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ASYMPTOTIC RELATIONS BETWEEN PERTURBED LINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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A generalization of the concept of asymptotic equivalence of two systems of ordinary differential equations is investigated. This extension of asymptotic equivalence is novel in two ways. First, the dimensions of the linear asymptotic subspaces of the differential equations are utilized. Secondly, the two Banach spaces L^∞ and L_0^∞, that are implicitly used in the usual definition of asymptotic equivalence, are replaced by two (arbitrary) Banach spaces that are stronger that $L(X)$. The main theorem establishes a functional asymptotic relationship between the solutions of two perturbed linear differential equations that utilizes the above modifications.

Consider the systems of ordinary differential equations

$$y' = A(t)y + f_1(t, y) , \tag{1}$$

$$x' = A(t)x + f_2(t, x) , \tag{2}$$

where x and y are vectors in an n-dimensional vector space X, $A(t)$ is an $n \times n$ matrix defined on $J = [0, \infty)$, and $f_1(t, y)$ and $f_2(t, x)$ are n-dimensional vector functions defined on $J \times X$.

The equations (1) and (2) are said to be *asymptotically equivalent* if for each bounded solution $y = y(t)$ of (1) there exists a bounded solution $x = x(t)$ of (2) such that

$$\lim_{t\to\infty} [x(t) - y(t)] = 0 ; \tag{3}$$

and, conversely, for each bounded solution $x = x(t)$ of (2) there exists a bounded solution $y = y(t)$ of (1) such that (3) holds. If two nonlinear systems are just known to be asymptotically equivalent then very little information can be obtained about the dimensions of the corresponding asymptotic subspaces of solutions. To remedy this situation, we formulate in §2, a definition that takes advantage of the dimensions of certain linear asymptotic manifolds of solutions of equations (1) and (2). A general Banach space setting for the equivalence is exploited in this new definition.

The principal tool used in this work is a result of P. Hartman and N. Onuchic [6] on the asymptotic integration of ordinary differential equations.

As corollaries to our main theorem, several recent results on the asymptotic equivalence of perturbed linear systems are obtained. In particular, we obtain extensions of results in the articles by F. Brauer and J. S. W. Wong [1] and T. G. Hallam [4]. Applications in new directions are also given. Results related to this problem that are found by using the same methods may be found in the papers [8] and [9] by N. Onuchic.

2. **Preliminary material and definitions.** In this section we state definitions, indicate notation, and summarize the results needed in our development of this problem. Additional details may be found in [6].

The symbol $\|\cdot\|$ will denote a norm on X. The symbol $\beta = \beta(J, R)$ denotes a Banach space of real-valued functions defined on J with the norm of $\phi \in \beta$ denoted by $|\phi|_\beta$. By $B = \beta(J, X)$ we represent the space of measurable functions $x = x(t)$ defined on J with values in X such that $\|x(t)\| \in \beta$, and with $|x(t)|_B = |\,\|x(t)\|\,|_\beta$. We let $L = L(J, R)$ denote the space of locally Lebesgue integrable real-valued functions defined on J, with the topology of convergence in the mean of order one on bounded subintervals of J, and let $L(X) = L(J, X)$ represent the space of measurable functions x from J to X such that $\|x(t)\| \in L(J, R)$. A Banach space B is *stronger* than $L(X)$ if B is algebraically contained in $L(X)$ and convergence in B implies convergence in $L(X)$. Every Banach space of measurable functions from J to X used below will be tacitly assumed to be stronger than $L(X)$.

The class $H = H(R)$ consists of all Banach spaces $\beta = \beta(J, R)$ of measurable functions from J to R with the four properties

(i) β is stronger than $L(J, R)$;

(ii) if $\phi \in \beta$, ψ is measurable, and $|\psi(t)| \leqq |\phi(t)|$, then $\psi \in \beta$ and $|\psi|_\beta \leqq |\phi|_\beta$;

(iii) if h_I is the characteristic function of the interval I, then $h_I \in \beta$ for all intervals $I = [0, T]$, for $T > 0$;

(iv) β is lean at infinity; that is, if $\phi \in \beta$, then $h_{[0,T]}\phi \rightarrow \phi$ as $T \rightarrow \infty$.

For example, the spaces $L^p(J, R)$, $1 \leqq p < \infty$, are contained in $H(R)$. The same is true of $L_0^\infty(J, R)$, the subspace of $L^\infty(J, R)$ whose elements x satisfy the condition $\operatorname{ess\,lim}_{t\to\infty} x(t) = 0$. However, $L^\infty(J, R)$ itself is not contained in $H(R)$.

Another useful class of Banach spaces in $H(R)$ consists of spaces β defined in the following manner. Let $\psi = \psi(t) > 0$ be a measurable function on J such that ψ and $1/\psi$ are locally bounded on J. The space $\beta = L_{\psi,0}(J, R)$ contains all measurable functions $\phi = \phi(t)$ such that $\phi/\psi \in L_0^\infty(J, R)$ with $|\phi|_\beta = |\phi/\psi|_{L^\infty}$.

We represent by $H(X)$ the class of all Banach spaces $B = \beta(J, X)$ where $\beta(J, R)$ is in $H(R)$.

In the equations

(4) $$z' = A(t)z ,$$

(5) $$w' = A(t)w + b(t) ,$$

$A(t)$ is a locally Lebesgue integrable $n \times n$ matrix defined on J, and $b \in L(X)$.

We assume that the Banach spaces in the sequel consist of measurable functions from J to X unless the contrary is specified. For such a Banach space D, let X_{0D} denote the set of initial points $x(0) \in X$ of solutions $x = x(t)$ of equation (4) which are in D. Let X_{1D} be any subspace of X complementary to X_{0D} and P_{0D} the projection of X onto X_{0D} which annihilates X_{1D}. A pair of Banach spaces (B, D) is called *admissible for* $A(t)$ if for every $b \in B$, (5) has at least one solution $w = w(t)$ in D. A function $x = x(t)$ from $[T, \infty)$ to X, $T \geqq 0$, is *asymptotically in the Banach space* B if there is a function $\tilde{x} = \tilde{x}(t)$ from J to X which is in B and such that $\tilde{x}(t) = x(t)$ for $t \geqq T$. Whenever the function x is asymptotically in B and x is also a solution of a differential equation then we will say that x is an *asymptotic* B *solution* of the differential equation.

Let p and q be integers that satisfy the inequalities $0 \leqq p \leqq n$, $0 \leqq q \leqq n$. Equations (1) and (2) are (p, q)-*asymptotically related with respect to the ordered pair* $\{D_1, D_2\}$ *of Banach spaces* if the following two conditions hold:

(i) There exists a family F_p of asymptotic D_1 solutions of (1) which depends upon at least p parameters.

(ii) For each solution $y = y(t)$ of (1) in F_p, there corresponds a family F_q of solutions $x = x(t)$ of (2), which depends upon at least q parameters, such that $y - x$ is asymptotically in D_2 for each $x \in F_q$.

We adopt the convention that a family which depends upon 0 parameters must consist of at least one member.

If the family of all bounded solutions of (1) is a p-parameter family and the family of all bounded solutions of (2) is a q-parameter family then the concept of asymptotic equivalence may be formulated as follows: Equations (1) and (2) are $(p, 0)$-asymptotically related with respect to $\{L^\infty, L_0^\infty\}$ and equations (2) and (1) are $(q, 0)$-asymptotically related with respect to $\{L^\infty, L_0^\infty\}$.

Let $C = C(X)$ denote the space of continuous functions from J to X with the compact open topology. Let $\Sigma_{D,\rho}$ be the closed ball of radius ρ in D and let $S_{D,\rho} = \Sigma_{D,\rho} \cap C$ and $\bar{S}_{D,\rho}$ be the closure of $S_{D,\rho}$ in $C(X)$.

A basic requirement imposed upon the functions f_i of (1) and (2) is

(6) $f_i(t, x)(i = 1, 2)$ is a measurable function of t for fixed $x \in X$ and a continuous function of x for fixed $t \in J$. Furthermore, the continuity of f_i in x is uniform for t in compact subintervals of J.

The matrix $A(t)$ will always satisfy the condition

$$A(t) \text{ is locally Lebesgue integrable on } J\,. \tag{7}$$

The next lemma is a consequence of Theorem 1.1 of [6].

LEMMA. *Suppose that equation* (1) *has the following properties.*

(i) *Condition* (6) *is satisfied for the function* f_1 *and condition* (7) *is satisfied.*

(ii) *The Banach space* $B = \beta(J, X)$ *is in* $H(X)$ *and the pair* (B, D) *is admissible for* $A(t)$.

(iii) *There exists a constant* $\rho > 0$ *and a function* $r_\rho = r_\rho(t) \in \beta(J, R)$ *such that*

$$\|f_1(t, y(t))\| \leqq r_\rho(t) \tag{8}$$

for all $t \in J$ *and all* $y \in \bar{S}_{D,\rho}$.

Let $\xi_0 \in X_{0D}$. *Then, there exist positive constants* C_0, K *depending only upon* $A(t)$, B, D, X_{1D} *(but not on* f_1 *nor* ξ_0*) such that whenever* $\|\xi_0\|$ *is sufficiently small and* T *is sufficiently large so that*

$$C_0\|\xi_0\| + K|h_{[T,\infty)}r_\rho|_\beta \leqq \rho\,,$$

then (1) *has an asymptotic* D *solution* $y = y(t)$ *valid on* $[T, \infty)$ *with the properties*

(iv) $P_{0D}Z^{-1}(T)y(T) = \xi_0$, *where* $Z(t)$ *is the fundamental matrix of* (4) *with* $Z(0) = I_n$; *and*

(v) *The solution* y *has an extension* $\tilde{y}$ *which is valid on* J *and is a solution of*

$$y' = A(t)y + h_{[T,\infty)}(t)f_1(t, y) \tag{9}$$

such that $\tilde{y}(T) = y(T)$ *and* $|\tilde{y}(t)|_D \leqq \rho$.

Proof. The only hypotheses of Theorem 1.1 of [6] which are not explicitly given aboven above are the conditions denoted by (b) and (c) in [6]. For (9), condition (b) states that the transformation

$$y(t) \to \tilde{f}(t, y(t)) \equiv h_{[T,\infty)}(t)f_1(t, y(t))$$

is a continuous map of the subset $\bar{S}_{D,\rho}$ of $C(X)$ into B. This is a consequence of the facts that $B \in H(X)$ and f_1 satisfies conditions (6) and (8). To establish this, let $\varepsilon > 0$ be given; then $\tau \geqq 0$ can be chosen sufficiently large so that $|h_{[\tau,\infty)}(t)r_\rho(t)|_\beta < \varepsilon/4$. By virtue of (6), there exists a $\delta = \delta(\varepsilon, \tau) > 0$ such that if $y_1, y_2 \in \bar{S}_{D,\rho}$ and $|y_1(t) - y_2(t)| < \delta$ for $t \in [0, \tau]$ then

$$|h_{[0,\tau]}(t)[\tilde{f}_1(t, y_1(t)) - \tilde{f}_1(t, y_2(t))]|_B < \varepsilon/2 .$$

Therefore, if y_1 and y_2 are in $\bar{S}_{D,\rho}$ with

$$|h_{[0,\tau]}(t)[y_1(t) - y_2(t)]| < \delta ,$$

then

$$\begin{aligned} |\tilde{f}_1(t, y_1(t)) - \tilde{f}_1(t, y_2(t))|_B &\leqq |h_{[0,\tau]}(t)[\tilde{f}_1(t, y_1(t)) - \tilde{f}_1(t, y_2(t))]|_B \\ &\quad + |h_{[\tau,\infty)}(t)\tilde{f}_1(t, y_1(t))|_B \\ &\quad + |h_{[\tau,\infty)}(t)\tilde{f}_1(t, y_2(t))|_B \\ &< \varepsilon/2 + 2|h_{[\tau,\in)}(t)r_\rho(t)|_\beta \\ &< \varepsilon . \end{aligned}$$

This shows that (b) is satisfied.

For (9), condition (c) states that there exists a constant $r > 0$ such that $|\tilde{f}_1(t, x(t))|_B \leqq r$ for $x(t) \in \bar{S}_{D,\rho}$. This follows by taking $r = |h_{[T,\infty)}r_\rho|_\beta$.

Theorem 1.1 of [6] may now be applied to system (9) to establish the existence of a D solution $\tilde{y} = \tilde{y}(t)$ of (9) which satisfies $|\tilde{y}(t)|_D \leqq \rho$ and $P_{0D}\tilde{y}(0) = \xi_0$. Since $\tilde{f}_1(t, y) = 0$ for $0 \leqq t < T$, it follows that $\tilde{y}(T) = Z(T)\tilde{y}(0)$ and hence

$$P_{0D}\tilde{y}(0) = P_{0D}Z^{-1}(T)\tilde{y}(T) = \xi_0 .$$

The function $y(t) = \tilde{y}(t)$, $t \geqq T$, is an asymptotic D solution such that $P_{0D}Z^{-1}(T)y(T) = \xi_0$. This completes the proof of the lemma.

REMARK 1. If $m = \dim X_{0D}$, this lemma implies the existence of a family of asymptotic D solutions of (1) which depends at least upon m parameters.

REMARK 2. Condition (iii) of the Lemma is sometimes difficult to establish. In the instance where $D = L^\infty$ or $D = L_0^\infty$, a simple sufficient condition for (iii) is that there exists a $\lambda_\rho = \lambda_\rho(t)$ in $\beta(J, R)$ such that $||f(t, x)|| \leqq \lambda_\rho(t)(t \in J; ||x|| \leqq \rho)$. A sufficient condition for (iii), without specifying D, is that there exists a $\lambda = \lambda(t) \in \beta(J, R)$ such that $||f(t, x)|| \leqq \lambda(t)(t \in J; x \in X)$. Theorem 4 below gives a sufficient condition for inequality (8) to hold whenever B and D are certain spaces of L^p type.

3. Asymptotic relations. This section contains our main result and some applications.

THEOREM 1. *Suppose that equations* (1) *and* (2) *satisfy the following conditions.*

(i) *Assumptions* (6) *and* (7) *hold.*

(ii) *The space* $B_i = \beta_i(J, X)$ *is in* $H(X)$ *and the pair* (B_i, D_i) *is admissible,* $i = 1, 2$.

(iii) *There exist constants* $\rho_i > 0$ *and functions*

$$r_{\rho_i} \equiv r_{\rho_i}(t) \in \beta_i(J, R)$$

such that

$$||f_1(t, y(t))|| \leqq r_{\rho_1}(t) \tag{10}$$

for all $t \in J$ *and all* $y \in \bar{S}_{D_1,\rho_1}$; *and*

$$||f_2(t, u(t) + y(t)) - f_1(t, y(t))|| \leqq r_{\rho_2}(t) \tag{11}$$

for all $t \in J$, *all* $y \in \bar{S}_{1D,\rho_1}$, *and all* $u \in \bar{S}_{D_2,\rho_2}$.

(iv) *The dimensions of* X_{0D_1} *and* X_{0D_2} *are* p *and* q *respectively. Then, under these hypotheses, equations* (1) *and* (2) *are* (p, q)*-asymptotically related with respect to* $\{D_1, D_2\}$.

Proof. The existence of a family F_p of asymptotic D_1 solutions of (1), which depends upon at least p parameters, is an immediate consequence of the Lemma. Let $y = y(t)$ be a solution of (9) where T and $y \in F_p$ are as given by the Lemma. It follows that $|y|_{D_1} \leqq \rho_1$.

The change of variable $u = x - y(t)$ leads to the differential equation

$$u' = A(t)u + f(t, u) \qquad (t \geqq T; u \in X) \tag{12}$$

where

$$f(t, u) = f_2(t, u + y(t)) - f_1(t, y(t)) \; . \tag{13}$$

It is convenient to consider equation (12) as

$$u' = A(t)u + h_{[T,\infty)}(t) f(t, u) \qquad (t \in J) \; .$$

From (11) it follows that for all $u \in \bar{S}_{D_2,\rho_2}$,

$$||h_{[T,\infty)}(t) f(t, u(t))|| \leqq h_{[T,\infty)}(t) r_{\rho_2}(t), \qquad (t \in J) \; .$$

The lemma now implies that there exists a family F_q of asymptotic D_2 solutions of (12) that depends upon at least q parameters. For each $u \in F_q$, $x(t) = u(t) + y(t)$ is a solution of equation (2); hence, there is a family G_q of solutions of (2), which depends at least upon q parameters, such that $x - y$ is asymptotically in D_2 for each $x \in G_q$. This completes the proof of Theorem 1.

COROLLARY 1. *Suppose that conditions* (i), (ii), *and* (iv) *of Theorem* 1 *are satisfied for the spaces* $B_i = B = \beta(J, X) \in H(X)$, $i = 1, 2$. *In addition, suppose that there exists a* $\lambda = \lambda(t) \in \beta(J, R)$ *such that* $\|f_i(t, x)\| \leqq \lambda(t)$ *for all* $t \in J$ *and all* $x \in X$, $i = 1, 2$. *Then, equations* (1) *and* (2) *are* (p, q)*-asymptotically related with respect to* $\{D_1, D_2\}$ *and equations* (2) *and* (1) *are* (q, p)*-asymptotically related with respect to* $\{D_2, D_1\}$.

Proof. Taking into account Remark 2, we have that (10) is satisfied with $r_{\rho_1} = \lambda$ and (11) is satisfied with $r_{\rho_2} = 2\lambda$. Since the hypotheses are symmetric for both the cases $i = 1, 2$, Theorem 1 yields the desired conclusion.

For the particular case in which $D_1 = L^\infty$ and $D_2 = L_0^\infty$ we obtain the following results.

THEOREM 2. *Suppose that equations* (1) *and* (2) *satisfy the following conditions*

(i) *Assumption* (i) *of Theorem* 1 *holds.*

(ii) *The function* $V(t, r)$ *is nonnegative for* $(t, r) \in J \times J$, *nondecreasing in* r *for fixed* t, *and*

$$\|f_i(t, x)\| \leqq V(t, \|x\|), \; (i = 1, 2; t \in J; x \in X) \; . \tag{14}$$

(iii) *The space* $B = \beta(J, X)$ *is in* $H(X)$; $V(t, r)$ *is in* $\beta(J, R)$ *for each fixed* $r > 0$; *and,* (B, L_0^∞) *is admissible for* $A(t)$.

(iv) *The dimensions of* X_{0L^∞} *and* $X_{0L_0^\infty}$ *are* p *and* q *respectively. Then, under these hypotheses, equations* (1) *and* (2) *are* (p, q)*-asymptotically related with respect to* $\{L^\infty, L_0^\infty\}$.

Proof. Theorem 2 is a consequence of Theorem 1 provided we show that the inequality (14) implies that the inequalities (10) and (11) hold. Let $\rho_i > 0$ $(i = 1, 2)$ be given. For all $y \in \bar{S}_{L^\infty, \rho_1}$,

$$\|f_1(t, y(t))\| \leqq V(t, \rho_1);$$

therefore, (10) is satisfied because $V(t, \rho_1)$ is in $\beta(J, R)$. To see that (11) holds, we only need observe that for all $u \in \bar{S}_{L_0^\infty, \rho_2}$ and all $y \in \bar{S}_{L^\infty, \rho_1}$

$$\|f_2(t, u(t) + y(t)) - f_1(t, y(t))\| \leqq 2V(t, \rho_1 + \rho_2) \; .$$

As Corollaries to the above theorem, we obtain extensions of some results of Brauer and Wong [1, Theorem 1] and Hallam [4, Theorem 2].

COROLLARY 2. *Suppose that equations* (1) *and* (2) *satisfy the following conditions.*

(i) *There exist supplementary projections* P_1, P_2 *and a constant* $K > 0$ *such that*

$$(15)\qquad \begin{aligned} &\|Z(t)P_1Z^{-1}(s)\| \leqq K,\ 0 \leqq s \leqq t\ ;\\ &\|Z(t)P_2Z^{-1}(s)\| \leqq K,\ 0 \leqq t \leqq s\ . \end{aligned}$$

(ii) *The function* $f_1 \equiv 0$, f_2 *satisfies* (6) *and*

$$(16)\qquad \|f_2(t, x)\| \leqq V(t, \|x\|),\ (t \in J;\ x \in X)$$

where $V(t, r)$ *satisfies hypothesis* (ii) *of Theorem* 2.

(iii) $\int_0^\infty V(t, r)dt < \infty$ *for all* $r > 0$.

(iv) *Assumption* (7) *and condition* (iv) *of Theorem* 2 *hold. Then, under these hypotheses, the equations* (1) *and* (2) *are* (p, q)*-asymptotically related with respect to* $\{L^\infty, L_0^\infty\}$.

Proof. It is known [7, p. 331] that if (L^1, L^∞) is admissible for $A(t)$ then (L^1, L_0^∞) is also admissible for $A(t)$. It is also known that (L^1, L^∞) is admissible if and only if (15) is satisfied. The corollary now follows from Theorem 2 with $B = L^1$.

COROLLARY 3. *Suppose that the assumptions* (ii) *and* (iv) *of Corollary* 2 *hold. Suppose that there exist sypplementary projections* P_1, P_2 *and a constant* $K > 0$ *such that*

$$(17)\qquad \begin{aligned} \rho(t) \equiv &\left(\int_0^t \|Z(t)P_1Z^{-1}(s)\|^\tau ds\right)^{1/\tau}\\ &+ \left(\int_t^\infty \|Z(t)P_2Z^{-1}(s)\|^\tau ds\right)^{1/\tau} \leqq K\ ,\\ &(t \in J;\ 1 < \tau < \infty)\ . \end{aligned}$$

Let $\int_0^\infty [V(t, r)]^\sigma dt < \infty$, *for all* $r > 0$, *where* $\sigma^{-1} + \tau^{-1} = 1$. *Then, equations* (1) *and* (2) *are* (p, q)*-asymptotically related with respect to* $\{L_0^\infty, L_0^\infty\}$.

Proof. Conti [2, Theorem 1] has shown that condition (17) is necessary and sufficient for the pair (L^σ, L^∞) to be admissible for $A(t)$. However, condition (17) is equivalent to the stronger result that (L^σ, L_0^∞) is admissible for $A(t)$. To verify this statement, we note that (17), $b \in L^\sigma$, and the Hölder inequality imply that

$$w(t) = \int_0^t Z(t)P_1Z^{-1}(s)b(s)ds - \int_t^\infty Z(t)P_2Z^{-1}(s)b(s)ds \tag{18}$$

is a solution of (5) with $||w(t)|| \leqq K|b|_{L^\sigma}, t \in J$.

It will now be shown that $\lim_{t\to\infty} w(t) = 0$. For a given $\varepsilon > 0$, $T_1 = T_1(\varepsilon)$ can be chosen so that

$$|h_{[T_1,\infty)}b|_{L^\sigma} < \varepsilon/2K. \tag{19}$$

Since $\lim_{t\to\infty} ||Z(t)P_1|| = 0$ (see [4, p. 359]), $T_2 \geqq T_1$ may be chosen so that

$$||Z(t)P_1|| < \varepsilon/2\int_0^{T_1}||Z^{-1}(s)b(s)||ds, \ t \geqq T_2. \tag{20}$$

Therefore, for $t \geqq T_2$ it follows from (19) and (20) that

$$\begin{aligned} \left\|\int_0^t Z(t)P_1Z^{-1}(s)b(s)ds\right\| &\leqq \int_0^{T_1}||Z(t)P_1||\,||Z^{-1}(s)b(s)||ds \\ &+ \left(\int_{T_1}^t ||Z(t)P_1Z^{-1}(s)||^\tau ds\right)^{1/\tau} \\ &\cdot \left(\int_{T_1}^t ||b(s)||^\sigma ds\right)^{1/\sigma} \\ &< \varepsilon. \end{aligned}$$

This shows that the first term of w in (18) tends to zero as t approaches infinity. The last term in (18) also tends to zero as t approaches infinity since

$$\left\|\int_t^\infty Z(t)P_2Z^{-1}(s)b(s)ds\right\| \leqq K\left(\int_t^\infty ||b(s)||^\sigma ds\right)^{1/\sigma}.$$

The corollary now follows from Theorem 2.

REMARK 4. The asymptotic equivalence result analogous to Corollary 2 is Theorem 2 of [4]. It yields an $(p, 0)$-asymptotic relation with respect to $\{L_0^\infty, L_0^\infty\}$. It should be noted that we have taken the weight functions ψ, ϕ of [4] as $\psi = \phi = 1$. The proof that (17) implies (L^σ, L_0^∞) is admissible for $A(t)$ is essentially contained in [4].

We point out that the statement—equations (1) and (2) are (p, q)-asymptotically related with respect to $\{D, D\}$—yields only a natural correspondence between the solutions of (1) and (2). Namely, that correspondence given by $y - x$ is asymptotically in D whenever $y \in D$. This means that x is asymptotically in D; hence, the above statement says that (1) $\{(2)\}$ has a family of asymptotic D solutions depending upon at least $p\{q\}$ parameters.

The above corollaries extend known results. The next results are applications of Theorem 1 in a new direction. For this purpose consider the Banach space $D = L^\eta \cap L_0^\infty$, $1 \leqq \eta < \infty$, where $x \in D$ implies $x \in L^\eta$ and $x \in L_0^\infty$ with $|x|_D = \max[|x|_{L^\eta}, |x|_{L_0^\infty}]$; see [7, p. 336]. Later in the paper we will utilize the Banach space $D = L^\eta \cap L^\infty$ which is defined in an analogous manner.

THEOREM 3. *Suppose that equations* (1) *and* (2) *satisfy the following conditions.*

(i) *Assumption* (i) *of Theorem* 1 *holds.*

(ii) *There exist supplementary projections P_1, P_2 and positive constants α, K such that*

$$\begin{aligned} &\|Z(t)P_1Z^{-1}(s)\| \leqq Ke^{-\alpha(t-s)}, \quad 0 \leqq s \leqq t\,; \\ &\|Z(t)P_2Z^{-1}(s)\| \leqq K, \qquad 0 \leqq t \leqq s\,. \end{aligned} \tag{21}$$

(iii) *There exists a nonnegative function $\lambda = \lambda(t)$ measurable on J such that*

$$\|f_i(t, x)\| \leqq \lambda(t) \quad (i = 1, 2; t \in J; x \in X)\,. \tag{22}$$

and

$$\int^\infty t\lambda(t)dt < \infty\,. \tag{23}$$

(iv) *The dimensions of X_{0L^∞}, $X_{0L^r}(r \geqq 1)$, and $X_{0,D}(D = L^s \cap L_0^\infty$, $s \geqq 1)$ are p, q, and m respectively.*
Then, under these hypotheses, equations (1) *and* (2) *are (p, m)-asymptotically related with respect to $\{L^\infty, L^s \cap L_0^\infty\}$ and equations* (2) *and* (1) *are (q, m)-asymptotically related with respect to $\{L^r, L^s \cap L_0^\infty\}$.*

Proof. Let $\phi = \phi(t)$ be chosen so that ϕ is a positive continuous function satisfying

$$\int^\infty t\lambda(t)\phi(t)dt < \infty$$

and

$$\lim_{t\to\infty} \phi(t) = \infty\,.$$

It will be shown that $(L_{\psi,0}, L^k \cap L_0^\infty)(k \geqq 1)$ is admissible for $A(t)$ where $\psi = \lambda\phi$.

If $b \in L_{\psi,0}$, then $w(t)$, as defined by (18), is a solution of (5) since (21) implies that there is a positive constant K_1 such that

$$\|Z(t)P_2Z^{-1}(s)b(s)\| \leqq K\|b(s)\| \leqq K_1\lambda(s)\phi(s)$$

where

$$\int^{\infty} \lambda(s)\phi(s)ds < \infty .$$

Also, from (21), it follows that

$$\|w(t)\| \leqq K_1\int_0^t e^{-\alpha(t-s)}\lambda(s)\phi(s)ds + K_1\int_t^{\infty} \lambda(s)\phi(s)ds . \tag{24}$$

We obtain from (24) that $w \in L_0^\infty$ because

$$\int^{\infty} \lambda(s)\phi(s)ds < \infty$$

and

$$\lim_{t\to\infty} e^{-\alpha t}\int_0^t e^{\alpha s}\lambda(s)\phi(s)ds = 0$$

(see [3], Lemma 1).

To see that $w = w(t)$ is also in L^k, $k \geqq 1$, it is sufficient to show that w is in L^1 since we have established that $w \in L_0^\infty$. We note that an integration in (24) leads to

$$\begin{aligned}\int_0^t \|w(\tau)\| d\tau \leqq\ & K_1\int_0^t d\tau\int_0^\tau e^{-\alpha(\tau-s)}\lambda(s)\phi(s)ds \\ & + K_1\int_0^t d\tau\int_\tau^\infty \lambda(s)\phi(s)ds .\end{aligned} \tag{25}$$

The integral $\int_0^\infty d\tau\int_\tau^\infty \lambda(s)\phi(s)ds$ converges because $\int^\infty t\lambda(t)\phi(t)dt$ converges. An integration by parts gives

$$\begin{aligned}\int_0^t e^{-\alpha\tau}\left[\int_0^\tau e^{\alpha s}\lambda(s)\phi(s)ds\right]d\tau =\ & -\alpha^{-1}e^{-\alpha t}\int_0^t e^{\alpha s}\lambda(s)\phi(s)ds \\ & + \alpha^{-1}\int_0^t \lambda(s)\phi(s)ds .\end{aligned} \tag{26}$$

The right side of (26) is bounded which implies that the first integral in the right side of (25) converges as $t \to \infty$. This implies that $w \in L^1$.

The corollary follows from Theorem 1 provided that $\lambda \in L_{\psi,0}(J, R)$; but, $\lambda/\psi = \phi^{-1}$ and $\lim_{t\to\infty} \phi^{-1}(t) = 0$.

REMARK 5. A sufficient condition for (21) is that A be a constant matrix where all eigenvalues of A with zero real part have linear elementary divisors.

THEOREM 4. *Suppose that equations* (1) *and* (2) *satisfy the following conditions.*

(i) *The function* $f_1 \equiv 0$; f_2 *satisfies* (6); *and* (7) *is satisfied.*

(ii) *There exist positive functions* $a = a(t)$ *and* $b = b(t)$ *that are continuous on* J *and a constant* $\delta > 0$ *such that*

$$||f_2(t, x)|| \leqq a(t)||x||^\delta + b(t), (t \in J; x \in X) . \tag{27}$$

(iii) *The pair* (L^σ, D) *is admissible for* $A(t)$, *where* $1 \leqq \sigma < \infty$, *and* $D = L^\eta \cap L^\infty$ *with* $\sigma\delta \leqq \eta < \infty$.

(iv) *The function* $a \in L^\alpha(J, R)$ *where* $\alpha = \eta\sigma/(\eta - \sigma\delta)$ *and the function* $b \in L^\sigma(J, R)$.

(v) *Let* $p = \dim X_{0L^\eta}$ *and* $q = \dim X_{0D}$.

Then, under these hypotheses, equations (1) *and* (2) *are* (p, q)*-asymptotically related with respect to* $\{L^\eta, D\}$.

Proof. First, we observe that $\bar{S}_{D,\rho} = S_{D,\rho}$. This equality is evident if it is shown that $x \in \bar{S}_{D,\rho}$ implies that $x \in D$ and $|x|_D \leqq \rho$. If $x \notin D$ then either $x \notin L^\eta$ or $x \notin L^\infty$. Since $x \in \bar{S}_{D,\rho}$, there exists a sequence $\{x_n\}$ with $x_n \in S_{D,\rho}$, $n = 1, 2, \cdots$ such that $\{x_n\}$ converges uniformly to x on compact subintervals of J.

Suppose that $x \notin L^\eta$; then, corresponding to any $\varepsilon > 0$ there is a $T = T(\varepsilon) > 0$ such that

$$\left[\int_0^T ||x(t)||^\eta dt\right]^{1/\eta} > \rho + \varepsilon .$$

Corresponding to the positive number $1/2T$, there exists an N such that whenever $n \geqq N$ then $||x_n(t) - x(t)|| < \varepsilon/2T^{1/\eta}$, $t \in [0, T]$. Therefore, for $n \geqq N$,

$$\begin{aligned} \rho + \varepsilon < \left[\int_0^T ||x(t)||^\eta dt\right]^{1/\eta} &\leqq \left[\int_0^T || x_n(t) ||^\eta dt\right]^{1/\eta} \\ &\quad + \left[\int_0^T || x(t) - x_n(t) ||^\eta dt\right]^{1/\eta} \\ &\leqq \rho + \varepsilon/2 . \end{aligned}$$

This contradiction shows that $x \in L^\eta$. Because $|x_n|_{L^\infty} \leqq \rho$, it is clear that $x \in L^\infty$; hence $x \in D = L^\eta \cap L^\infty$.

It remains to show that $|x|_D \leqq \rho$. It is immediate that $|x|_{L^\infty} \leqq \rho$. The argument given in the above paragraph shows that $|x|_{L^\eta} \leqq \rho$. Thus, $|x|_D \leqq \rho$.

For any two functions u, y, (27) implies that

$$||f_2(t, u(t) + y(t))|| \leqq a(t)[||u(t) + y(t)||]^\delta + b(t) . \tag{28}$$

Let $\rho_1 > 0$ and $\rho_2 > 0$ be given; then for $u \in \bar{S}_{D,\rho_2}$ and $y \in \bar{S}_{L^\eta,\rho_1}$, we assert that the right side of (28) is in L^σ. The Hölder inequality and the fact that $D = L^\eta \cap L^\infty$ implies that

$$|f_2(t, u(t) + y(t))|_{L^\sigma} \leqq |a(t)|_{L^\alpha}[|u(t)|_{L^\eta} + |y(t)|_{L^\eta}]^\delta + |b(t)|_{L^\sigma} .$$

Since

$$u \in \bar{S}_{D,\rho_2} \text{ and } y \in \bar{S}_{L^\eta,\rho_1} \text{ then } |u|_D \leqq \rho_2 \text{ and } |y|_{L^\eta} \leqq \rho_1 .$$

The result now follows from Theorem 1.

REMARK 6. If f_2 satisfies the global Lipschitz condition

$$\|f_2(t, x_1) - f_2(t, x_2)\| \leqq a(t)\|x_1 - x_2\| \quad (t \in J, x \in X)$$

where $a \in L^\alpha$ and $f_2(t, 0) \in L^\sigma$ then f_2 satisfies condition (27) with $\delta = 1, a = a(t)$, and $b(t) = f_2(t, 0)$.

REMARK 7. A sufficient condition for $(L^\sigma, L^\eta \cap L^\infty)$ to be admissible is that $\rho(t)$ as defined in (17) be in $L^\eta \cap L^\infty$. Indeed, for each $b \in L^\sigma$, the solution $w(t)$ of (5) defined by (17) satisfies the estimate

$$\|w(t)\| \leqq |b|_{L^\sigma}\rho(t) \quad (t \in J) .$$

(Related as well as supplementary comments on this topic may be found in [5].) Hence, if $\rho \in L^\eta(J, R)$ and (17) is satisfied, then $(L^\sigma, L^\eta \cap L_0^\infty)$ admissible for $A(t)$.

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