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ON THE ABSOLUTE HAUSDORFF SUMMABILITY OF A FOURIER SERIES

S. N. LAL AND SIYA RAM

In this paper a theorem on the absolute Hausdorff summability of a series associated with a Fourier series has been established. This theorem unifies and extends various known results.

1. Let μ_n be a sequence of real or complex numbers and write

$$\Delta^0\mu_n = \mu_n, \Delta^p\mu_n = \Delta^{p-1}\mu_n - \Delta^{p-1}\mu_{n+1}, \quad p \geqq 1 .$$

If S_n denotes the sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n$, the transformation

$$t_m = \sum_{n=0}^{m} \binom{m}{n}(\Delta^{m-n}\mu_n)S_n$$

defines the sequence $\{t_m\}$ of (H, μ) means or the Hausdorff means [3, 12] of the sequence $\{S_n\}$. The series $\sum a_n$ is said to be summable (H, μ) to the sum s if $\lim_{m\to\infty} t_m = s$ and is said to be absolutely summable (H, μ) or summable $|H, \mu|$ if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < C^1 .$$

In order that (H, μ) should be a convergence preserving transformation it is necessary and sufficient that μ_n should be a moment constant, that is, there exists a function $\chi(x)$ of bounded variation in $0 \leqq x \leqq 1$, such that

$$\mu_n = \int_0^1 x^n d\chi(x) , \qquad n = 0, 1, 2, \cdots .$$

We may suppose without loss of generality that $\chi(0) = 0$. If also, $\chi(1) = 1$ and $\chi(+0) = \chi(0) = 0$, so that $\chi(x)$ is continuous at the origin, then μ_n is a regular moment constant and (H, μ) is a regular Hausdorff transformation [3]. It is known that the sequence to sequence Hausdorff transformation is absolute convergence preserving or absolutely regular if and only if it is a convergence preserving or regular transformation of the same type [4, 8, 9].

In the case in which

$$\chi(x) = 1 - (1 - x)^\delta , \quad \delta > 0 ,$$

[1] Throughout the paper C denotes a positive constant not necessarily the same at each occurrence.

the method (H, μ) reduces to the well known Cesàro method (C, δ) [3, 12].

2. Let $f(t)$ be a periodic function with period 2π and integrable in the Lebesgue sense in $(-\pi, \pi)$. Let the Fourier series of $f(t)$ be

$$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t) ,$$

it being assumed that the constant term is zero.

The εth forward and backward fractional integrals of a function $g(x)$, which is Lebesgue integrable in $(0, 1)$, are respectively defined as

$$g_\varepsilon^+(x) = \frac{1}{\Gamma(\varepsilon)}\int_0^x (x-u)^{\varepsilon-1} g(u)du ,$$

and

$$g_\varepsilon^-(x) = \frac{1}{\Gamma(\varepsilon)}\int_x^1 (u-x)^{\varepsilon-1} g(u)du .$$

These integrals exist almost everywhere for $\varepsilon > 0$.

We write

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\} ;$$

$$\Phi_0(t) = \phi(t) ;$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)du , \quad \alpha > 0 ;$$

$$\phi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t) , \quad \alpha \geqq 0 ;$$

$$M(n, x, t) = \sum_{\nu=1}^{n} \nu\binom{n}{\nu}\varepsilon(\nu)x^\nu(1-x)^{n-\nu} \cos \nu t ;$$

$$N(n, x, t) = \sum_{\nu=1}^{n} \binom{n}{\nu}\varepsilon(\nu)x^\nu(1-x)^{n-\nu} \sin \nu t ;$$

$$L(\chi; n, t) = \frac{2}{\pi}\sum_{\nu=1}^{n} \binom{n}{\nu}\varepsilon(\nu) \sin \nu t\int_0^1 x^\nu(1-x)^{n-\nu}d\chi(x) ;$$

$$I(\chi; n, u) = \frac{1}{\Gamma(1-\alpha)}\int_u^\pi (t-u)^{-\alpha}\frac{d}{dt}L(\chi; n, t)dt ;$$

$$J(\chi; n, u) = \frac{1}{\Gamma(1+\alpha)}\int_0^u v^\alpha \frac{d}{dv}I(\chi; n, v)dv .$$

3. In this paper we establish the following:

THEOREM. *Let $\varepsilon(t)$ be a positive and monotonic nondecreasing* *function of* t *such that*

$$\sum_{n=[1/t]+1}^{\infty} \frac{\varepsilon(n)}{n^{1+\gamma-\alpha}t^{\gamma-\alpha}} = O\left(\varepsilon\left(\frac{k}{t}\right)\right), \quad (0 < t < \pi,\ k > \pi) \tag{3.1}$$

and

$$\int_0^{\pi} \varepsilon\left(\frac{k}{t}\right) |d\phi_\alpha(t)| < C\,. \tag{3.2}$$

If

$$\begin{aligned} &\textit{either (a)} \quad \chi(u) = g_{1+\gamma}^{+}(u) + C\,, \\ &\textit{or} \quad (b) \quad \chi(u) = g_{1+\gamma}^{-}(u) + C\,, \end{aligned} \qquad (0 \leqq \alpha < \gamma < 1)$$

for some function $g(u)$ *which is Lebesgue integrable in* $(0, 1)$, *then the series* $\sum \varepsilon(n)A_n(t)$ *is summable* $|H, \mu|$ *at the point* $t = x$, *it being assumed that the transformation* (H, μ) *is convergence preserving.*

Taking $\varepsilon(t) = 1$ and $\alpha = 0$ the above theorem reduces to a recent result on the absolute Hausdorff summability of a Fourier series ([11], Theorem 1) which in turn includes[2] a result of Bosanquet ([1], Theorem 1) on the absolute Cesàro summability of a Fourier series and the case $0 < \alpha < 1$ covers another result on the absolute Cesàro summability of a Fourier series ([2], Theorem 1). Also for $\alpha = 0$ choosing $\varepsilon(t) = t^\beta$ and $\gamma = \beta + \delta$ ($\beta > 0$, $\delta > 0$) we get another result ([10], Theorem 1) on the absolute Hausdorff summability which is known to include a theorem on the absolute Cesàro summability of the series $\sum n^\beta A_n(t)$ due to Mohanty ([6], Theorem 1). Further choosing $\varepsilon(t) = \log(1 + t)$ we get (cf. [7])

THEOREM A. *If*

$$\int_0^{\pi} \log\frac{k}{t} |d\phi_\alpha(t)| < C \quad (k > \pi)$$

and

$$\begin{aligned} &\textit{either (a)} \quad \chi(u) = g_{1+\gamma}^{+}(u) + C\,, \\ &\textit{or} \quad (b) \quad \chi(u) = g_{1+\gamma}^{-}(u) + C\,, \end{aligned} \qquad (0 \leqq \alpha < \gamma < 1)$$

for some function $g(u)$ *which is Lebesgue integrable in* $(0, 1)$, *then the series* $\sum \log(n + 1)A_n(t)$ *is summable* $|H, \mu|$, *at the point* $t = x$, *it being assumed that the transformation* (H, μ) *is convergence preserving.*

4. We require the following lemmas for the proof of our theorem.

[2] While this paper was in press, a paper due to B. Kuttner and N. Tripathi (Quart. J. Math., **22** (1971), 229-308) appeared in which it is shown that Tripathi's theorem can be deduced from the result of Bosanquet.

LEMMA 1 [4]. *Let* $\{t_n\}$ *and* $\{S_n\}$ *be the partial sums of the series* $\sum b_n$ *and* $\sum a_n$ *respectively. Then the sequence to sequence transformation*

$$t_m = \sum_{n=0}^{m} \binom{m}{n} (\Delta^{m-n}\mu_n) S_n$$

can be put in the series to series form as

$$b_m = \frac{1}{m}\sum_{n=1}^{m} \binom{m}{n} (\Delta^{m-n}\mu_n) n a_n ,$$
$$b_0 = a_0 .$$

LEMMA 2 [5]. *If* $g(x)$ *and* $h(x)$ *are Lebesgue integrable in* $(0, 1)$, *then for* $\varepsilon > 0$

$$\int_0^1 g_\varepsilon^+(x) h(x) dx = \int_0^1 g(x) h_\varepsilon^-(x) dx .$$

LEMMA 3.

$$I_1 \equiv \int_0^x N(n, 1 - v, t) dv = O\left(\frac{\varepsilon(n)}{nt}\right)$$

and

$$I_2 \equiv \int_0^x M(n, 1 - v, t) dv = O\left(\frac{\varepsilon(n)}{t}\right) .$$

uniformly for x *in* $(0, 1)$.

Proof. By Abel's transformation, we have

$$\begin{aligned} I_1 &= \int_0^x \left(\sum_{\nu=1}^{n} \binom{n}{\nu} \varepsilon(\nu)(1 - \nu)^\nu v^{n-\nu} \sin \nu t \right) dv \\ &= \int_0^x \Bigg[\sum_{\nu=1}^{n-1} \Delta_\nu \left\{ \binom{n}{\nu} \varepsilon(\nu)(1 - v)^\nu v^{n-\nu} \right\} \sum_{r=1}^{\nu} \sin rt \\ &\quad + \binom{n}{n} \varepsilon(n)(1 - v)^n \sum_{\nu=1}^{n} \sin \nu t \Bigg] dv \\ &= O\left(\frac{1}{t}\right) \sum_{\nu=1}^{n-1} | \Delta_\nu(\varepsilon(\nu) p_\nu(x)) | + O\left(\frac{\varepsilon(n)}{nt}\right) , \end{aligned}$$

where

$$p_\nu(x) = \binom{n}{\nu} \int_0^x v^{n-\nu}(1 - v)^\nu dv , \quad (1 \leqq \nu \leqq n - 1) .$$

We observe that $\varepsilon(\nu) p_\nu(x)$ $(1 \leqq \nu \leqq n - 1)$ is a nondecreasing function of ν for fixed x, since by hypothesis $\varepsilon(\nu)$ is non-decreasing and

$$
\begin{aligned}
p_\nu(x) &= \binom{n}{\nu}\int_0^x v^{n-\nu}(1-v)^\nu dv \\
&= \binom{n}{\nu}\Big[-\frac{v^{n-\nu}(1-v)^{\nu+1}}{\nu+1}\Big]_0^x + \binom{n}{\nu}\frac{(n-\nu)}{(\nu-1)}\int_0^x v^{n-\nu-1}(1-v)^{\nu+1}dv \\
&= \binom{n}{\nu+1}\int_0^x v^{n-\nu-1}(1-v)^{\nu+1}dv - \frac{1}{(\nu+1)}\binom{n}{\nu}x^{n-\nu}(1-x)^{\nu+1} \\
&= p_{\nu+1}(x) - \frac{1}{(\nu+1)}\binom{n}{\nu}x^{n-\nu}(1-x)^{\nu+1} \\
&< p_{\nu+1}(x)\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_1 &= O\Big(\frac{1}{t}\Big)\sum_{\nu=1}^{n-1}[\varepsilon(\nu+1)p_{\nu+1}(x) - \varepsilon(\nu)p_\nu(x)] + O\Big(\frac{\varepsilon(n)}{nt}\Big) \\
&= O\Big(\frac{\varepsilon(n)}{t}\Big)\int_0^x (1-v)^n dv + O\Big(\frac{\varepsilon(n)}{nt}\Big) \\
&= O\Big(\frac{\varepsilon(n)}{nt}\Big)\,.
\end{aligned}
$$

I_2 can be similarly estimated. Hence the lemma.

LEMMA 4. *For* $0<\gamma<1$,

$$J_1 \equiv \int_0^{1-x}(1-x-u)^{\gamma-1}N(n, 1-u, t)du = O\Big(\frac{\varepsilon(n)}{n^\gamma t^\gamma}\Big)$$

and

$$J_2 \equiv \int_0^{1-x}(1-x-u)^{\gamma-1}M(n, 1-u, t)du = O\Big(\frac{\varepsilon(n)}{n^{\gamma-1}t^\gamma}\Big)$$

uniformly for x *in* $(0, 1)$.

Proof. Since

$$
\begin{aligned}
|N(n, 1-u, t)| &\leqq \varepsilon(n)\sum_{\nu=1}^{n}\binom{n}{\nu}(1-u)^\nu u^{n-\nu} \\
&\leqq \varepsilon(n)
\end{aligned}
\tag{4.1}
$$

we have

$$
\begin{aligned}
J_1 &= \int_0^{1-x}(1-x-u)^{\gamma-1}N(n, 1-u, t)du \\
&= O(\varepsilon(n))\int_0^{1-x}(1-x-u)^{\gamma-1}du \\
&= O\Big(\frac{\varepsilon(n)}{n^\gamma t^\gamma}\Big)\,,
\end{aligned}
\tag{4.2}
$$

if $x > 1 - 1/nt$.

On the other hand if $x < 1 - 1/nt$, write

$$
\begin{aligned}
J_1 &= \int_0^{1-x} (1-x-u)^{\gamma-1} N(n, 1-u, t)du \\
&= \int_0^{1-x-1/nt} (1-x-u)^{\gamma-1} N(n, 1-u, t)du \\
&\quad + \int_{1-x-1/nt}^{1-x} (1-x-u)^{\gamma-1} N(n, 1-u, t)du \\
&= J_{1,1} + J_{1,2} ,
\end{aligned}
\tag{4.3}
$$

say. Since $\gamma < 1$ and $(1-x-u)^{\gamma-1}$ is an increasing function of u,

$$
\begin{aligned}
J_{1,1} &= \frac{1}{(nt)^{\gamma-1}} \int_\eta^{1-x-1/nt} N(n, 1-u, t)du , \quad \left(0 \leqq \eta \leqq 1 - x - \frac{1}{nt}\right) \\
&= O\left(\frac{\varepsilon(n)}{n^\gamma t^\gamma}\right)
\end{aligned}
\tag{4.4}
$$

by the application of the estimate I_1 of Lemma 3.

And using the estimate (4.1) we have

$$
J_{1,2} = O(\varepsilon(n)) \int_{1-x-1/nt}^{1-x} (1-x-u)^{\gamma-1} du = O\left(\frac{\varepsilon(n)}{n^\gamma t^\gamma}\right) . \tag{4.5}
$$

A combination of the estimates in (4.4) and (4.5), in view of (4.3), yields

$$
J_1 = O\left(\frac{\varepsilon(n)}{n^\gamma t^\gamma}\right) \tag{4.6}
$$

when $x < 1 - 1/nt$. Hence, in view of the estimates in (4.2) and (4.6), the first part of the lemma follows. The second part follows on similar lines.

5. **Proof of the theorem.** In view of the definition and Lemma 1, the absolute Hausdorff summability of the series $\sum \varepsilon(n)A_n(x)$ is equivalent to the absolute convergence of the series $\sum_{n=1}^\infty b_n$, where

$$
b_n = \frac{1}{n} \sum_{\nu=1}^n \nu \binom{n}{\nu} (\Delta^{n-\nu}\mu_\nu)\varepsilon(\nu)A_\nu(x) .
$$

We first consider the case $\alpha = 0$.

Since

$$
\begin{aligned}
A_\nu(x) &= \frac{2}{\pi}\int_0^\pi \phi(t) \cos \nu t \, dt \\
&= -\frac{2}{\pi}\int_0^\pi \frac{\sin \nu t}{\nu} d\phi(t) ,
\end{aligned}
$$

and the transformation (H, μ) is convergence preserving,

$$\begin{aligned} b_n &= -\frac{2}{n\pi}\int_0^\pi d\phi(t)\int_0^1\left(\sum_{\nu=1}^{n}\binom{n}{\nu}\varepsilon(\nu)x^\nu(1-x)^{n-\nu}\sin\nu t\right)d\chi(x) \\ &= -\frac{2}{n\pi}\int_0^\pi d\phi(t)\int_0^1 N(n, x, t)d\chi(x)\ , \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{n=1}^{\infty}|b_n| \leqq \frac{2}{\pi}\int_0^\pi |d\phi(t)|\Bigg[\sum_{n=1}^{[1\backslash t]}\frac{1}{n}\left|\int_0^1 N(n, x, t)d\chi(x)\right| \\ + \sum_{n=[1/t]+1}^{\infty}\frac{1}{n}\left|\int_0^1 N(n, x, t)d\chi(x)\right|\Bigg]\ . \end{aligned}$$

Since

$$\int_0^\pi \varepsilon\left(\frac{k}{t}\right)|d\phi(t)| < C\ ,$$

it is clear that we have to show that uniformly in $0 < t \leqq \pi$,

$$\sum\nolimits_1 \equiv \sum_{n=1}^{[1\backslash t]}\frac{1}{n}\left|\int_0^1 N(n, x, t)d\chi(x)\right| = O\left(\varepsilon\left(\frac{k}{t}\right)\right)\ , \tag{5.1}$$

and

$$\sum\nolimits_2 \equiv \sum_{n=[1/t]+1}^{\infty}\frac{1}{n}\left|\int_0^1 N(n, x, t)d\chi(x)\right| = O\left(\varepsilon\left(\frac{k}{t}\right)\right)\ . \tag{5.2}$$

Clearly

$$\begin{aligned} |N(n, x, t)| &\leqq \sum_{\nu=1}^{n}\binom{n}{\nu}\varepsilon(\nu)x^\nu(1-x)^{n-\nu}\nu t \\ &\leqq nt\varepsilon(n)\sum_{\nu=1}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu} \\ &\leqq nt\varepsilon(n)\ , \end{aligned}$$

and therefore

$$\begin{aligned} \sum\nolimits_1 &= \sum_{n=1}^{[1/t]}\frac{1}{n}\left|\int_0^1 N(n, x, t)d\chi(x)\right| \\ &\leqq \sum_{n=1}^{[1\backslash t]}\frac{nt\varepsilon(n)}{n}\int_0^1 |d\chi(x)| \\ &= O\left(\varepsilon\left(\frac{k}{t}\right)\right)\ , \end{aligned}$$

the function $\chi(x)$ being of bounded variation in $(0, 1)$. This completes the proof of the estimate in (5.1). We now proceed to establish (5.2).

Putting

$$\chi(x) = g^+_{1+\gamma}(x) + C$$

we have

$$\begin{aligned}\textstyle\sum_2 &\equiv \sum_{n=[1/t]+1}^{\infty} \frac{1}{n}\left|\int_0^1 N(n, x, t)d\chi(x)\right| \\ &= \sum_{n=[1/t]+1}^{\infty} \frac{1}{n}\left|\int_0^1 g^+_\gamma(x)N(n, x, t)dx\right| \\ &= \sum_{n=[1/t]+1}^{\infty} \frac{1}{n}\left|\int_0^1 g(x)N^-_\gamma(n, x, t)dx\right| \end{aligned} \tag{5.3}$$

(by the application of Lemma 2)

$$\leqq \int_0^1 |g(x)|\Big(\sum_{n=[1/t]+1}^{\infty} \frac{1}{n}|N^-_\gamma(n, x, t)|\Big)dx ,$$

where $N^-_\gamma(n, x, t)$ means the γth backward fractional integral of $N(n, x, t)$ regarded as a function of x. By the application of Lemma 4 we get

$$\begin{aligned} N^-_\gamma(n, x, t) &= \frac{1}{\Gamma(\gamma)}\int_x^1 (u - x)^{\gamma-1}N(n, u, t)du \\ &= O\Big(\frac{\varepsilon(n)}{n^\gamma t^\gamma}\Big) , \end{aligned} \tag{5.4}$$

and therefore

$$\begin{aligned}\textstyle\sum_2 &= O(1) \sum_{n=[1/t]+1}^{\infty} \frac{\varepsilon(n)}{n^{1+\gamma}t^\gamma}\int_0^1 |g(x)|dx \\ &= O(1) \sum_{n=[1/t]+1}^{\infty} \frac{\varepsilon(n)}{n^{1+\gamma}t^\gamma} \\ &= O\Big(\varepsilon\Big(\frac{k}{t}\Big)\Big) ,\end{aligned}$$

and this completes the proof of (5.2). The proof of the estimate in (5.2) in the case when

$$\chi(x) = g^-_{1+\gamma}(x) + C$$

follows on similar lines.

We now consider the case $0 < \alpha < 1$. We have

$$\begin{aligned} b_n &= \frac{1}{n}\int_0^\pi \phi(t)\frac{d}{dt}\Big(\frac{2}{\pi}\sum_{\nu=1}^{n}\binom{n}{\nu}\varepsilon(\nu)\sin \nu t\int_0^1 x^\nu(1 - x)^{n-\nu}d\chi(x)\Big)dt \\ &= \frac{1}{n}\int_0^\pi \phi(t)\frac{d}{dt}L(\chi; n, t)dt \\ &= \frac{1}{n}\int_0^\pi \frac{d}{dt}L(\chi; n, t)\Big(\frac{1}{\Gamma(1 - \alpha)}\int_0^t (t - u)^{-\alpha}d\Phi_\alpha(u)\Big)dt \\ &= \frac{1}{n\Gamma(1 - \alpha)}\int_0^\pi d\Phi_\alpha(u)\int_u^\pi (t - u)^{-\alpha}\frac{d}{dt}L(\chi; n, t)dt \end{aligned} \tag{5.5}$$

$$= \frac{1}{n}\int_0^{\pi} I(\chi;\, n,\, u)d\Phi_\alpha(u)$$
$$= \frac{1}{n}\Phi_\alpha(\pi)I(\chi;\, n,\, \pi) - \frac{1}{n}\Phi_\alpha(\pi)J(\chi;\, n,\, \pi) + \frac{1}{n}\int_0^{\pi} J(\chi;\, n,\, u)d\phi_\alpha(u)\ ,$$

and therefore in order to prove the theorem we have to show that

$$(5.6) \qquad \sum_{n=1}^{\infty} n^{-1}|I(\chi;\, n,\, \pi)| < C\ ,$$

$$(5.7) \qquad \sum_{n=1}^{\infty} n^{-1}|J(\chi;\, n,\, \pi)| < C$$

and

$$(5.8) \qquad \sum_{n=1}^{\infty} n^{-1}|J(\chi;\, n,\, u)| = O\left(\varepsilon\left(\frac{k}{u}\right)\right)\ ,$$

uniformly in $0 < u \leqq \pi$.

We show that all the above estimates are true with

$$\chi(x) = g_{1+\gamma}^{+}(x) + C\ .$$

The method of proof for $\chi(x) = g_{1+\gamma}^{-}(x) + C$ will be similar.

For sake of brevity we write g^+ for $g_{1+\gamma}^{+}$. We have

$$I(g^+;\, n,\, u) = \frac{1}{\Gamma(1-\alpha)}\int_u^{u+n^{-1}} (t-u)^{-\alpha}\frac{d}{dt}L(g^+;\, n,\, t)dt$$
$$+ \frac{1}{\Gamma(1-\alpha)}\int_{u+n^{-1}}^{\pi} (t-u)^{-\alpha}\frac{d}{dt}L(g^+,\, n,\, t)dt$$
$$= I_1(g^+;\, n,\, u) + I_2(g^+;\, n,\, u)\ ,$$

say. Now

$$I_1(g^+;\, n,\, u) = \frac{2}{\pi\Gamma(1-\alpha)}\int_u^{u+n^{-1}} (t-u)^{-\alpha}dt\int_0^1 g_\gamma^{+}(x)$$
$$\times \left(\sum_{\nu=1}^{n}\nu\binom{n}{\nu}\varepsilon(\nu)x^\nu(1-x)^{n-\nu}\cos \nu t\right)dx$$
$$= \frac{2}{\pi\Gamma(1-\alpha)}\int_u^{u+n^{-1}} (t-u)^{-\alpha}dt\int_0^1 g_\gamma^{+}(x)M(n,\, x,\, t)dx$$
$$= \frac{2}{\pi\Gamma(1-\alpha)}\int_u^{u+n^{-1}} (t-u)^{-\alpha}dt\int_0^1 g(x)M_\gamma^{-}(n,\, x,\, t)dx\ [3]$$
$$= \frac{2}{\pi\Gamma(1-\alpha)\Gamma(\gamma)}\int_u^{u+n^{-1}} (t-u)^{-\alpha}dt\int_0^1 g(x)dx$$

[3] $M_\gamma^{-}(n, x, t)$ is γth backward fractional integral of $M(n, x, t)$ regarded as a function of x.

$$\times \int_0^{1-x} (1 - x - v)^{\gamma-1} M(n, 1 - v, t) dv$$

$$= O\left(\frac{\varepsilon(n)}{n^{\gamma-1}}\right) \int_u^{u+n^{-1}} t^{-\gamma}(t - u)^{-\alpha} dt \int_0^1 |g(x)| dx$$

$$= O\left(\frac{\varepsilon(n)}{n^{\gamma-\alpha} u^{\gamma}}\right),$$

by the application of the estimate of J_2 in Lemma 4 and using the fact that $g(x)$ is Lebesgue integrable in (0, 1).

Also

$$I_2(g^+; n, u) = \frac{1}{\Gamma(1-\alpha)} \int_{u+n^{-1}}^{\pi} (t - u)^{-\alpha} \frac{d}{dt} L(g^+; n, t) dt$$

$$= \frac{n^{\alpha}}{\Gamma(1-\alpha)} \int_{u+n^{-1}}^{\zeta} \frac{d}{dt} L(g^+; n, t) dt, \quad (u + n^{-1} < \zeta < \pi)$$

where

$$L(g^+; n, t) = \frac{2}{\pi} \int_0^1 \left(\sum_{\nu=1}^{n} \binom{n}{\nu} \varepsilon(\nu) x^{\nu} (1 - x)^{n-\nu} \sin \nu t \right) g_{\gamma}^{+}(x) dx$$

$$= \frac{2}{\pi} \int_0^1 g_{\gamma}^{+}(x) N(n, x, t) dx$$

$$= \frac{2}{\pi} \int_0^1 g(x) N_{\gamma}^{-}(n, x, t) dx$$

$$= \frac{2}{\pi \Gamma(\gamma)} \int_0^1 g(x) dx \int_0^{1-x} (1 - x - v)^{\gamma-1} N(n, 1 - v, t) dv$$

$$= O\left(\frac{\varepsilon(n)}{n^{\gamma} t^{\gamma}}\right) \int_0^1 |g(x)| dx$$

$$= O\left(\frac{\varepsilon(n)}{n^{\gamma} t^{\gamma}}\right),$$

by the application of the estimate of J_1 in Lemma 4 and the Lebesgue integrability of $g(x)$ in (0, 1). Hence

$$I_2(g^+; n, u) = O\left(\frac{\varepsilon(n)}{n^{\gamma-\alpha} u^{\gamma}}\right).$$

Thus we have shown that

$$I(g^+; n, u) = O\left(\frac{\varepsilon(n)}{n^{\gamma-\alpha} u^{\gamma}}\right). \tag{5.9}$$

Using this estimate we have

$$\sum_{n=1}^{\infty} n^{-1} |I(g^+; n, \pi)| \leqq C \sum_{n=1}^{\infty} \frac{\varepsilon(n)}{n^{1+\gamma-\alpha}} < C,$$

and this completes the proof of (5.6).

If in particular we suppose that $\phi(t) = 1$ for all t, in which case $\phi_\alpha(t) = 1$ for all t and $b_n = 0$ for every n, we obtain from (5.5) and the estimate in (5.9)

$$O = O\Big(\frac{\varepsilon(n)}{n^{1+\gamma-\alpha}}\Big) - \frac{J(g^+; n, \pi)}{n}$$

and therefore

$$J(g^+; n, \pi) = O\Big(\frac{\varepsilon(n)}{n^{\gamma-\alpha}}\Big) . \tag{5.10}$$

Hence

$$\sum_{n=1}^{\infty} n^{-1}|J(g^+; n, \pi)| \leqq C\sum_{n=1}^{\infty} \frac{\varepsilon(n)}{n^{1+\gamma-\alpha}} < C ,$$

and this establishes the estimate in (5.7). Now it remains to establish (5.8). We note that

$$\begin{aligned} I(\chi; n, u) &= \frac{1}{\Gamma(1-\alpha)}\Big(\int_u^{u+n^{-1}} + \int_{u+n^{-1}}^{\pi}\Big)(t-u)^{-\alpha}\frac{d}{dt}L(\chi; n, t)dt \\ &= O(n\varepsilon(n))\int_u^{u+n^{-1}}(t-u)^{-\alpha}dt\int_0^1\Big(\sum_{\nu=1}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}\Big)|d\chi(x)| \\ &\quad + \frac{2}{\pi}\cdot\frac{n^\alpha}{\Gamma(1-\alpha)}\sum_{\nu=1}^{n}\nu\binom{n}{\nu}\varepsilon(\nu)\int_0^1 x^\nu(1-x)^{n-\nu}d\chi(x) \\ &\quad \times\int_{u+n^{-1}}^{\zeta}\cos\nu t\,dt \qquad\qquad (u+n^{-1} < \zeta < \pi) \\ &= O(n^\alpha\varepsilon(n)) + O(n^\alpha\varepsilon(n))\int_0^1|d\chi(x)|\sum_{\nu=1}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu} \\ &= O(n^\alpha\varepsilon(n)) , \end{aligned}$$

and therefore

$$\begin{aligned} J(\chi; n, u) &= \frac{1}{\Gamma(1+\alpha)}[v^\alpha I(\chi; n, v)]_0^u - \frac{\alpha}{\Gamma(1+\alpha)}\int_0^u v^{\alpha-1}I(\chi; n, v)dv \\ &= O(n^\alpha u^\alpha\varepsilon(n)) + O(n^\alpha\varepsilon(n))\int_0^u v^{\alpha-1}dv \qquad (5.11) \\ &= O(n^\alpha u^\alpha\varepsilon(n)) . \end{aligned}$$

Also

$$\Gamma(\alpha+1)[J(\chi; n, \pi) - J(\chi; n, u)] = [v^\alpha I(\chi; n, v]_u^\pi - \alpha\int_u^\pi v^{\alpha-1}I(\chi; n, v)dv$$

and therefore

$$
\begin{aligned}
J(g^+;\, n,\, u) &= J(g^+;\, n,\, \pi) - \frac{1}{\Gamma(1+\alpha)}[v^\alpha I(g^+;\, n,\, v]_u^\pi \\
&\quad + \frac{\alpha}{\Gamma(\alpha+1)}\int_u^\pi v^{\alpha-1} I(g^+;\, n,\, v)dv \\
&= O\Big(\frac{\varepsilon(n)}{n^{\gamma-\alpha}}\Big) + O\Big(\frac{\varepsilon(n)}{n^{\gamma-\alpha}u^{\gamma-\alpha}}\Big) + O\Big(\frac{\varepsilon(n)}{n^{\gamma-\alpha}}\Big)\int_u^\pi v^{-\gamma+\alpha-1}dv \\
&= O\Big(\frac{\varepsilon(n)}{n^{\gamma-\alpha}u^{\gamma-\alpha}}\Big)
\end{aligned}
\tag{5.12}
$$

using the estimates in (5.9) and (5.10). Now by the application of the estimate in (5.11) and (5.12) we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{-1}[J(g^+;\, n,\, u)| &= \sum_{n=1}^{[1/u]} n^{-1}|J(g^+;\, n,\, u)| + \sum_{n=[1/u]+1}^{\infty} n^{-1}|J(g^+;\, n,\, u)| \\
&= O(u^\alpha)\sum_{n=1}^{[1\backslash u]} n^{\alpha-1}\varepsilon(n) + O(1)\sum_{n=[1/u]+1}^{\infty}\frac{\varepsilon(n)}{n^{1+\gamma-\alpha}u^{\gamma-\alpha}} \\
&= O\Big(\varepsilon\Big(\frac{k}{u}\Big)\Big),
\end{aligned}
$$

uniformly in $0 < u \leqq \pi$. This completes the proof of the estimate in (5.8). Hence the theorem.

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BANARAS HINDU UNIVERSITY