A TOPOLOGICAL LEMMA AND APPLICATIONS TO REAL FUNCTIONS

CLIFFORD E. WEIL

In working with functions of Baire class one having the Darboux property, one of the most useful tools has been a theorem due to Baire that says a function of Baire class one has a point of continuity on every closed set relative to the closed set. The lemma mentioned in the title can be used in many instances more efficiently than Baire's theorem as is shown in § 4. It is concerned with sets rather than functions and hence more basic than Baire's Theorem, and easier to prove requiring only one application of Baire's category theorem.

1. **Definitions and conventions.** Euclidean n-space will be denoted by E_n and K will designate a fixed, closed subset of E_n. For each x in K and r positive, $B(x, r) = \{y \in K : |x - y| < r\}$; that is, the open ball of radius r about x. For any subset H of K, $\operatorname{cl} H$ will be its closure and $\operatorname{int} H$ will denote its interior relative to K. Finally $m(E)$ will be the Lebesgue measure of E.

DEFINITION 1. A G_δ subset H of K will be called a ball closed G_δ set if whenever $B(x, r) \subset H$, $\{y \in K : |x - y| = r\} \subset H$.

Clearly every closed set is a ball closed G_δ set. However, not every ball closed G_δ set is closed. For example, in $[0, 1]$ the set

$$H = \bigcup_{n=2}^{\infty} [n^{-1}, n^{-1} + 2^{-n}]$$

is a ball closed G_δ set that is not closed because it does not contain 0. Not every G_δ set is a ball closed G_δ set. In particular it is fairly easy to prove (using one of the techniques used in the proof of the lemma) that the only nonempty open subset of K that is a ball closed G_δ set is K itself. This fact will be needed.

DEFINITION 2. A function $f : E_n \to E_k$ has the Denjoy property if for each open subset G of E_k either $f^{-1}(G) = \varnothing$ or $m(f^{-1}(G)) > 0$.

2. **The lemma.**

LEMMA. *If H_1 and H_2 are two nonempty disjoint, ball closed G_δ subsets of K, then there is an $x \in K - (H_1 \cup H_2)$, and an open neighborhood O of x (open relative to K) such that $O \cap H_1 = \varnothing$ or $O \cap H_2 = \varnothing$.*

Proof. Suppose, on the contrary, that every open neighborhood of every point $x \in K - (H_1 \cup H_2)$ intersects both H_1 and H_2. The first step of the proof is establishing the following fact. Every open set intersecting $H_1 - \text{int } H_1$ also intersects $H_2 - \text{int } H_2$, and conversely, every open set intersecting $H_2 - \text{int } H_2$ also intersects $H_1 - \text{int } H_1$. It will suffice to prove the first part; the proof of the second being similar.

Let $x \in H_1 - \text{int } H_1$, U an open set with $x \in U$ and $r > 0$ so that $B(x, r) \subset U$. Since $x \notin \text{int } H_1$, there is a $y \in K - H_1$ with $|x - y| < r/2$. Either $y \in H_2$ or $y \in K - H_2$ in which case $y \in K - (H_1 \cup H_2)$; so since $B(x, r/2)$ is an open set containing y, by assumption $B(x, r/2)$ contains a point of H_2. In either case there is a point $z \in B(x, r/2) \cap H_2$. If $z \notin \text{int } H_2$ it is the desired point. If not to find the desired point let

$$s = \sup \{t: B(z, t) \subset H_2\} \, .$$

Since $x \in H_1$ and $H_1 \cap H_2 = \varnothing$, $s \leqq r/2$ (actually $s < r/2$, but that fact is not needed). A simple compactness argument then shows that there is a point w with $|z - w| = s$ such that $w \in K - \text{int } H_2$. But since H_2 is a ball closed G_δ set, $w \in H_2$. Moreover,

$$|x - w| \leqq |x - z| + |z - w| < r/2 + s \leqq r \, .$$

Hence

$$w \in U \cap (H_2 - \text{int } H_2) \, .$$

Let

$$F = \text{cl} \left[(H_1 - \text{int } H_1) \cup (H_2 - \text{int } H_2) \right] \, .$$

The fact just verified says that both H_1 and H_2 are dense in F. Now H_1 and H_2 are G_δ sets; so write

$$H_1 = \bigcap_{n=1}^{\infty} U_n$$

and

$$H_2 = \bigcap_{n=1}^{\infty} V_n$$

where U_n and V_n are open sets (relative to K) for each $n = 1, 2, \cdots$. Then for each $n = 1, 2, \cdots$, $U_n \cap F$ and $V_n \cap F$ are open subsets of F and each contains a dense subset of F. The intersection of two such sets is a dense set; so $U_n \cap V_n \cap F$ is an open, dense subset of F. So by the Baire category theorem,

$$\bigcap_{n=1}^{\infty} (U_n \cap V_n \cap F)$$

is a dense subset of F. But

$$\bigcap_{n=1}^{\infty} (U_n \cap V_n \cap F) = \left(\bigcap_{n=1}^{\infty} U_n\right) \cap \left(\bigcap_{n=1}^{\infty} V_n\right) \cap F$$
$$= H_1 \cap H_2 \cap F = \varnothing \ .$$

So $F = \varnothing$; that is $H_1 - \text{int } H_1 = \varnothing$ and $H_2 - \text{int } H_2 = \varnothing$. That is, H_1 and H_2 are both open, ball closed G_δ sets which are nonempty. By the remark following the definition of ball closed G_δ sets, $H_1 = K = H_2$ contrary to the assumption that they are disjoint.

One immediate consequence of the lemma that will be used in the next section and in §6 is that a closed subset of K cannot be decomposed into two, nonempty disjoint, ball closed G_δ sets.

3. **Applications to Baire one, Darboux functions.** A function $f\colon E_1 \to E_1$ of Baire class one having the Darboux property has a connected graph (see [1] Theorem 4.1). An easy proof of this fact can be given by first observing that if G is an open subset of E_2, then $\{x\colon (x, f(x)) \in G\}$ is an F_σ set. If the graph of f is disconnected by two open sets O_1 and O_2, then the two sets $E_i = \{x\colon (x, f(x)) \in O_i\}$, $i = 1, 2$, are disjoint F_σ subsets of E_1 whose union is E_1. Hence they are also G_δ sets. That f has the Darboux property implies that E_1 and E_2 are ball closed G_δ sets. By the lemma either $E_1 = \varnothing$ or $E_2 = \varnothing$. A similar argument can be used to give a short proof of Zahorski's classification of Baire one, Darboux functions (see [1] or [10]).

In [6] (also [1] Theorem 6.1) it is shown that a function $f\colon E_1 \to E_1$ is of Baire class one and has the Darboux property if and only if for each real number a, the sets

$$E_a = \{x\colon f(x) \geqq a\}$$

and

$$E^a = \{x\colon f(x) \leqq a\}$$

are ball closed G_δ subsets of E_1. The "if" part follows at once from the remark after the lemma, for if on a closed interval I, f attains values larger than a and smaller than a, but does not attain the value a, then E_a and E^a would be a decomposition of I into two nonempty disjoint ball closed G_δ sets.

The following corollary was certainly attainable using Baire's Theorem, but seems to have escaped detection. It will be the crux of many of the remaining applications.

COROLLARY. *If $f\colon E_1 \to E_1$ is a function of Baire class one*

having the Darboux property, then for each open interval (a, b) with $f^{-1}(a, b) \neq \varnothing$, there is an interval I such that $f^{-1}(a, b) \cap I \neq \varnothing$ and on I, f is strictly larger than a or strictly smaller than b.

Proof. By the above result of Neugebauer's the two sets E_b and E^a are ball closed G_δ sets, and it may be assumed that they are nonempty for otherwise the conclusion is immediate. The lemma then says that there is an interval I and an $x \in I$ such that $x \notin E_b \cup E^a$ (that is $x \in f^{-1}(a, b)$) and $I \cap E_b = \varnothing$ or $I \cap E^a = \varnothing$ which is the desired result.

4. **Applications to derivatives.** The proceeding corollary gives an easy proof of the following theorem of Denjoy's (see [2]).

THEOREM. *If $f: E_1 \to E_1$ is everywhere differentiable, then f' has the Denjoy property.*

Proof. The function f' is of Baire class one and has the Darboux property; so if $f'^{-1}(a, b) \neq \varnothing$ then by the corollary there is an interval I such that $f'^{-1}(a, b) \cap I \neq \varnothing$ and on I, f' is strictly larger than a or strictly smaller than b.

Thus the theorem will follow by proving the special case that if $f'(x) > 0$ for all $x \in I$ and $f'(x) \geqq 1$ for a.e. $x \in I$, then $f'(x) \geqq 1$ for every $x \in I$. This can be established in two ways. Using the Lebesgue theory it follows that f' is integrable and for every

$$x < y, \, x, \, y \in I, \, f(y) - f(x) \geqq \int_x^y f'(t)dt \geqq y - x \, ;$$

that is $(f(y) - f(x))/(y - x) \geqq 1$. Letting $y \to x$ yields $f'(x) \geqq 1$ for every $x \in I$. Using more primitive methods a standard nested interval argument will prove that if the lower Dini derivate of a function is $\geqq 1$ for each $x \in I$, then each difference quotient of the function is $\geqq 1$. Next let $E = \{x \in I : f'(x) < 1\}$ and for each $n = 1, 2, \cdots$, let G_n be an open set, $G_n \supset E$, with $m(G_n) < 1/n$. Define

$$f_n(x) = m((-\infty, x) \cap G_n) + f(x) \, .$$

It is easy to show that the lower Dini derivate of f_n is $\geqq 1$ for each x in I and hence that for each $x, y \in I$, $x \neq y$,

$$(f_n(y) - f_n(x))/(y - x) \geqq 1 \, .$$

But clearly $\{f_n\}$ converges pointwise to f. Thus

$$(f(y) - f(x))/(y - x) \geqq 1 \, .$$

And as before it follows that $f'(x) \geqq 1$ for each $x \in I$.

The Denjoy property is possessed by L_p-derivatives, approximate derivatives and kth Peano derivatives. For definitions and proofs [3] and [7]. The properties that all of these generalized derivatives have which yield an immediate proof of the result is that they are Baire one functions, have the Darboux property, and if bounded above or below on an interval they are ordinary derivatives on that interval (see [3], [4], and [7]). So by the corollary if one of the generalized derivatives inverses an open interval to a nonempty set E, then there is an $x \in E$ and an interval I containing x such that the generalized derivative is bounded above or below on I and hence is an ordinary derivative on I. So by Denjoy's theorem

$$m(E \cap I) > 0 .$$

Actually a property slightly stronger than the Denjoy property was proved here. Even though the set where the derivative is not an ordinary derivative may have positive measure, it does not contain the inverse image of any open interval. It was shown that if the inverse image of an open interval is not empty, then it meets an interval on which the generalized derivative is an ordinary derivative. (This fact was first proved for approximate derivatives by Muk-hopadhyay. He gave a standard proof using Baire's theorem). In particular, the generalized derivative, when restricted to the set where it is an ordinary derivative, has the Denjoy property. For standard proofs of this fact see [3] and [9].

5. **An application to functions of several variables.** In [5] Mišik introduced a Darboux property for functions of several variables.

DEFINITION 3. A function $f: E_n \to E_1$ has the Darboux property if whenever it attains a value larger than a and a value smaller than a on a closed ball $\{y \in E_n : |x - y| \leqq r\}$, then it attains a on the open ball $\{y \in E_n : |x - y| < r\}$.

The lemma will now be used to prove Mišik's theorem concerning this Darboux property.

THEOREM. *A function* $f: E_n \to E_1$ *of Baire class one has the Darboux property if and only if for each real number a, the two sets*

$$E_a = \{x : f(x) \geqq a\}$$

and

$$E^a = \{x: f(x) \le a\}$$

are ball closed G_δ subsets of E_n.

Proof. The "only if" part is straightforward and does not need the lemma.

For the "if" part, let B be a closed ball on which f attains a value larger than a and one smaller than a. Then f must attain a value larger than a on the interior of B, for if not the interior of B would be part of E^a and then by hypothesis so would B contrary to assumptions on B. Likewise f attains values larger than a on B. Let K be a closed ball contained in the *interior* of B, on which f attains values larger than a and smaller. Then $K \cap E_a$ and $K \cap E^a$ are nonempty ball closed G_δ subsets of K and their union is K. But a closed subset of E_n cannot be decomposed into nonempty, disjoint, ball closed G_δ subsets. Thus $(K \cap E_a) \cap (K \cap E^a) \ne \varnothing$. If x is in this intersection, then x is interior to B because K is and $f(x) = a$ because $x \in E_a \cap E^a$.

6. **The Darboux property and partial derivatives.** This work is concluded by an example, a theorem and some remarks concerning how Mišik's Darboux property relates to partial derivatives. It is shown that a partial derivative even of a continuous function need not have the property, but if the function is differentiable then the partial derivatives, (in fact, all directional derivatives) do satisfy the condition.

DEFINITION 4. If $\nu \in E_n$, $|\nu| = 1$, $f: E_n \to E_1$, $x \in E_n$, then the directional derivative of f at x in the direction ν is denoted by $\partial_\nu f(x)$ and defined by

$$\partial_\nu f(x) = \lim_{h \to 0} (f(x + h\nu) - f(x))/h \ .$$

(In E_2 this notation is a bit clumsy; so there $\partial_x f$ and $\partial_y f$ are used to denote the usual partial derivatives.)

DEFINITION 5. Let $f: E_n \to E_1$, let $\nu^1, \cdots, \nu^n$ denote the usual basis of unit vectors in E_n, and let $x \in E_n$. Then the gradient of f at x is denoted by grad $f(x)$ and is the vector in E_n whose kth coordinate is $\partial_{\nu^k} f(x)$.

DEFINITION 6. Let $F: E_n \to E_1$ and let $x \in E_n$. Then f is said to be differentiable at x if $f(y) = f(x) + (y - x) \cdot \operatorname{grad} f(x) + o(|y - x|)$ as $y \to x$.

EXAMPLE. Let h be a differentiable function of one variable that is 0 on $(-\infty, -2]$, constant on $[2, \infty)$ and has derivative 1 on $[-1, 1]$. Define $f: E_2 \to E_1$ by

$$f(x, y) = \begin{cases} \sqrt{y}\, h(x/\sqrt{y}) & \text{if } y > 0 \\ 0 & \text{if } y \leqq 0 \,. \end{cases}$$

Since h is bounded, f is continuous at each point $(x, 0)$. The continuity elsewhere is immediate. It is easily shown that

$$\partial_x f(x, y) = \begin{cases} h'(x/\sqrt{y}) & \text{if } y > 0 \\ 0 & \text{if } y \leqq 0 \,. \end{cases}$$

In particular $\partial_x f(x, y) = 1$ if $x^2 \leqq y$ and $y > 0$, but $\partial_x(0, 0) = 0$. It follows then that $E = \{(x, y): \partial_x f(x, y) \geqq 1\}$ contains the open ball $B((0, r), r)$ for any $0 < r < 1/2$, but $(0, 0)$, which is on the boundary of such a ball, is not in E. Thus E is not a ball closed G_δ set.

THEOREM. *If f is differentiable, $\nu \in E_n$ $|\nu| = 1$, $a \in E_1$, then the sets $E_\nu^a = \{x: \partial_\nu f(x) \leqq a\}$ and $E_{\nu,a} = \{x: \partial_\nu f(x) \geqq a\}$ are ball closed G_δ sets.*

Proof. It suffices to prove that $E_{\nu,0}$ is a ball closed G_δ set. Let $B(y, r) \subset E_{\nu,0}$ and let x be such that $|y - x| = r$. For $0 < s < 1$ the two points

$$x + s(y - x) = x_1$$

and

$$x + s(y - x) + sr\nu/2 = x_2$$

both lie in $B(y, r)$ since

$$|x_1 - y| = (1 - s)|y - x| < |y - x| = r$$

and

$$|x_2 - y| \leqq (1 - s)|y - x| + rs/2 = (1 - s/2)r < r\,.$$

The line segment joining x_1 and x_2 lies entirely in $B(y, r)$. So by the Mean Value Theorem for each $0 < s < 1$, there is a $0 < t < s$ with

$$f(x_2) - f(x_1) = (sr/2)\partial_\nu f(x + s(y - x) + tr\nu/2)\,.$$

But $x + s(y - x) + tr\nu/2 \in B(y, r)$; so

$$(f(x_2) - f(x_1))/(sr/2) \geqq 0\,.$$

Now since f is differentiable at x,

$$\begin{aligned}
f(x_2) - f(x_1) &= f(x) + (s(y-x) + sr\nu/2) \cdot \operatorname{grad} f(x) \\
&\quad + o(|\,s(y-x) + sr\nu/2\,|) \\
&\quad - f(x) - s(y-x) \cdot \operatorname{grad} f(x) - o(|\,s(y-x)\,|) \\
&= (sr\nu/2 \cdot \operatorname{grad} f(x) + o(|\,s(y-x) + sr\nu/2\,|) \\
&\quad - o(|\,s(y-x)\,|) \\
&= (sr/2)\partial_\nu f(x) + o(s) \quad \text{as } s \to 0
\end{aligned}$$

(since y, x, r, and ν are fixed the two "o" terms are just $o(s)$). But $(f(x_2) - f(x_1))/(sr/2) \geqq 0$. Thus $\partial_\nu f(x) + o(1) \geqq 0$ as $s \to 0$, and hence $\partial_\nu f(x) \geqq 0$.

That $E_{\nu,0}$ is a G_δ set follows since f is continuous and hence $\partial_\nu f$ is of Baire class one. Hence $E_{\nu,0}$ is a ball closed G_δ set.

The converse is far from true. There are functions f having directional derivatives in every direction and such that for each ν and for each real number a, $E_{\nu,a}$ and E_ν^a are ball closed G_δ sets, but f is not differentiable. The situation is even worse. It is not hard to see that if f is differentiable, then $E_{\nu,a}$ and E_ν^a have a much stronger closure property than just being ball closed G_δ sets. Suppose that some x in E_n is the vertex of a "cone" in E_n whose interior lies in $E_{\nu,a}$ (resp. E_ν^a). Then $x \in E_{\nu,a}$ (resp. E_ν^a). This property on the sets $E_{\nu,a}$ and E_ν^a is still not enough to guarantee that f is differentiable. For example, it is possible to construct a function $f(x, y)$ which is zero except for points (x, y) satisfying $y > 0$ and $y^2 < x < 4y^2$, has continuous partial derivatives except at $(0, 0)$ where all directional derivatives exist, however, and are zero, but f is not differentiable at $(0, 0)$ (If $h(t)$ is infinitely differentiable, with support $[0, 1]$, then $f(x, y) = xh((x - y^2)/3y^2)$ for $y > 0$ and $f(x, y) = 0$ for $y \leqq 0$ is such a function.) If (x, y) is the vertex of a "cone" whose interior lies in $E_{\nu,a}$(resp. E_ν^a) and if $(x, y) \neq (0, 0)$, then $\partial_\nu f$ is continuous at (x, y); so $x \in E_{\nu,a}$(resp. $x \in E_\nu^a$). If $(0, 0)$ is the vertex of such a "cone," then that cone does not lie in the support of f; so there is an (x, y) in that cone with $\partial_\nu f(x, y) = 0$. Thus $0 \geqq a$ (resp. $0 \leqq a$). Since $\partial_\nu f(0, 0) = 0$, it follows that $(0, 0) \in E_{\nu,a}$ (resp. $(0, 0) \in E_\nu^a$). Consequently the sets $E_{\nu,a}$ and E_ν^a contain the vertex of any cone whose interior they contain, but by choice f is not everywhere differentiable.

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MICHIGAN STATE UNIVERSITY