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ON THE FUNDAMENTAL UNIT OF A PURELY CUBIC FIELD

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Let $a = D^3 + d$, where a, D, d are rational integers with $D, a > 0$, $|d| > 1$, and $d \mid 3D^2$. It is proved that the fundamental unit of the field $Q(\omega)$, where $\omega = \sqrt[3]{a}$, is $(\omega - D)^3/d$ with only six exceptions.

1. Introduction. The purpose of this paper is to establish the following result:

THEOREM 1. *Let $a = D^3 + d$, where $a, D, d \in Z$, with $a, D > 0$, $|d| > 1$, and a cubefree. Then $\varepsilon = (\omega - D)^3/d$, where $\omega = \sqrt[3]{a}$, is a unit of $K = Q(\omega)$ if and only if $d \mid 3D^2$. Moreover, in this case $\varepsilon = \eta$, the fundamental unit of K, except for $(D, d) = (2, -6)$, $(1, 3)$, $(2, 2)$, $(3, 1)$, and $(5, -25)$, where $\varepsilon = \eta^2$, and $(2, -4)$, where $\varepsilon = \eta^3$.*

Here, Z, Q denote respectively the rational integers and the field of rationals.

Theorem 1 is an extension of a result of Stender [4], who showed that when

$$(1) \qquad a = D^3 + d\,, \qquad d \mid D,\ d > 1$$

$$(2) \qquad a = D^3 + 3d\,, \qquad d \mid D,\ 3d \leqq D,\ d > 0$$

$$(3) \qquad a = D^3 + 3D\,, \qquad D \geqq 2\,,$$

$$(4) \qquad a = D^3 - d\,, \qquad d \mid D,\ 4 < 4d \leqq D\,,$$

or

$$(5) \qquad a = D^3 - 3d\,, \qquad d \mid D,\ 12d \leqq D,\ d > 0$$

$\varepsilon = (\omega - D)^3/(\omega^3 - D^3) = \eta$, except for $(D, d) = (2, 2)$ in (1), where $e = \eta^2$. The case $d = 1$ in (1) and (4) had already been settled by Nagell [2], who proved that $\varepsilon = \eta$ with the single exception of $a = 28$, when $\varepsilon = \eta^2$. The method of proof used here follows [4].

2. Preliminaries. We now make the assumption that $d \mid 3D^2$. Since a is cubefree we put $a = mn^2$ with m squarefree. Also, d is cubefree, as $d \mid 3a$.

Let $\bar{a} = m^2n$, $\bar{\omega} = \sqrt[3]{\bar{a}}$, and ζ be the fundamental unit of the ring $R = [1, \omega, \bar{\omega}]$. It is well known that if $a \not\equiv \pm 1 \pmod 9$, an integral basis for K is $\langle 1, \omega, \bar{\omega} \rangle$ (a field of the first kind). However, if $a \equiv \pm 1 \pmod 9$, an integral basis for K is given by

$$\langle(1 + m\omega + n\bar{\omega})/3,\ \omega,\ \bar{\omega}\rangle$$

(a field of the second kind) and each integer of K is representable in the form $(x + y\omega + z\bar{\omega})/3$. If $\zeta \neq \eta$ then K is of the second kind and $\zeta = \eta^2$ [3].

Now, if $\vartheta \in K \cap (0, 1)$ and if $\varepsilon = \vartheta^t$, t a natural number, then it is easily seen [4] that for $\varepsilon', \varepsilon''$ the (complex) conjugates of ε, $\vartheta = (x + y\omega + z\bar{\omega})/3$ implies that

$$(6) \qquad \begin{cases} |x| < \sigma \\ |y| < \sigma/\omega \\ |z| < \sigma/\bar{\omega}\,, \end{cases}$$

where

$$\sigma = 1 + 2\,|\varepsilon'|^{1/t}\,.$$

3. We observe that $\varepsilon = 1 + (3D^2/d)\omega - (3D/d)\omega^2$ satisfies the equation $t^3 - 3t^2 + (3 + 27D^3a/d^2)t - 1 = 0$. Hence ε is a unit of K if and only if $d^2 \mid 27D^3a$. Putting $x = 3D^3/d = p/q$ with $(p, q) = 1$, we can write the quotient as $3x(x + 3)$, i.e., $3p(p + 3q)/q^2$. It follows that ε is a unit if and only if $q^2 = 1$, i.e., $d \mid 3D^3$. But since a is cubefree, this is equivalent to $d \mid 3D^2$.

LEMMA 1. *If* $(D, d) \neq (2, -6)$ *then*

$$1 < |\varepsilon'| < \begin{cases} 6D\omega^2/d & \text{if } d > 0 \\ 6D^2\omega/|d| & \text{if } d < 0\,. \end{cases}$$

Proof. Since $(\omega - D)^3 = d - 3D\omega(\omega - D)$, we see that $(\omega - D)^3 < d$ if and only if $d > 0$ and hence $0 < \varepsilon < 1$. Therefore $\varepsilon + 2 < 3$ and since $\omega > 3/2$,

$$\begin{aligned} 1 < |\varepsilon'|^2 &= \frac{1}{4}\Big(2 - \frac{3D^2}{d}\omega + \frac{3D}{d}\omega^2\Big)^2 + \frac{3}{4}\Big[\frac{3D}{d}\omega(D + \omega)\Big]^2 \\ &< \Big(\frac{3D}{d}\omega\Big)^2\Big[D^2 + \frac{3}{4}(D + \omega)^2\Big] \end{aligned}$$

and the result follows.

PROPOSITION 1. *If* $(D, d) \neq (2, -6), (1, 3)$ *then* ε *is not a square in* R.

Proof. We first assume that $d \mid D^2$. This implies that $d \mid a$ and hence we may write $d = uv^2$ where $u \mid m$, $v \mid n$. Putting $D^2 = de$, $n = vr$ and assuming that $\varepsilon = (x + y\omega + z\bar{\omega})^2$, we obtain, by equating coefficients in the basis $\langle 1, \omega, \bar{\omega}\rangle$,

$$x^2 + 2mrvyz = 1 \tag{7}$$

$$az^2/v^2r^2 + 2xy = 3e \tag{8}$$

$$rvy^2 + 2xz = -3r(D/uv)\,. \tag{9}$$

Since (7) implies $(x, r) = 1$ we see from (9) that $r \mid 2z$ and hence $r^2 \leqq 4z^2$.

If $d > 0$, so that $u, e > 0$, (7) and (9) respectively imply that $yz \leqq 0$ and $xz < 0$. Since $y = 0$ implies that $r = 3r^2(m/u)$, we conclude that $xy > 0$. It therefore follows from (8) that $Du < 12$. The pairs (D, d) for which this inequality holds (and which are not considered in [4]) are seen to be $(2, 4)$, $(3, 9)$, $(5, 25)$, $(6, 4)$, $(6, 9)$, $(6, 36)$, $(10, 4)$, $(10, 25)$, $(10, 100)$, and $(11, 121)$. In each case it is immediate that (7), (8), and (9) cannot all be satisfied. We prove this for the pair $(6, 9)$, the other proofs being similar: here we obtain $x^2 + 30yz = 1$, $z^2 + 2xy = 12$, and $15y^2 + 2xz = -30$. These lead to $xy < 6$ and hence $|30yz| < 24$.

If $d < 0$ then we see from (7), (8) that $yz \leqq 0$ and $xy < 0$. Since $z = 0$ implies that $9de^3 = 16$, we conclude that $xz > 0$. Hence $|y| < (3D/|d|)^{1/2}$, while (9) implies that $8|xy| \geqq (a/v^2) - 12e$. Combining these with (6) and Lemma 1 and assuming that $D \geqq 5$ we obtain after a straightforward calculation that $(D-1)u < 13$. It then follows directly that none of the thirteen pairs (D, d) for which this last inequality holds can satisfy (7), (8), and (9).

Considering separately $D \leqq 5$, $d < 0$, we obtain η directly by the algorithm of Berwick [1] which has been programmed by the author. The results show that for $(D, d) = (2, -2)$, $(3, -9)$, $(4, -4)$, and $(4, -2)$ we have $\varepsilon = \zeta = \eta$, for $(2, -4)$ $\varepsilon = \zeta^3 = \eta^3$, and for $(5, -25)$ $\varepsilon = \zeta = \eta^2$. The proposition is therefore true in these cases also.

In general, $d \mid 3D^2$ but we may now assume $d \nmid D^2$ so that $d = 3d_0$, where $d_0 \mid D^2$. Replacing d by d_0 and proceeding as before, we obtain for $d > 0$, $Du < 4$, and for $d < 0$, $(D-3)u < 9$. Here it is easily seen that only in the cases $(D, d) = (2, -6)$, $(1, 3)$ is ε a square in R.

PROPOSITION 2. *If $(D, d) \neq (2, -4)$ then ε is not a cube in R.*

Proof. $\varepsilon^{1/3} = (\omega - D)/\sqrt[3]{d} \in K$ if and only if $\sqrt[3]{d} \in K$. Since d is cubefree, this would imply that $\sqrt[3]{d}$ generates K. It then follows by considering traces that $|d| = a$ or $\bar{a}$, which forces us to conclude that $(D, d) = (2, -4)$.

PROPOSITION 3. *If $(D, d) \neq (2, -6)$, $(2, -4)$, $(1, 3)$, or $(5, -25)$, then $\varepsilon = \zeta$.*

Proof. Let $\zeta = (x + y\omega + z\bar{\omega})$ and suppose that $\varepsilon = \zeta^t$, $t > 1$.

By Propositions 1 and 2, t is not divisible by 2 or 3. Hence for $d > 0$ we obtain from (6) and Lemma 1 that $|y| < 1/3 + 2/3\ (6/D^2d)^{1/5} < 1$. For $d < 0$ the cases $D \leqq 5$ have already been considered in the proof of Proposition 1. We may therefore assume that $D > 5$ and hence $|y| < 1/3 + 2/3\ (6D^2/|d|a\omega)^{1/5} < 1$. Thus $y = 0$, and expanding $(x + z\bar{\omega})^t$ we find that

$$1 = \sum_{k=0}^{[t/3]} \binom{t}{3k} x^{t-3k} z^{3k} (\bar{a})^k$$

and since each term in the sum is divisible by x, $x = \pm 1$. But then $1 = N(\pm 1 + z\bar{\omega}) = \pm 1 + \bar{a}z^3$, together with $\bar{a} > 2$, yields a contradiction.

4. *Proof of Theorem* 1. By Proposition 3 we may assume K is of the first kind. Therefore it suffices to prove that $9\varepsilon \neq (x + y\omega + z\bar{\omega})^2$ for integral x, y, z. We see that here $d \mid D^2$, for otherwise $a = D^3 + 3d_0$, where $d_0 \mid D^2$, $3 \nmid d_0$, and since $D^3 \equiv 0, \pm 1 \pmod 9$, we have $a \not\equiv \pm 1 \pmod 9$.

Proceeding as in the proof of Proposition 1 we obtain

$$\begin{cases} x^2 + 2mrvyz = 9 \\ az^2/v^2r^2 + 2xy = 27e \\ rvy^2 + 2xz = -3r(D/uv)\,. \end{cases} \tag{10}$$

Since $3 \mid r$ implies that $a = mn^2 \equiv 0 \pmod 9$ we again find that $(x, r) = 1$. Here, we obtain for $d > 0$, $Du < 108$, while if $d < 0$ and $D > 5$ we have $(D - 1)u < 123$. The result now follows by individually considering each of the fifty-three pairs (D, d) to which these inequalities give rise, the equations (10) having no solution in these cases.

REFERENCES

1. W. E. H. Berwick, *The classification of ideal numbers that depend on a cubic irrationality*, Proc. London Math. Soc., (2) **12** (1913) 393-429.

2. T. Nagell, *Über die Einheiten in reinen kubischen Zahlkörpern*, Videnskapsselskapet Skrifter, I. Mat.-naturv. Klasse 1923, Nr. 11.

3. ———, *Über einige kubische Gleichungen mit zwei Unbestimmten*, Math. Zeitschr. **24** (1926) 422-447.

4. H. Stender, *Über die Grundeinheit fur spezielle unendlich Klassen reiner kubischer Zahlkörper*, Abh. Math. Sem. Univ. Hamburg, **33** (1969) 203-215.

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