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A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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This paper proves that there is a (weak) solution u (not necessarily unique) to the generalized Dirichlet problem (with null boundary data) for the equation $Au + pu = h$. Here A is a strongly and uniformly elliptic operator of order $2m$ on a bounded open set $\Omega \subseteqq \boldsymbol{R}^n$. Also A is "normal": roughly, $AA^* = A^*A$. The functions p and h are bounded and continuous, but are allowed to depend on $x (x \in \Omega)$, u, and the generalized derivatives of u up to order m. The values of p are restricted to lie in a closed disk of the complex plane which contains the negative of no weak eigenvalue of A.

In [4], E. Landesman and A. Lazer proved that the boundary value problem

$$Lu + p\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)u = h\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right) \text{ on } D$$
$$u = 0 \text{ on } \partial D$$

has a (not necessarily unique) weak solution u. Here D is any bounded open subset of $\boldsymbol{R}^n$ with boundary ∂D. Here L is any linear, uniformly and strongly elliptic, self-adjoint, second order partial differential operator with only second order terms and with real-valued, bounded measurable coefficients for its corresponding Dirichlet bilinear form. Here p and h are any real-valued, bounded, continuous functions. It is assumed that there exist constants γ_N and γ_{N+1} such that $\alpha_N < \gamma_N \leqq p(z) \leqq \gamma_{N+1} < \alpha_{N+1}$ for every z in $D \times \boldsymbol{R}^{n+1}$ (here α_N and α_{N+1} are the negatives of successive weak eigenvalues of L).

The present paper may perhaps best be viewed as a generalization of [4]. Although other generalizations are made, the main result is that the assumption that L is self-adjoint can be replaced by the assumption that L is "normal": roughly, $LL^* = L^*L$. Two examples at the end of the present paper show in what sense the result is best-possible and show that uniqueness can not be expected.

As in [4], the final existence result is proved using Schauder's theorem. In the solving of a preliminary linear problem, a contraction mapping and the fact that the spectral radius of a normal operator is equal to its norm replace the argument in [4] based on the maximun characterization of the eigenvalues and a comparison result for self-adjoint operators.

2. NOTATION. Let Ω be a bounded open subset of $\boldsymbol{R}^n$. Let

$C_0^\infty(\Omega)$ denote the set of all infinitely differentiable complex-valued functions with compact support in Ω. Let $L_2(\Omega)$ denote the Hilbert space of all complex-valued square-integrable functions on Ω, with inner product $(\ ,\ )$ and norm $\|\ \|$. Let $H^{(m)}(\Omega)$ denote the Hilbert space of all complex-valued functions on Ω whose distribution derivatives (using $C_0^\infty(\Omega)$ test functions) of order 0 through m are in $L_2(\Omega)$. The inner product and norm of this space will be denoted by $(\ ,\ )_m$ and $\|\ \|_m$ respectively. A multi-index is an n-tuple of nonnegative integers. If $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a multi-index, define

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} .$$

Here the indicated derivative is a distribution derivative. It will be used only when u is in $H^{(|\alpha|)}(\Omega)$. Let $H_0^{(m)}(\Omega)$ denote the Hilbert subspace of $H^{(m)}(\Omega)$ obtained by taking the closure of the set $C_0^\infty(\Omega)$ in $H^{(m)}(\Omega)$.

Let A be the *formal* differential operator given by

$$Au = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta} D^\beta u) ,$$

where the complex-valued functions $a_{\alpha\beta}$ are uniformly continuous in Ω for $|\alpha| = |\beta| = m$ and bounded and measurable otherwise. We assume that A is uniformly strongly elliptic and normalized, i.e., that there exists a constant $E_0 > 0$ such that for all vectors $\xi = (\xi_1, \cdots, \xi_n)$ with real entries, and for all x in Ω,

$$\mathrm{Re}\left\{ \sum_{\substack{|\alpha| = m \\ |\beta| = m}} a_{\alpha\beta}(x) \xi_1^{\alpha_1+\beta_1} \xi_2^{\alpha_2+\beta_2} \cdots \xi_n^{\alpha_n+\beta_n} \right\} \geqq E_0 |\xi|^{2m}$$

where Re takes the real part of any complex number and where $|\xi|$ denotes the length of ξ in $\boldsymbol{R}^n$.

For any φ and ψ in $H_0^{(m)}(\Omega)$, define

$$B[\varphi, \psi] = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (D^\alpha \varphi, a_{\alpha\beta} D^\beta \psi) .$$

We say that u is a solution of the generalized Dirichlet problem for $Au = f$ if and only if f is in $L_2(\Omega)$, u is in $H_0^{(m)}(\Omega)$, and

$$B[\varphi, u] = (\varphi, f) \text{ for every } \varphi \text{ in } H_0^{(m)}(\Omega) .$$

We say that λ is a weak eigenvalue for A corresponding to weak eigenfunction u if $u \neq 0$ is a solution of the generalized Dirichlet problem for $Au = \lambda u$.

With the assumptions on A made above, Garding's inequality holds (see S. Agmon [1], p. 102):

$$\text{(1)} \qquad \operatorname{Re} B[\phi, \phi] + \lambda_0(\phi, \phi) \geqq c_0 \|\phi\|_m^2 .$$

Here λ_0 and c_0 are real constants with $c_0 > 0$. The inequality holds for each ϕ in $C_0^\infty(\Omega)$ and hence (taking limits in $H^{(m)}(\Omega)$) for each ϕ in $H_0^{(m)}(\Omega)$. For each u in $H_0^{(m)}(\Omega)$, define

$$\|u\|_B = [\operatorname{Re} B(u, u) + \lambda_0(u, u)]^{1/2} .$$

An easy calculation shows that $\| \ \|_B$ is bounded above by a multiple of the $\| \ \|_m$ norm. Since Garding's inequality shows that it is also bounded below, these two norms on $H_0^{(m)}(\Omega)$ are equivalent.

We are assured by [1; p. 102] that the generalized Dirichlet problem for $Au = f - \lambda_0 u$ has for each f in $L_2(\Omega)$ a unique solution $T_0 f$ in $H_0^{(m)}(\Omega)$. The mapping $T_0: L_2(\Omega) \to H_0^{(m)}(\Omega)$ is linear and continuous.

Let $\mathscr{I}: H_0^{(m)}(\Omega) \to L_2(\Omega)$ denote the inclusion map and let $I: L_2(\Omega) \to L_2(\Omega)$ denote the identity map.

3. Preliminary lemmas.

Lemma 1, of interest in itself, greatly simplifies the proof of Theorem 2. Lemma 2 gives an elementary proof of the fact that the operator norm of a normal operator is equal to its spectral radius. Lemma 3 gives conditions under which a differential operator is "normal" in the sense required by this paper. Lemma 4 introduces an operator T and Lemma 5 finds an upper bound for $\|\mathscr{I} T\|$. These last two lemmas will be used immediately in Theorem 1.

LEMMA 1. *T_0 is compact as a map from $L_2(\Omega)$ to $H_0^{(m)}(\Omega)$.*

Proof. Let $\{f_k\}$ be a sequence in $L_2(\Omega)$ with $\|f_k\| \leqq r$. Since Ω is bounded, N. Dunford and J. Schwartz [3; p. 1693] assure us that $\mathscr{I}$ is compact. There is therefore a subsequence $\{g_l\}$ of $\{f_k\}$ such that $\{\mathscr{I} T_0 g_l\}$ converges in $L_2(\Omega)$. Use $f = g_l - g_k$ and $\phi = T_0 g_l - T_0 g_k$ and the definition of T_0 to obtain

$$\begin{aligned} \|T_0 g_l - T_0 g_k\|_B^2 &= \operatorname{Re} B[\phi, T_0 f] + \lambda_0(\phi, \phi) \\ &\leqq |B[\phi, T_0 f] + \lambda_0(\phi, \phi)| \\ &= |(\phi, f) - \lambda_0(\phi, T_0 f) + \lambda_0(\phi, \phi)| \\ &= |(\phi, f)| \leqq \|f\| \, \|\phi\| \\ &\leqq 2r \|T_0 g_l - T_0 g_k\| . \end{aligned}$$

Since $\{T_0 g_l\}$ is a Cauchy sequence in $L_2(\Omega)$, $\{T_0 g_l\}$ is a Cauchy sequency in $H_0^{(m)}(\Omega)$ with the $\| \|_B$ norm. Therefore it is Cauchy under the $\| \|_m$ norm.[1]

LEMMA 2. *If N is a normal operator in a Hilbert space with inner product $(,)$ and norm $\| \|$, then $\|N\|$, the operator norm of N, is equal to its spectral radius.*

Proof. For any x in the Hilbert space, $(N^2x, N^2x) = (N^*Nx, N^*Nx)$ and thus $\|N^2\| = \|N^*N\|$. But for any operator in a Hilbert space, $\|N^*N\| = \|N\|^2$ (see [3], p. 874). Thus $\|N^2\| = \|N\|^2$. By induction $\|N^p\| = \|N\|^p$ whenever p is a power of 2. The spectral radius of N is given by the expression

$$\lim_{p\to\infty} \|N^p\|^{1/p} \qquad \text{(see [3], p. 864)} .$$

Considering the subsequence involving only those p which are powers of 2, the result follows.[2]

LEMMA 3. *Let A be a differential operator with coefficients having enough continuous derivatives so that A^*, AA^*, and A^*A make sense classically on $C_0^\infty(\Omega)$. Suppose that $AA^* = A^*A$. Then $\mathscr{I}T_0$ is a normal operator.*

Proof. The discussion in [1; pp. 97–103] shows that the generalized Dirichlet problem for $A^*u = f - \lambda_0 u$ has for every f in $L_2(\Omega)$ a unique solution T_0^*f in $H_0^{(m)}(\Omega)$, where λ_0 is the same constant as was used to define T_0. For φ and ψ in $C_0^\infty(\Omega)$ the Dirichlet form for A is given by $B[\varphi, \psi] = B_A[\varphi, \psi] = (\varphi, A\psi)$. Similarly $B_{A^*}[\varphi, \psi] = (\varphi, A^*\psi)$. It follows easily that $\mathscr{I}T_0^*$ is the adjoint of $\mathscr{I}T_0$.

The Dirichlet form for $(A + \lambda_0)^*(A + \lambda_0)$ is given by

$$B_{(A+\lambda_0)^*(A+\lambda_0)}[\varphi, \psi] = (\varphi, (A + \lambda_0)^*(A + \lambda_0)\psi) = ([A + \lambda_0]\varphi, [A + \lambda_0]\psi) .$$

An easy calculation shows that the Dirichlet form for $(A + \lambda_0)(A + \lambda_0)^*$ is the same since $AA^* = A^*A$. If u is a solution of the generalized Dirichlet problem for $(A + \lambda_0)^*(A + \lambda_0)u = 0$, then

$$([A + \lambda_0]u, [A + \lambda_0]u) = 0 ,$$

so $(A + \lambda_0)u = 0$ and hence finally $u = 0$. By the Fredholm alternative the generalized Dirichlet problem for $(A + \lambda_0)^*(A + \lambda_0)u = f$ has a unique solution u in $H_0^{(2m)}(\Omega)$. It is easy to see that $\mathscr{I}T_0^*\mathscr{I}T_0 f = u = \mathscr{I}T_0\mathscr{I}T_0^*f$. Thus $\mathscr{I}T_0^*\mathscr{I}T_0 = \mathscr{I}T_0\mathscr{I}T_0^*$.

[1] The proof of this lemma is motivated by a similar calculation in [4; pp. 321, 322].

[2] The author wishes to thank Dr. S. Ebenstein for his elementary proof of Lemma 2.

LEMMA 4. *If γ_0 is a complex number such that $-\gamma_0$ is not a weak eigenvalue of A, then we may set $T = T_0[(\gamma_0 - \lambda_0)\mathscr{J}T_0 + I]^{-1}$ and have for every f in $L_2(\Omega)$ and every φ in $H_0^{(m)}(\Omega)$ that*

$$B[\varphi, Tf] + \overline{\gamma_0}(\varphi, Tf) = (\varphi, f) .$$

(Thus Tf is the unique weak solution of $Au + \gamma_0 u = f$.)

Proof. Since $-\gamma_0$ is not a weak eigenvalue of A, $(\lambda_0 - \gamma_0)^{-1}$ is not an eigenvalue of $\mathscr{J}T_0$. Since $\mathscr{J}T_0$ is compact, every nonzero complex number in its spectrum must be an eigenvalue. Therefore $(\lambda_0 - \gamma_0)^{-1}$ is not in the spectrum of $\mathscr{J}T_0$, so $[\mathscr{J}T_0 - (\lambda_0 - \gamma_0)^{-1}I]^{-1}$ (and hence $[(\gamma_0 - \lambda_0)\mathscr{J}T_0 + I]^{-1}$) exists and is continuous.

$$\begin{aligned} B[\varphi, Tf] &+ \bar{\gamma}_0(\varphi, Tf) \\ &= -\lambda_0(\varphi, T_0[(\gamma_0 - \lambda_0)\mathscr{J}T_0 + I]^{-1}f) + (\varphi, [(\gamma_0 - \lambda_0)\mathscr{J}T_0 + I]^{-1}f) \\ &\quad + \bar{\gamma}_0(\varphi, T_0[(\gamma_0 - \lambda_0)\mathscr{J}T_0 + I]^{-1}f) \\ &= (\varphi, [(\gamma_0 - \lambda_0)\mathscr{J}T_0 + I][(\gamma_0 - \lambda_0)\mathscr{J}T_0 + I]^{-1}f) = (\varphi, f) . \end{aligned}$$

LEMMA 5. *Assume that $\mathscr{J}T_0$ is a normal operator and that $|z - \gamma_0| \leqq c$ is a disk in the complex plane which contains the negative of no weak eigenvalue of A. Then $\|\mathscr{J}T\|c < 1$, where T is the map of the above lemma.*

Proof. Since $\mathscr{J}T_0$ is a normal operator, so is $[(\gamma_0 - \lambda_0)\mathscr{J}T_0 + I]^{-1}$. Since $\mathscr{J}T_0$ and this operator commute,

$$\mathscr{J}T = \mathscr{J}T_0[(\gamma_0 - \lambda_0)\mathscr{J}T_0 + I]^{-1}$$

is normal. Therefore $\|\mathscr{J}T\|$ is the same as the spectral radius of $\mathscr{J}T$. Since $\mathscr{J}T$ is compact, the spectral radius is the supremum of the norms of the eigenvalues of $\mathscr{J}T$. But λ is a weak eigenvalue of A if and only if $(\lambda + \gamma_0)^{-1}$ is an eigenvalue of $\mathscr{J}T$. Thus the weak eigenvalues of A have no accumulation point in the (finite) complex plane. Since $|-\lambda - \gamma_0| \geqq c + \varepsilon$ for some $\varepsilon > 0$ and every weak eigenvalue λ of A, $|(\lambda + \gamma_0)^{-1}| \leqq (c + \varepsilon)^{-1}$ so that every eigenvalue of $\mathscr{J}T$ has norm $\leqq (c + \varepsilon)^{-1}$. Thus $\|\mathscr{J}T\|c < 1$ as claimed.

4. The preliminary linear problem.

THEOREM 1. *Let D be a closed disk $\{z \in C; |z - \gamma_0| \leqq c\}$ in the complex plane which contains the negative of no weak eigenvalue of A. Let h be in $L_2(\Omega)$ and let p be a measurable function on Ω whose values lie in the disk D. Suppose that the operator $\mathscr{J}T_0$ associated with A is normal. Then the generalized Dirichlet problem* *for $Au + pu = h$ has a unique solution u in $H_0^{(m)}(\Omega)$. Moreover, there exists a constant M independent of p such that*

$$\operatorname{Re} B[u, u] + \lambda_0(u, u) \leqq M(h, h) .$$

Proof. We want $Au + pu = h$, or equivalently $Au + \gamma_0 u = h - (p - \gamma_0)u$. Thus we want $u = T(h - (p - \gamma_0)u)$, where T is the map of Lemmas 4 and 5. We prove that the map from $L_2(\Omega)$ into itself given by $u \to \mathscr{I} T[h - (p - \gamma_0)u]$ is a contraction map.

For any u_1 and u_2 in $L_2(\Omega)$,

$$\begin{aligned} &\mathscr{I} T[h - (p - \gamma_0)u_1] - \mathscr{I} T[h - (p - \gamma_0)u_2] \,|| \\ &\quad = ||\mathscr{I} T(p - \gamma_0)(u_1 - u_2)|| \leqq ||\mathscr{I} T|| c \, ||u_1 - u_2|| \,. \end{aligned}$$

Since $||\mathscr{I} T||c < 1$ by Lemma 5, the map is a contraction as claimed. Thus there exists a unique v in $L_2(\Omega)$ such that $v = \mathscr{I} T[h - (p - \gamma_0)v]$.

Let $Q = ||\mathscr{I} T||(1 - ||\mathscr{I} T||c)^{-1}$. Then $Q = ||\mathscr{I} T|| + ||\mathscr{I} T||cQ$. Since $||u|| \leqq Q||h||$ implies that

$$\begin{aligned} ||\mathscr{I} T[h - (p - \gamma_0)u]|| &\leqq ||\mathscr{I} T|| \, ||h|| + c||\mathscr{I} T|| \, ||u|| \\ &\leqq ||\mathscr{I} T|| \, ||h|| + c||\mathscr{I} T||Q||h|| \\ &= Q||h|| \,, \end{aligned}$$

it follows that for fixed h the ball $\{u \in L_2(\Omega); ||u|| \leqq Q||h||\}$ is mapped into itself by our contraction map. Therefore the fixed point v satisfies $||v|| \leqq Q||h||$. Since the $|| \;||_m$ norm and the $|| \;||_B$ norm are equivalent, and since

$$||v||_m = ||T[h - (p - \gamma_0)v]||_m \leqq ||T|| \, ||h - (p - \gamma_0)v|| \,,$$

(here $||T||$ is the operator norm of T: $L_2(\Omega) \to H_0^{(m)}(\Omega)$) it follows easily that there exists an M such that $||v||_B^2 \leqq M||h||^2$.

5. The nonlinear problem.

THEOREM 2. *Let D be a closed disk in the complex plane which contains the negative of no weak eigenvalue of A. Let $h(x, u, \partial u/\partial x_1, \cdots)$ and $p(x, u, \partial u/\partial x_1, \cdots)$ be continuous functions of their arguments, allowed to involve derivatives of u up to order m. Let $|h(x, u, \cdots)| \leqq r$ and assume that the values of p are always in the disk D. Assume that the operator $\mathscr{I} T_0$ associated with A is normal. Then the generalized Dirichlet problem for*

$$Au + p\Big(x, u, \frac{\partial u}{\partial x_1}, \cdots\Big)u = h\Big(x, u, \frac{\partial u}{\partial x_1}, \cdots\Big) \tag{3}$$

has a (not necessarily unique) solution u in $H_0^{(m)}(\Omega)$.

Proof. Define a map $G: H_0^{(m)}(\Omega) \to H_0^{(m)}(\Omega)$ as follows: for every u in $H_0^{(m)}(\Omega)$, let $G(u)$ be the unique solution v in $H_0^{(m)}(\Omega)$ of

$$v = \mathscr{I}T\left[h\left(x, u, \frac{\partial u}{\partial x_1}, \cdots\right) - \left(p\left(x, u, \frac{\partial u}{\partial x_1}, \cdots\right) - \gamma_0\right)v\right],$$

where γ_0 is the center of the disk D and T is the operator of Lemmas 4 and 5. It is clear that a fixed point of G would furnish a solution for the generalized Dirichlet problem for (3). We will show that G is continuous and compact from a bounded, closed, convex subset S of $H_0^{(m)}(\Omega)$ into itself. Schauder's theorem (see, for example, J. Cronin [2], p. 131) then assures us a fixed point.

Since $|h(x, u, \cdots)| \leqq r$, $(h, h) \leqq R = r^2 \operatorname{meas}(\Omega) < \infty$. Using the constant M of Theorem 1, $\|G(u)\|_B^2 \leqq MR$ for all u in $H_0^{(m)}(\Omega)$. Thus if we take $S = \{u \in H_0^{(m)}(\Omega); \|u\|_B^2 \leqq MR\}$, S is a bounded, closed, convex set of $H_0^{(m)}(\Omega)$ and $G(S) \subseteqq S$.

Now we show that G is continuous. Let $\{u_k\}$ be a sequence in $H_0^{(m)}(\Omega)$ converging to u. The sequence $\{h(x, u_k, \cdots) - (p(x, u_k, \cdots) - \gamma_0)G(u_k)\}$ is clearly bounded in $L_2(\Omega)$, so since T is compact (Lemma 1 shows that T_0 is compact, and T is T_0 composed with a continuous map) there is a subsequence of $\{G(u_k)\}$ which converges in $H_0^{(m)}(\Omega)$ to a limit v. Then taking limits with the corresponding subsequence of $\{u_k\}$,

$$v = \mathscr{I}T[h(x, u, \cdots) - (p(x, u, \cdots) - \gamma_0)v],$$

so that $v = G(u)$. Since any subsequence of $\{G(u_k)\}$ has a subsequence converging in $H_0^{(m)}(\Omega)$ to $G(u)$, $\{G(u_k)\}$ itself converges in $H_0^{(m)}(\Omega)$ to $G(u)$, proving continuity.

Now we show that G is compact. Let $\{u_k\}$ be a bounded sequence *in* $H_0^{(m)}(\Omega)$. *Then the sequence* $\{h(x, u_k, \cdots) - (p(x, u_k, \cdots) - \gamma_0)G(u_k)\}$ is bounded in $L_2(\Omega)$, so the fact that T is compact assures us a subsequence of $\{G(u_k)\}$ which converges in $H_0^{(m)}(\Omega)$.

6. Examples and a remark.

EXAMPLE 1. If the disk D includes the negative of a weak eigenvalue λ of A, let v be a weak eigenfunction of A^* corresponding to the weak eigenvalue $\bar{\lambda}$. If $h(x)$ is any bounded continuous function on Ω such that $(h, v) \neq 0$, then the generalized Dirichlet problem for $Au + \lambda u = h$ has no solution, since the Fredholm alternative applies [1, p. 102]. It is in this sense that Theorem 2 is best possible.

EXAMPLE 2. Suppose that there is a weak eigenvalue λ of A which corresponds to a continuous weak eigenfunction v with $|v(x)| \leqq 1$ for every x in Ω. Let γ_0 be the center of the disk D and let $p = \gamma_0$

identically. Let $h = h(u)$ be a bounded C^∞ function of u with $h(u) = \gamma_0 u + \lambda u$ for $|u| \leqq 1$. Then v and $v/2$ are two distinct solutions of the generalized Dirichlet problem for $Au + pu = h$. This shows that we cannot expect a unique solution to problems of the type discussed in this paper.

REMARK. Consider the generalized Dirichlet problem for $Au = f(x, u, \partial u/\partial x_1, \cdots)$, where f is a continuous function of its arguments, involving derivatives of u up to order m. Under what circumstances can we write $f = -pu + h$, where $|h| \leqq r$ and the values of p lie in a closed disk D with center γ_0 and radius c? Clearly $|f + \gamma_0 u| \leqq c|u| + r$ is a necessary condition. It is interesting to note that this condition is also sufficient. To see this, given an f satisfying this growth condition, define p to be the closest point in D to $-f/u$ for any values of the arguments with $|u| \geqq 1$. Then extend p so as to be defined also for $|u| < 1$, so as to be continuous overall, and so as to have each of its values in D. Then set $h = f + pu$. (For $|u| \geqq 1$ we have $|h| \leqq r$, but for $|u| < 1$, although h as given in the above construction is bounded, we are not assured that $|h| \leqq r$.)

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