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ON THE SPECTRAL RADIUS FORMULA IN BANACH ALGEBRAS

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B will always denote a commutative semi-simple Banach algebra with a unit element. If $f \in B$ then $r(f)$ denotes its spectral radius. A sequence $F = (f_j)_1^\infty$ is called a spectral null sequence if $\|f_j\| \leqq 1$ for each j, while $\lim_{j\to\infty} r(f_j) = 0$. If $F = (f_j)$ is a spectral null sequence we put $r_N(F) = \limsup_{j\to\infty} \|f_j^N\|^{1/N}$ for each $N \geqq 1$. Finally we define the complex number $r_N(B) = \sup\{r_N(F)\colon F$ is a spectral null sequence in $B\}$. In general $r_N(B) = 1$ for all $N \geqq 1$ and the aim of this paper is to study the case when $r_N(B) < 1$ for some N.

We say that B satisfies a bounded inverse formula if there exists some $0 < \varepsilon < 1$ and a constant K_0 such that for all f in B satisfying $\|f\| \leqq 1$ and $r(f) \leqq \varepsilon$, it follows that $\|(e-f)^{-1}\| \leqq K_0$. In Theorem 3.1. we prove that B satisfies a bounded inverse formula if and only if $r_N(B) < 1$ for some N.

In § 1 we give a criterion which implies that B is a sup-norm algebra. In § 2 we introduce the so called infinite product of B which will enable us to study spectral null sequences in § 3.

1. Sup-norm algebras.

Recall that B is a sup-norm algebra if there exists a constant K such that $\|f\| \leqq Kr(f)$ for all f in B. Clearly this happens if and only if $r_1(B) = 0$. Next we give an example where $r_1(B) = 1$ while $r_2(B) = 0$.

Let $B = C^1[0, 1]$ be the algebra of all continuously differentiable functions on the closed unit interval. If $f \in B$ we put $\|f\| = \sup\{|f(x)| + |f'(y)|\colon 0 \leqq x, y \leqq 1\}$. The maximal ideal space M_B can be identified with $[0, 1]$, so the spectral radius formula shows that $r(f) = \sup\{|f(x)|\colon 0 \leqq x \leqq 1\}$. From this we easily deduce that $r_2(B) = 0$. In fact we also notice that $\|f^n\| \leqq n\|f\|(r(f))^{n-1}$ holds for all $n \geqq 2$. We will now prove that this estimate is sharp.

THEOREM 1.1. *Let the norm in B satisfy $\|f^n\| \leqq qn\|f\|r(f)^{n-1}$ for some $q < 1$ and some $n \geqq 2$. Then B is a sup-norm algebra and there is a constant $K(n, q)$ such that $\|f\| \leqq K(n, q)r(f)$ for all $f \in B$.*

LEMMA 1.2. *Let $n \geqq 3$ and suppose that $\|f^n\| \leqq K\|f\|r(f)^{n-1}$ for all f in B and some constant K. Then there is a constant $K(n)$ such that $\|f^2\| \leqq K(n)K\|f\|r(f)$.*

Proof. Notice that all the inequalities above are homogeneous. Hence it is sufficient to consider the case when $||f|| = 1$. If now $r(f) = \varepsilon$, then we must prove that $||f^2|| < K(n)K\varepsilon$ for some $K(n)$.

Under the hypothesis, we note that

$$|| (k\varepsilon + f)^n || \leqq K || k\varepsilon + f ||(\varepsilon + k\varepsilon)^{n-1} \leqq K\varepsilon^{n-1}(1 + n)^n$$

for all $0 \leqq k \leqq n$.

Now consider the inhomogeneous system of equations

$$\sum_{j=0}^{n} \binom{n}{j}(k\varepsilon)^{n-j}f^j = (k\varepsilon + f)^n , \qquad 0 \leqq k \leqq n ,$$

which we wish to solve for the f^j. The determinant of the system is $\varepsilon\varepsilon^2 \cdots \varepsilon^n K_0(n)$, and the determinants of the minors can be expressed similarly. Using Cramer's rule to solve this system for f^2, we obtain the estimate $||f^2|| \leqq K(n)K\varepsilon$, as required.

Proof of Theorem 1.1. Firstly we choose $\varepsilon > 0$ so small that $1 - \varepsilon^n > 2n\varepsilon^n + q$. Next we introduce the power series $\phi(z) = \varepsilon + a_1z + a_2z^2 + \cdots$, which satisfies $(\phi(z))^n = \varepsilon^n + z$ for all $|z| < \varepsilon^n$. Notice that $na_1\varepsilon^{n-1} = 1$ holds. If $0 < x < \varepsilon^n$ we put

$$A_v(x) = x^v(|a_{vn}| + \cdots + |a_{vn+n-1}|) .$$

Then it is clear that the sum $U(x) = A_1(x) + A_2(x) + \cdots$ is finite, while $\lim U(x) = 0$ as $x \to 0$.

Note that from Lemma 1.2. there is a constant $K(n)$ such that $||f^k|| \leqq K(n)r(f)$ for all $2 \leqq k \leqq n - 1$ and all f in B satisfying $||f|| \leqq 1$. It follows that there is a constant $K(n, \varepsilon)$ such that $|| a_2f^2 + \cdots + a_{n-1}f^{n-1} || \leqq K(n, \varepsilon)r(f)$ for all f satisfying $||f|| \leqq 1$.

Now we choose $\delta > 0$ so small that $n\delta^{n-1} < \varepsilon^n$ and $U(n\delta^{n-1}) + K(n, \varepsilon)\delta < \varepsilon$ holds.

Suppose now that B is not a sup-norm algebra. Then we can choose f in B such that $||f|| = 1$ while $r(f) < \delta$. The assumption shows that $||f^n|| \leqq qn\delta^{n-1} \leqq n\delta^{n-1} \leqq \varepsilon^n$. Hence $||f^{vn+k}|| \leqq ||f^n||^v ||f^k|| \leqq (n\delta^{n-1})^v \to$ for all $v \geqq 1$ and all $k = 0 \cdots (n - 1)$. It follows that we can define the element $g = \phi(f) = \varepsilon + a_1f + a_2f^2 + \cdots$ in B.

We get $||g|| \leqq \varepsilon + |a_1| + ||a_2f^2 + \cdots + a_{n-1}f^{n-1}|| + U(n\delta^{n-1}) \leqq 2\varepsilon + |a_1|$. We also have $r(g) \leqq (r(\varepsilon^n + f))^{1/n} \leqq (\varepsilon^n + \delta)^{1/n}$.

It follows that $1 - \varepsilon^n \leqq ||\varepsilon^n + f|| = ||g^n|| \leqq qn||g||r(g)^{n-1} \leqq qn(2\varepsilon + (n\varepsilon^{n-1})^{-1})(\varepsilon^n + \delta)^{1-1/n} = Z(\delta)$.

Clearly $Z(\delta)$ tends to $2qn\varepsilon^n + q$ as $\delta \to 0$. The original choice of ε shows that $1 - \varepsilon^n \leqq Z(\delta)$ cannot hold for sufficiently small values of δ. This proves that B must be a sup-norm algebra and the proof gives a lower bound for δ, once we have fixed ε.

2. The infinite product of a Banach algebra. Firstly we introduce the infinite product.

DEFINITION 2.1. Put $B_\infty = \{(f_j)_1^\infty: (f_j)$ is a sequence in B such that $\sup_j \|f_j\| < \infty$ while $\lim_{j\to\infty} r(we - f_j) = 0$ for some $w \in C^1\}$.

Clearly B_∞ is a Banach algebra if to each $F = (f_j)$ we define $\|F\| = \sup_j \|f_j\|$. If $F = (f_j)$ and if $N \geqq 1$, then we put $\pi_N(F) = (g_j)$, where $g_j = 0$ for $j \leqq N$ and $g_j = f_j$ for $j > N$.

A complex-valued homomorphism H on B_∞ is free if $H(F) = H(\pi_N(F))$ for all $N \geqq 1$ and each $F \in B_\infty$. The part at infinity in M_{B_∞} consists of the points determined by free homomorphisms. We denote this set by M_∞.

To each $N \geqq 1$ we have an idempotent e_N in B_∞, whose Nth component is e while all the other components are zero. The set $\Delta_N = \{x \in M_{B_\infty}: \hat{e}_N(x) = 1\}$ is a clopen (closed and open) subset of M_{B_∞}. We can identify Δ_N with M_B. For if $x \in M_B$ we get a point $T_N(x)$ in Δ_N satisfying $\hat{F}(T_N(x)) = \hat{f}_N(x)$ for all $F = (f_j)$. It is easily seen that T_N is a homeomorphism from M_B onto Δ_N.

If we put $\Delta = \bigcup \Delta_N$: $N \geqq 1$, then it is easily seen that $\Delta = M_{B_\infty} \backslash M_\infty$. Here Δ is open and hence M_∞ is closed. The set M_∞ contains a distinguished point m_∞, determined by the complex-valued homomorphism which sends $F = (f_j)$ into the complex number w satisfying $\lim_{j\to\infty} r(we - f_j) = 0$.

With the notations above the following result is evident.

LEMMA 2.2. *Let V be an open neighborhood of m_∞ in M_{B_∞}. Then there is an integer N such that $\Delta_j \subset V$ for all $j > N$.*

LEMMA 2.3. *Let $b\Delta$ be the topological boundary of Δ in M_{B_∞}. Then $b\Delta = \{m_\infty\}$.*

Proof. Lemma 2.2. means that the clopen sets Δ_N converge to $\{m_\infty\}$. Then it is a trivial topological fact that m_∞ is the only boundary point of Δ.

The result below was motivated by Theorem 2 in [2].

THEOREM 2.4. *The set M_∞ is a closed and connected subset of M_{B_∞}.*

Proof. We already know that M_∞ is closed. Suppose next that S and T are disjoint closed subsets whose union is M_∞, such that $m_\infty \in S$. Then Lemma 2.3. implies that $S \cup \Delta$ is clopen in M_{B_∞}. By Shilov's idempotent Theorem there is $E \in B_\infty$ such that $\hat{E} = 0$ on $S \cup \Delta$ while $\hat{E} = 1$ on T. In particular $\hat{E} = 0$ on each Δ_j, which implies that the jth component is zero. Since this holds for all j we conclude that $E = 0$, and T is empty. Hence M_∞ is connected.

The next result gives a useful characterization of M_∞. This result is due to the referee.

THEOREM 2.5. *Let I be the closed ideal of all F in B_∞ for which $\lim \| \pi_N(F) \| = 0$ as $N \to \infty$. Then M_∞ is the maximal ideal space of B_∞/I.*

Proof. A point m in M_{B_∞} induces a complex-valued homomorphism on B_∞/I if and only if $\hat{F}(m) = 0$ for all $F \in I$. Clearly each idempotent e_N belongs to I. This proves that if m annihilates I, then m must belong to M_∞. Conversely, if $m \in M_\infty$ then $\hat{F}(m) = \pi_N(F)\hat{}(m)$ for all $N \geqq 1$. Hence $| \hat{F}(m) | \leqq \lim_{N \to \infty} \| \pi_N(F) \| = 0$ follows if $F \in I$. This proves that every point in M_∞ annihilates I.

If $F = (f_j)$ is in B_∞ we put $r_N(F) = \limsup_{j \to \infty} \| f_j^N \|^{1/N}$ for each $N \geqq 1$. Let us also put $| F |_\infty = \sup \{ | \hat{F}(m) | : m \in M_\infty \}$. With these notations the following result is a direct consequence of Theorem 2.5.

PROPOSITION 2.6. *If $F \in B_\infty$, then $| F |_\infty = \lim_{N \to \infty} r_N(F)$.*

3. Spectral null sequences.

THEOREM 3.1. *The following conditions on B are equivalent:*

(a) *$r_N(B) < 1$ for some $N \geqq 1$.*

(b) *B satisfies a bounded inverse formula.*

(c) *There is a constant K_q such that if $f \in B$ satisfies $\| f \| \leqq 1$ and $r(f) = q < 1$, then $\| (e - f)^{-1} \| \leqq K_q(1 - q)^{-1}$.*

Proof. Since (c) $\to$ (b) we only prove that (a) $\to$ (b) and (b) $\to$ (a). Firstly we assume that $r_N(B) < 1$ for some $N \geqq 1$. Then we get some $\varepsilon > 0$ and $a < 1$ such that $\| f^N \| \leqq a^N$ for all f satisfying $\| f \| \leqq 1$ and $r(f) \leqq \varepsilon$.

Let then $\| f \| \leqq 1$ while $r(f) \leqq q < 1$. Let s be the positive integer satisfying $q^s < \varepsilon \leqq q^{s-1}$. It follows that $\| f^{Ns} \| a^N$ and hence $\| f^{kNs} \| \leqq a^{kN}$ for all $k \geqq 1$. Using this fact we see that if $R = \sum f^j : j \geqq sN$, then $\| R \| \leqq sNa^N(1 - a^N)^{-1}$.

We have $(e - f)^{-1} = e + f + \cdots + f^{Ns-1} + R$. Since $\| f \| \leqq 1$ we get $\| (e - f)^{-1} \| \leqq sN + \| R \| \leqq K_0 s$. Finally $\varepsilon \leqq q^{s-1}$ which implies that $s \leqq K_1(1 - q)^{-1}$. Hence (c) follows with $K_q = K_0 K_1$.

Now we assume that (b) holds in B. Suppose that $r_N(B) = 1$ for all N. To each $j \geqq 1$ we can choose f_j such that $\| f_j \| = 1$ and $r(f_j) < (j + 1)^{-1}$, while $\| f_j^j \|^{1/j} > 1 - 1/j$.

Let us consider $F = (f_j)$ in B_∞. Since $\lim_{j \to \infty} \| F^j \|^{1/j} = 1$, it follows that there is some $w \in C^1$ satisfying $|w| = 1$ while $we - F$ is not invertible in B_∞.

Consider the elements $g_j = (e - f_j/w)^{-1}$ which exist for all $j \geqq 1$. Clearly (b) implies that $\|g_j\| \leqq K$ for some fixed constant K. Since $\lim_{j\to\infty} r(f_j) = 0$ it follows that the element $G = (g_j)$ exists in B_∞. Now $(we - F)Gw^{-1} = e$ in B_∞ which shows that $we - F$ is invertible, a contradiction. Hence $r_N(B) < 1$ must hold for some N.

Let us observe that a spectral null sequence $F = (f_j)$ simply is an element of B_∞ for which $\|F\| \leqq 1$ and $\hat{F}(m_\infty) = 0$. The following result is a direct consequence of Proposition 2.6.

THEOREM 3.2. *The following two conditions on* B *are equivalent*:
(a) $\lim r_N(B) = 0$ *as* $N \to \infty$.
(b) $M_\infty = \{m_\infty\}$.

Finally we study spectral null sequences satisfying polynomial conditions.

THEOREM 3.3. *Let* p *be a polynomial of the form* $z^s\ (1 + a_1z + \cdots + a_tz^t)$, *with* $s > 1$. *Then there exist constants* K *and* c *with the following property*: *If* $f \in B$ *satisfies* $\|f\| \leqq 1$, $\|p(f)\| \leqq \varepsilon$ *and* $r(f) \leqq \varepsilon$, *where* $\varepsilon \leqq c$, *then* $\|f^s\| \leqq K\varepsilon$.

Proof. For each $\varepsilon > 0$ we put $S(\varepsilon) = \{f \in B: \|f\| \leqq 1,\ \|p(f)\| \leqq \varepsilon$ and $r(f) \leqq \varepsilon\}$. Suppose the constants c and K do not exist. Then there is a decreasing sequence (ε_j), with $\lim_{j\to\infty} \varepsilon_j = 0$, while $S(\varepsilon_j)$ contains an element f_j for which $\|f_j^s\| > j\varepsilon_j$.

We may assume that $1 > |a_1|\varepsilon_1 + \cdots + |a_t|\varepsilon_1^t$ holds. This implies that the elements $u_j = e + a_1f_j + \cdots + a_tf_j^t$ are invertible in B.

Now $p(f_j) = f_j^s u_j$ and hence $j\varepsilon_j < \|f_j^s\| \leqq \|p(f_j)\|\,\|u_j^{-1}\| \leqq \varepsilon_j \|u_j^{-1}\|$. This means that $\|u_j^{-1}\| > j$ for all j, so the element $G = (u_j)$ is not invertible in B_∞.

Now we obtain a contradiction by proving that G must be invertible in B_∞. Since $\lim_{j\to\infty} \|p(f_j)\| = 0$ it follows that $\lim \|p(\pi_N(G))\| = 0$ as $N \to \infty$. Then Proposition 2.6. shows that $p(G)$ must vanish on M_∞.

Hence the set $\hat{G}(M_\infty)$ is contained in the finite set of zeros of p. Using Theorem 2.4. we see that $\hat{G}(M_\infty)$ is connected. It follows that $\hat{G}(M_\infty) = \{\hat{G}(m_\infty)\}$. Clearly $\hat{G}(m_\infty) = 1$ holds and hence $\hat{G}$ does not vanish on M_∞. The choice of ε_1 shows that $\hat{G} \neq 0$ on Δ too. This proves that G is invertible in B_∞ which gives the desired contradiction.

Finally we raise some problems. We do not know if the condition that $r_N(B) < 1$ for some $N > 2$ implies that $r_2(B) < 1$. We also ask if the condition that $r_N(B) < 1$ for some $N \geqq 2$ implies that $\lim r_J(B) = 0$ as $J \to \infty$.

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