P(F) - \pi/4\, w_2^2$.

Further, if G is rectangular the extremal figures relative to G are shown for (i), (ii) and (iii) in Figure 1 *(i), (ii) and (iii) respectively; in these cases, equality holds.*

By Bender's Lemma [1], only rectangular lattices and admissible convex figures symmetric about the lines $x' = 1/2$, $y' = 1/2$ need be considered (x' and y' are coordinates relative to the lattice); in the remainder of this paper only such figures and lattices will be considered.

DEFINITION. Let G be a (rectangular) lattice and denote by R the set of all admissible rhombi whose vertices lie on the lines $x' =$

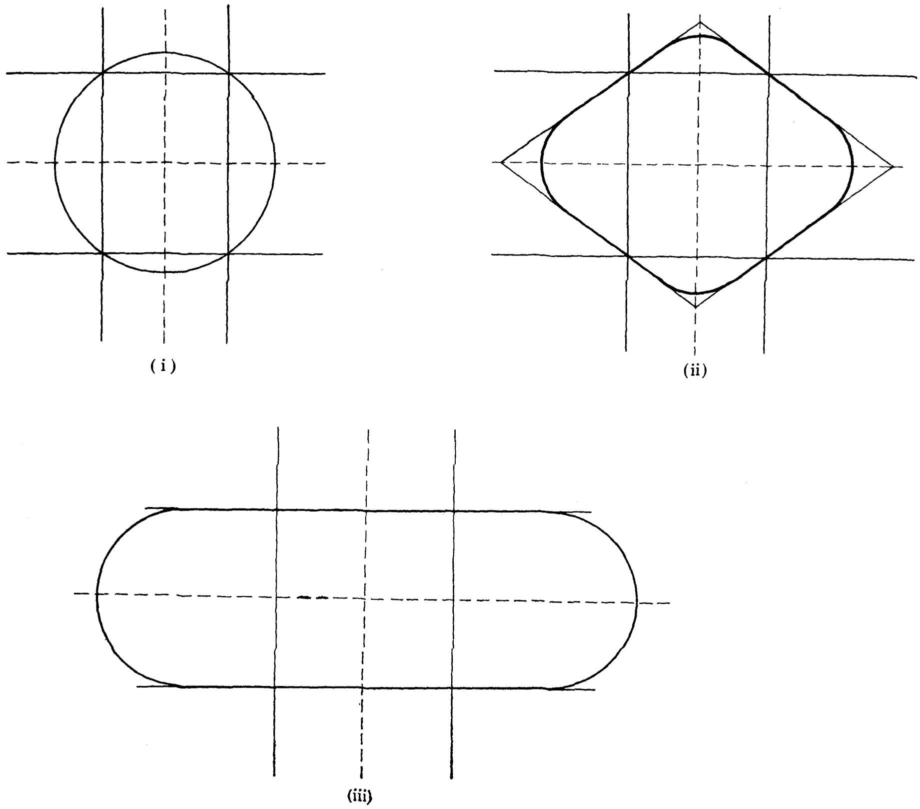

FIGURE 1

1/2, $y' = 1/2$ and each of whose sides pass through at least one lattice point of G (see Figure 2). $R(\varphi)$ denotes the rhombus in R with base angle φ (Figure 2) where $0 \leqq \varphi \leqq \pi$ ($\varphi = 0$ and $\varphi = \pi$ yield the two infinite strips).

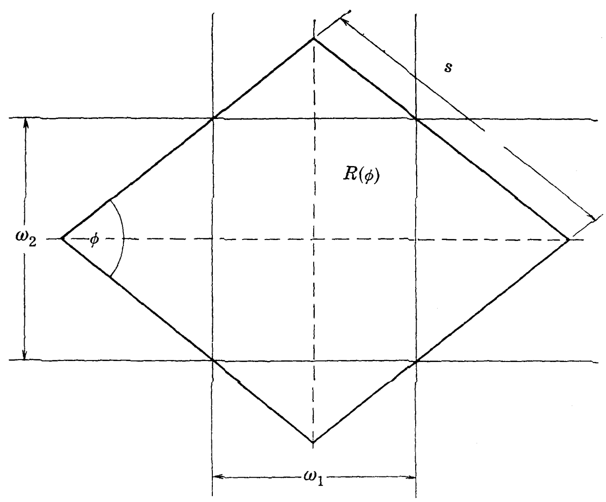

FIGURE 2

LEMMA 1. *Every figure F is contained in at least one rhombus $R(\varphi)$ of the set R.*

Proof. Let g be one of the four lattice points which contain the intersection of the lines $x' = 1/2$ and $y' = 1/2$. Consider the following two cases: (i) g is a boundary point of F, (ii) g is not a boundary point of F.

(i) Since F is convex and g is a boundary point, there exists a line of support S of F at the point G. Construct the three remaining lines symmetric to S about the lines $x' = 1/2$ and $y' = 1/2$. By the symmetry of F, all four of these lines are support lines of F and the rhombus formed contains F and belongs to R.

(ii) Since g is exterior to F, there exists a line S' which separates g and F. Construct S through g parallel to S'. Clearly S lies in the exterior of F and the proof is completed as in (i).

Proof of the Theorem. The inequalities are proven by finding the admissible convex figure of perimeter L which encloses the maximum area (extremal figure). The problem has been reduced to rectangular lattices and symmetric figures which are contained in rhombi of R. Denote by $Y(L, \varphi)$ the extremal figure of perimeter L contained in $R(\varphi)$. The existence, uniqueness, form, etc., of the extremal figure are discussed in references [2] and [4], pp. 124–5. For fixed L, define q by $A(Y(L, q)) = sup_\varphi A(Y(L, \varphi))$. The maximum area is thus attained by the extremal figure contained in the rhombus $R(q)$. Since any figure F is contained in $R(\varphi)$ for some φ (Lemma 1), $A(F) \leqq A(Y(L, \varphi)) \leqq A(Y(L, q))$. The inequalities (ii) and (iii) are nothing other than $A(Y(L, q))$ expressed in terms of L and the lattice constants; (i) means simply that $Y(L, q)$ is a circle. In (ii) and (iii), $Y(L, q)$ contains lattice points on its boundary; otherwise, it is easy to construct a figure of perimeter L having larger area. Hence, for a rectangular lattice, the inequalities of the theorem are the "sharpest possible".

In the remainder of the proof, $A(Y(L, \varphi))$ and $A(Y(L, q))$ are determined.

$Y(L, \varphi)$

From Figure 2, it follows for $0 < \varphi < \pi$

$$S = \frac{1}{2} w_1 \sec \frac{\varphi}{2} + \frac{1}{2} w_2 \csc \frac{\varphi}{2} \tag{1}$$

or

$$S \sin \varphi = w_1 \sin \frac{\varphi}{2} + w_2 \cos \frac{\varphi}{2} \tag{2}$$

$Y(L, \varphi)$ is the parallel figure of radius r taken about a concentric subrhombus of $R(\varphi)$ (see reference 3, p. 124–5). Denoting by v the length of a side of the subrhombus,

$$A(Y(L, \varphi)) = v^2 \sin\varphi + 4rv + \pi r^2 \quad \text{(see Figure 3).} \tag{3}$$

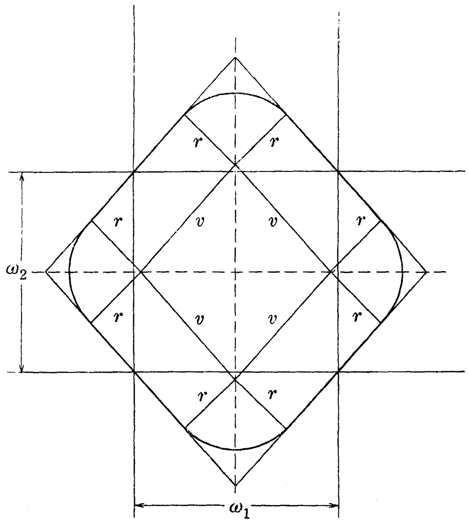

FIGURE 3

The perimeter of $Y(L, \varphi)$ is given by

$$P(Y(L, \varphi)) = L = 2\pi r + 4v\,. \tag{4}$$

Use (4) to eliminate r from (3). After simplification it follows that

$$\frac{L^2}{4\pi} - A(Y(L, \varphi)) = v^2\left(\frac{4}{\pi} - \sin\varphi\right). \tag{5}$$

The right side of (5) is, of course, the classical isoperimetric deficit. From Figure 3,

$$S\cos\frac{\varphi}{2} = v\cos\frac{\varphi}{2} + r\csc\frac{\varphi}{2}\,. \tag{6}$$

Using equation (4), eliminate r from equation (6); use the resulting equation to eliminate v from equation (5), and finally, use equation (2) to eliminate S:

$$A(Y(L, \varphi)) = \frac{L^2}{4\pi} - \frac{\left(\frac{L}{\pi} - \left(w_1 \sin\frac{\varphi}{2} + w_2\cos\frac{\varphi}{2}\right)\right)^2}{\frac{4}{\pi} - \sin\varphi}\,. \tag{7}$$

Equation (7) is valid for $0 < \varphi < \pi$. If $\varphi = 0$ or π, the extremal figure consists of two parallel lines connected by semicircles [3]. Calculation shows that the area agrees in both cases with (7). Hence, (7) is valid for $0 \leqq \varphi \leqq \pi$.

$Y(L, q)$

From equation (7), $A(Y(L, \varphi))$ is a single valued continuous function of L and φ which possesses neither a singularity nor a cusp. To find $Y(L, q)$, the isoperimetric deficit

$$D = \frac{\left(\frac{L}{\pi} - \left(w_1 \sin\frac{\varphi}{2} + w_2 \cos\frac{\varphi}{2}\right)\right)^2}{\frac{4}{\pi} - \sin\varphi} \tag{8}$$

must be minimized.

If $L \leqq \pi(w_1^2 + w_2^2)^{1/2}$, the solution is trivial; viz. the circle. In the remainder of the proof, it is assumed that $L > \pi(w_1^2 + w_2^2)^{1/2}$. Setting $dD/d\varphi = 0$, the condition for an extremum becomes:

$$L \cos\varphi = (4w_1 + \pi w_2)\cos\frac{\varphi}{2} - (4w_2 + \pi w_1)\sin\frac{\varphi}{2}\,. \tag{9}$$

The value (s) of φ which yield an extremum of D must be either 0, π or a root of equation (9).

The case $w_1 = w_2$ will be treated separately; if not otherwise stated, it is assumed that $w_2 > w_1$.

LEMMA 2. *The absolute minimum of $D(L, \varphi)$ lies in the interval $0 \leqq \varphi \leqq \pi/2$.*

Proof. Consider an arbitrary rhombus $R(\varphi) \in R$. From the midpoint of $R(\varphi)$ mark off the distance $1/2\ w_2$ along the line $x' = 1/2$; at this point construct the perpendicular d. From similar triangles,

$$\frac{d}{S\sin\frac{\varphi}{2}} = \frac{S\cos\frac{\varphi}{2} - \frac{1}{2}w_2}{S\cos\frac{\varphi}{2}}\,.$$

Using equation (1), eliminate S and solve for d:

$$d = \frac{1}{2}w_2 - \frac{1}{2}(w_2 - w_1)\tan\frac{\varphi}{2}\,. \tag{10}$$

If $R(\varphi)$ is rotated through $90°$ about its midpoint, it will not contain a lattice point (the boundary included) if $d < 1/2\ w_1$. Applying this condition to equation (10), it follows that (11) $\tan \varphi/2 > 1$. Hence, $R(\varphi)$ does not contain a lattice point when rotated about its midpoint through 90° if $\varphi > \pi/2$. Suppose $Y(L, \varphi)$ is the extremal figure of the rhombus $R(\varphi)$ where $\varphi > \pi/2$. Rotate $R(\varphi)$ through 90° about its midpoint. By the preceding argument, $R(\varphi)$ and thus $Y(L, \varphi)$ contains no lattice point (boundary included). Thus, $Y(L, \varphi)$ cannot be $Y(L, q)$.

Thus, q is either 0 or a root of equation (8) $(w_1 \neq w_2)$.

LEMMA 3. *For* $0 \leqq \varphi \leqq \pi/2$ *equation* (9) *has*
(i) *exactly one root if* $L < 4w_1 + \pi w_2$
(ii) *exacly one root* (viz., $\varphi = 0$) *if* $L = 4w_1 + \pi w_2$
(iii) *no roots if* $L > 4w_1 + \pi w_2$

Proof. Form the auxillary functions $y_1 = L \cos \varphi$ and

$$y_2 = (4w_1 + \pi w_2) \cos \frac{\varphi}{2} - (4w_2 + \pi w_1) \sin \frac{\varphi}{2}$$
$$= ((4w_1 + \pi w_2)^2 + (4w_2 + \pi w_1)^2)^{1/2} \sin \left(\beta - \frac{\varphi}{2}\right)$$

where

$$\tan \beta = \frac{4w_1 + \pi w_2}{\pi w_1 + 4w_2} .$$

Clearly, $38° < \beta < 45°$. The roots of equation (9) are the points of intersection of y_1 and y_2. Divide the problem into three parts
(i) $y_1(0) < y_2(0)$; i.e., $L < 4w_1 + \pi w_2$
(ii) $y_1(0) = y_2(0)$; i.e., $L = 4w_1 + \pi w_2$
(iii) $y_1(0) > y_2(0)$; i.e., $L > 4w_1 + \pi w_2$
y_1 and y_2 are cosine and sine curves; the lemma follows from the elementary properties of these curves.

From Lemma 3, it follows for (iii) and (ii) that $q = 0$. In case (i), $D'(0)$ is negative and q must be the (single) root of equation (9). Thus, the extremal figures have been found and inserting the value of q into equation (7) gives the theorem (for $w_1 \neq w_2$).

The Solution for $w_1 = w_2 = w$.

This is the most important single case; viz., the square lattice. Geometrically it is obvious that equation (7) and therefore (8) are symmetric about $\pi/2$; viz., $R(\varphi)$ is identical with $R(\pi - \varphi)$ except for a rotation of $\pi/2$ about the midpoint. Hence $Y(L, \varphi)$ is identical with $Y(L, \pi - \varphi)$, except for a rotation of $\pi/2$ about its midpoint. φ can therefore be restricted to the interval $0 \leqq \varphi \leqq \pi/2$. In this case, equation (9) becomes:

$$(12) \qquad \left(\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}\right)\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2} - \frac{(4+\pi)w}{L}\right) = 0$$

Equation (12) has two roots

$$(13) \qquad \varphi = \frac{\pi}{2}\,,$$

$$(14) \qquad \sin\varphi = \left(\frac{w}{L}\right)^2 (4+\pi)^2 - 1\,.$$

(i) $0 < L \leqq \sqrt{2}\,\pi w$. The circle is admissible and $A(B) \leqq (P^2(B))/4\pi$.
(iii) $L > (4+\pi)$w. Equation (14) does not yield an admissible root; since $D(\varphi)$ is strictly increasing, $q = 0$. The extremal figure is of the form shown in Figure 1 (iii) and $A(B) \leqq 1/2wP(B) - 1/4\,\pi w^2$

(ii) $\sqrt{2}\,\pi w < L \leqq (4+\pi)w$. Case (ii) decomposes into two cases:
(iia) $\sqrt{2}\,\pi w < L \leqq \sqrt{2}\,(4+\pi)w$
(iib) $\sqrt{2}\,(4+\pi)w < L \leqq (4+\pi)w$

Case (*ii*a) If $L \leqq 1/\sqrt{2}\,(4+\pi)w$, equation (14) offers no solution; since $D(\varphi)$ is strictly decreasing, $q = \pi/2$. Thus, for all L in (iia), the extremal figure is contained in $R(\pi/2)$. The desired inequality becomes $A(B) \leqq \dfrac{1}{(4-\pi)}(-1/4L^2 + \sqrt{8}\,wL - 2\pi w^2)$. Note that there is no analogy if $w_1 \neq w_2$.

Case (*iib*) q occurs in $(0, \pi/2)$ and is given by equation (14). The extremal figure has the form shown in Figure 1 (ii) and $A(B) \leqq (L^2)/(4+\pi)4 + w^2$.

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