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COMMUTANTS OF SOME HAUSDORFF MATRICES

B. E. RHOADES

Let $B(c)$ denote the Banach algebra of bounded operators over c, the space of convergent sequences. Let Γ and Δ denote the subalgebras of $B(c)$ consisting, respectively, of conservative and conservative triangular infinite matrices, and C the Cesaro matrix of order one. In this paper we investigate Com(C) in Γ and $B(c)$, Com(H) in Γ and $B(c)$ for certain Hausdorff matrices H, and some related questions.

Let $B(c)$ denote the Banach algebra of bounded operators over c, the space of convergent sequences. Let Γ and Δ denote the subalgebras of $B(c)$ consisting, respectively, of conservative and conservative triangular infinite matrices. It is well known (see, e.g. [3, p. 77]) that the commutant of C, the Cesaro matrix of order one, in Δ is the family $\mathscr{H}$ of conservative Hausdorff matrices. The same proof yields the result that if H is any conservative Hausdorff triangle with distinct diagonal elements, then Com$(H) = \mathscr{H}$ in Δ. In this paper we investigate Com(C) in Γ and $B(c)$, Com(H) in Γ and $B(c)$ for certain Hausdorff matrices H, and some related questions.

The spaces of bounded, convergent, and absolutely convergent sequences shall be denoted by m, c, and l. U will denote the unilateral shift, and we shall use $A \leftrightarrow B$ to indicate that the operators A and B commute. An infinite matrix A is said to be triangular if it has only zero entries above the main diagonal, and a triangle if it is triangular and has no zeros on the main diagonal. An infinite matrix A is conservative; i.e., $A\colon c \rightarrow c$ if and only if

$$\| A \| = \sup_n \sum_k | a_{nk} | < \infty , \qquad a_k = \lim_n a_{nk}$$

exists for each k, and $\lim_n \sum_k a_{nk}$ exists.

The proof [2, p. 249] that Com$(C) = \mathscr{H}$ in Δ, uses the associativity of matrix multiplication. If Com(C) is to remain unchanged in the larger algebra Γ, it is necessary that Com(C) contain only triangular matrices. We are thus led to the following result, where e_k denotes the coordinate sequence with a 1 in the kth position and zeros elsewhere.

THEOREM 1. *Let A be a conservative triangle, B an infinite matrix with finite norm, $B \leftrightarrow A$. Then B is triangular if and only if*

$$(1)\qquad t(A - a_{nn}I) = 0$$

and $t \in l$ *imply* t *lies in the span of* $(e_0, e_1, \cdots, e_n)$, $n = 0, 1, 2, \cdots$.

The conditions in (1) are merely a reformulation of the fact that B is triangular. For, if $B \leftrightarrow A$, then we obtain the system

$$(2)\qquad \sum_{j=k}^{\infty} b_{nj}a_{jk} = \sum_{j=0}^{n} a_{nj}b_{jk};\ n, k = 0, 1, 2, \cdots .$$

Define $t^n = \{b_{nk}\}_{k=0}^{\infty}$, $n = 0, 1, 2, \cdots$; i.e., t^n is the n-th row of B. With $n = 0$, (2) can be written in the form $t^0(A - a_{00}I) = 0$. Thus $b_{0k} = 0$ for $k > 0$. By induction, one can then show that $b_{nk} = 0$ for $k > n$, and hence B is triangular.

To prove the converse, suppose (1) fails to hold for all n. Let N be the smallest such n. Then (1) has a nonzero solution outside the span of $(e_0, e_1, \cdots, e_N)$ and B is not triangular.

A matrix A is said to be of type M if it is not a right zero divisor over l: i.e., $tA = 0$ and $t \in l$ imply $t = 0$. Therefore, an equivalent formulation of (1) is that $(U^*)^{n+1}(A - a_{nn}I)U^{n+1}$ be of type M for each $n = 0, 1, 2, \cdots$.

Let $\mathscr{D}$ denote the set of conservative Hausdorff triangles with distinct diagonal entries, $\mathscr{A}$ the algebra of all matrices with finite norm.

COROLLARY 1. *Let* $H \in \mathscr{D}$. *Then* $\mathrm{Com}(H)$ *in* $\Delta = \mathrm{Com}(H)$ *in* $\Gamma = \mathrm{Com}(H)$ *in* $\mathscr{A} = \mathscr{H}$ *if and only if* (1) *is satisfied.*

The last equality follows from the fact that every Hausdorff matrix with finite norm is automatically conservative.

A matrix A is said to be factorable if $a_{nk} = c_n d_k$ for each n and k. Examples of factorable triangular matrices are C, the Hausdorff matrices generated by $\{a/(n + a)\}$ for $a > 0$, and the weighted mean methods (see [2, p. 57]).

THEOREM 2. *If* A *is a factorable triangle and* $B \leftrightarrow A$ *then* B *is triangular.*

Proof. Set $n = k = 0$ in (2) to get

$$(3)\qquad \sum_{j=1}^{\infty} b_{0j}a_{j0} = 0 .$$

From (2) with $n = 0$, $k = 1$, we have

$$a_{00}b_{01} = \sum_{j=1}^{\infty} b_{0j}a_{j1} = \sum_{j=1}^{\infty} b_{0j}c_j d_1 = (d_1/d_0) \sum_{j=1}^{\infty} b_{0j}a_{j0} .$$

Since $a_{00} \neq 0$, $b_{01} = 0$ from (3). By induction one can show that $b_{nk} = 0$ for $k > n$.

COROLLARY 2. Com(C) in Δ = Com(C) *in* Γ = Com(C) *in* $\mathscr{A} = \mathscr{H}$.

Corollary 2 follows immediately from Theorem 2 since C is factorable.

COROLLARY 3. *If $A \in \Delta$, is factorable, and has exactly one zero on the main diagonal, then $B \leftrightarrow A$ implies B is triangular.*

Proof. Let N be such that $a_{NN} = 0$. If $N > 0$, then the proof of Theorem 2 forces $b_{nk} = 0$ for $k > n$, $n < N$. For $k > N$, $n = N$ in (2) we have

$$\sum_{j=k}^{\infty} b_{nj} a_{jk} = \sum_{j=0}^{N} a_{Nj} b_{jk} = a_{NN} b_{Nk} = 0 ,$$

or

$$-b_{Nk} c_k = \sum_{j=k+1}^{\infty} b_{Nj} c_j ,$$

since $d_k \neq 0$ for $k > N$. The above equation leads to $b_{Nk} c_k = 0$ which implies $b_{Nk} = 0$. By induction, $b_{nk} = 0$ for $n > N$, $k > n$.

If a factorable triangular matrix A contains at least two zeros on the main diagonal, then Com(A) in Δ need not equal Com(A) in Γ. This fact is a special case of the following. A necessary condition for any conservative triangle A to satisfy Com(A) in Δ = Com(A) in Γ is that A have distinct diagonal entries. For, suppose there exist integers i, k, $k > i \geqq 0$ such that $a_{ii} = a_{kk}$. Then the matrix $(U^*)^{i+1}(A - a_{ii}I)U^{i+1}$ has a zero on the main diagonal in the $(k-i)$th position and is therefore not of type M.

A necessary condition, therefore, for a conservative Hausdorff matrix H to satisfy Com(H) in Δ = Com(H) in Γ is that H have distinct diagonal entries. The condition, however, is not sufficient. Let $A = H + \lambda K$ where H is the Hausdorff matrix generated by $\mu_n = (n - a)/(-a)(n + 1)$, $a > 0$, K is the compact Hausdorff matrix generated by $\mu_0 = 1$, $\mu_n = 0$, $n > 0$, and λ is any real number satisfying $-(a + 1)/a < \lambda < 0$. We shall show that $B = U^*(A - a_{00}I)\, U$ is not of type M. Thus Com(A) in Γ will contain nontriangular matrices.

Let D by the Hausdorff matrix generated be

$$\nu_n = \frac{\lambda(n - \varepsilon)}{-\varepsilon(n + 1)} , \qquad \text{where } \varepsilon = \lambda/\delta,\ \delta = -\lambda - 1 - 1/a .$$

Since $a_{00} = 1 + \lambda$, a straightforward calculation verifies that D and $A - a_{00}I$ agree, except for terms in the first column. B is obtained by removing the first row and first column from $A - a_{00}I$. Therefore $B = U^*DU$. By Theorem 1 of [4], D is not of type M, and a suitable sequence t is $t_0 = 1$, $t_n = (-1)^n\varepsilon(\varepsilon - 1) \cdots (\varepsilon - n + 1)/n!$ $n > 0$. Therefore B is also not of type M.

For Com(H) in Δ to equal Com(H) in Γ it is not necessary that the Hausdorff matrix H be a triangle. Set $H = \bar{H} - \mu_0 I$, when $\bar{H}$ is any conservative Hausdorff matrix such that Com$(\bar{H})$ in $\Delta =$ Com$(\bar{H})$ in Γ.

We shall now examine Com(C) in $B(c)$.

Let e denote the sequence of all ones. If $T \in B(c)$ then one can define continuous linear functionals χ and χ_i by $\chi(T) = \lim Te - \sum_k \lim (Te_k)$ and $\chi_i(T) = (Te)_i - \sum_k (Te_k)_i$, $i = 1, 2, \cdots$. (See, e.g., [5, p. 241].) It is known [1, p. 8] that any $T \in B(c)$ has the representation

$$Tx = v \lim x + Bx \qquad \text{for each } x \in c, \tag{4}$$

where B is the matrix representation of the restriction of T to c_0 and v is the bounded sequence $v = \{\chi_i(T)\}$.

The second adjoint of T has the matrix representation

$$T^{**} = \begin{pmatrix} \chi(T) & a_1 & a_2 & \cdots \\ \chi_1(T) & b_{11} & b_{12} & \cdots \\ \chi_2(T) & b_{21} & b_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{5}$$

where the a_i's occur in the representation of

$$\lim \circ T \in c^* \quad \text{as} \quad (\lim T)(x) = \lim (Tx) = \chi(T) \lim x + \textstyle\sum_k a_k x_k .$$

See, e.g., [6, p. 357].

For the matrix C, each $\chi_i(C) = 0$, [5, p. 241] and $\chi(C) = 1$, so that

$$C^{**} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{6}$$

Since $C \leftrightarrow T$ if and only if $C^{**} \leftrightarrow T^{**}$, we may use (5) and (6) to obtain $(C^{**}T^{**})_{00} = (T^{**}C^{**})_{00} = \chi(T)$, and, for $n > 0$,

$$(C^{**}T^{**})_{n0} = \frac{1}{n}\sum_{k=1}^{n} \chi_k(T) = \chi_n(T) = (T^{**}C^{**})_{n0} . \tag{7}$$

The system (7) yields $\chi_n(T) = \chi_1(T)$, $n = 1, 2, 3, \cdots$. Thus $v = \chi_1(T)e$. Substituting in (4) with $\chi \in c_0$ we see that c must commute with B. Since B is a matrix and $B \in \mathscr{A}$, we may use Corollary 2 to obtain the following result.

THEOREM 3. *Let* $T \in B(c)$. *Then* $T \leftrightarrow C$ *if and only if* T *has the form* (4) *with* $v = \chi_1(T)e$ *and* $B \in \mathscr{H}$.

Note added in proof. The hypotheses of Theorem 1 can be modified without changing the details of the proof. For example, if A and B are any two bounded operators over l^p, $p > 1$, then the conclusion of Theorem 1 holds. In particular, since $C \in B(l^p)$ for $p > 1$, we get as a corollary that Com(C) in $B(l^p)$ consists only of those Hausdorff matrices that belong to $B(l^p)$. Another description of Com(C) in $B(l^2)$ appears in A. Shields and L. Wallen [Indiana Univ. Math. J., 20 (1971) 777-788].

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INDIANA UNIVERSITY