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LENGTH OF PERIOD OF SIMPLE CONTINUED FRACTION EXPANSION OF $\sqrt{d}$

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In this article, the length, $p(d)$, of the period of the simple continued fraction (s.c.f.) for $\sqrt{d}$ is discussed, where d is a positive integer, not a perfect square. In particular, it is shown that

$$p(d) < d^{1/2+\log 2/\log\log d+O(\log\log\log d/(\log\log d)^2)} .$$

In addition, some properties of the complete quotients of the s.c.f. expansion of $\sqrt{d}$ are developed.

It is well known that the s.c.f. expansion for $\sqrt{d}$ is periodic if d is a positive integer, not a perfect square. Throughout this paper, $p(d)$ will denote the length of this period. It is shown in [2] (page 294), that $p(d) < 2d$. Computer calculation of $p(d)$ originally suggested that $p(d) \leqq 2[\sqrt{d}]$. This was shown to be false for $d = 1726$, for which $p(d) = 88$ and $2[\sqrt{d}] = 82$. Further calculation revealed 3 more counterexamples for $d \leqq 3000$. They were $p(2011) = 94$ while $2[\sqrt{2011}] = 88$, $p(2566) = 102$ while $2[\sqrt{2566}] = 100$, and $p(2671) = 104$ while $2[\sqrt{2671}] = 102$.

This suggests as a conjecture that

$$p(d) = O(d^{1/2}) \quad \text{and} \quad p(d) \neq o(d^{1/2}) .$$

It follows from the corollary to Theorem 2 that

$$p(d) = O(d^{1/2+\varepsilon})$$

or more precisely, that

$$p(d) < d^{1/2+\log 2/\log\log d+O(\log\log\log d/(\log\log d)^2)} .$$

We will need the following results which are given in or follow from §§ 7.1–7.4 and 7.7 of [1].

(1) Any periodic s.c.f. is a quadratic irrational number, and conversely.

(2) The s.c.f. expansion of the real quadratic irrational number $(a + \sqrt{b})/c$ is purely periodic if and only if $(a + \sqrt{b})/c > 1$ and $-1 < (a - \sqrt{b})/c < 0$.

(3) Any quadratic irrational number ξ_0 may be put in the form $\xi_0 = (m_0 + \sqrt{d})/q_0$, where d, m_0, and q_0 are integers, $q_0 \neq 0$, $d \geqq 1$, d is not a perfect square, and $q_0 \mid (d - m_0^2)$. We may then define infinite

sequences m_i, q_i, a_i, and ξ_i by the equations $\xi_i = (m_i + \sqrt{d})/q_i$, $a_i = [\xi_i]$, $m_{i+1} = a_i q_i - m_i$, and $q_{i+1} = (d - m_{i+1}^2)q_i$. Then, for $i \geqq 0$, m_i, q_i, and a_i are integers, $q_i \neq 0$, and $q_i \mid (d - m_i^2)$. Also, for $i \geqq 1$, a_i and ξ_i are positive.

(4) In the notation of (3) above, we have for $i \geqq 0$, $\xi_i = \langle a_i, a_{i+1}, a_{i+2}, \cdots\rangle$. In particular, $\xi_0 = \langle a_0, a_1, a_2, \cdots\rangle$.

(5) There is a positive integer N such that, if $i > N$, then $q_i > 0$.

(6) There exist nonnegative integers j and k such that $j < k$, $m_j = m_k$, and $q_j = q_k$. We may choose j to be the smallest integer such that for some $k > j$, $m_j = m_k$ and $q_j = q_k$. We may then choose k to be the smallest integer such that $j < k$, $m_j = m_k$, and $q_j = q_k$. Then, if t is a nonnegative integer, then $m_{j+t} = m_{k+t}$, $q_{j+t} = q_{k+t}$, $a_{j+t} = a_{k+t}$, and $\xi_{j+t} = \xi_{k+t}$. Therefore, if $i < j$, then

$$\xi_i = \langle a_i, a_{i+1}, \cdots, a_{j-1}, \overline{a_j, \cdots, a_{k-1}}\rangle ,$$

while if $i \geqq j$, then $\xi_i = \langle\overline{a_{i'}, a_{i'+1}, \cdots, a_{k-2}, a_{k-1}, a_j, a_{j+1}, \cdots, a_{i'-1}}\rangle$, where i' is the integer such that $j \leqq i' \leqq k-1$ and $i \equiv i' \pmod{(k-j)}$. In particular, $\xi_0 = \langle a_0, a_1, \cdots, a_{j-1}, \overline{a_j, \cdots, a_{k-1}}\rangle$.

(7) If $\xi_0 = \sqrt{d}$ then we may take $m_0 = 0$ and $q_0 = 1$ in (3). In (6), we have $j = 1$ and $k = r + 1$ for some positive integer r. Then $\xi_0 = \langle a_0, \overline{a_1, \cdots, a_r}\rangle$ and, for $i \geqq 1$, $\xi_i = \langle\overline{a_{i'}, \cdots, a_r, a_1, \cdots, a_{i'-1}}\rangle$, where i' is such that $1 \leqq i' \leqq r$ and $i \equiv i' \pmod{r}$.

(8) In (7), if $t \geqq 0$ then $m_{1+t} = m_{r+1+t}$, $q_{1+t} = q_{r+1+t}$, $a_{1+t} = a_{r+1+t}$, and $\xi_{1+t} = \xi_{r+1+t}$. It follows from this that if $i \geqq 1$ and $s \geqq 0$, then $m_{i+rs} = m_i$, $q_{i+rs} = q_i$, $a_{i+rs} = a_i$, and $\xi_{i+rs} = \xi_i$.

Throughout this paper it will be assumed that d is a positive integer, not a perfect square. The period r of the s.c.f. expansion of $\sqrt{d}$ will be denoted by $p(d)$.

2. Preliminary results.

In this section, m_i, q_i, a_i, and ξ_i will refer to the sequences defined in (3)-(8) above, with $\xi_0 = \sqrt{d}$, $m_0 = 0$, and $q_0 = 1$.

LEMMA 1. *If $i \geqq 0$, then $q_i > 0$.*

Proof. From (5), there is an N such that, if $i > N$, then $q_i > 0$. Suppose $i \geqq 1$. Then there is an integer s such that $i + rs > N$. By (8), $q_i = q_{i+rs}$. But since $i + rs > N$, $q_{i+rs} > 0$. Therefore, $q_i > 0$. That is, if $i \geqq 1$, we are done. Since $q_0 = 1$, this result holds for $i = 0$ also, so the proof is complete.

THEOREM 1. *If $i \geqq 1$, then $0 < m_i < \sqrt{d}$ and $\sqrt{d} - m_i < q_i < \sqrt{d} + m_i$.*

Proof. From (7), if $i \geqq 1$, then $\xi_i = \langle\overline{a_{i'}, \cdots, a_r, a_1, \cdots, a_{i'-1}}\rangle$ so the s.c.f. for ξ_i is purely periodic. But $\xi_i = (m_i + \sqrt{d})/q_i$, so from (2), $(m_i + \sqrt{d})/q_i > 1$ and $-1 < (m_i - \sqrt{d})/q_i < 0$. Since, from Lemma 1, $q_i > 0$, we obtain $m_i + \sqrt{d} > q_i$ and $-q_i < m_i - \sqrt{d} < 0$. This yields $m_i < \sqrt{d}$ and $\sqrt{d} - m_i < q_i < \sqrt{d} + m_i$.

Thus $-m_i < m_i$ and $m_i > 0$, so the proof is complete.

For given d, let $T = T(d)$ be the set of ordered pairs (m, q) which satisfy $m < \sqrt{d}, \sqrt{d} - m < q < \sqrt{d} + m$, and $q \mid (d - m^2)$. That is, $T = \{(m, q) \mid m < \sqrt{d}, \sqrt{d} - m < q < \sqrt{d} + m, q \mid (d - m^2)\}$. Let $g(d) = c(T)$, the cardinality of T.

From (6) and (7) of Section 1, if $1 \leqq i < l \leqq r$ then $(m_i, q_i) \neq (m_l, q_l)$. Therefore, the set $U = \{(m_i, q_i) \mid 1 \leqq i \leqq r\}$ has exactly r elements. By Theorem 1, $U \subset T$ so $r = c(U) \leqq c(T) = g(d)$. Since $r = p(d)$, we obtain

LEMMA 2. $p(d) \leqq g(d)$.

3. An upper bound on $g(d)$.

THEOREM 2. $g(d) < d^{1/2+\log 2/\log\log d + O(\log\log\log d/(\log\log d)^2)}$.

Proof.

$$
\begin{aligned}
&g(d)\\
&= c(T) = c(\{(m, q) \mid 0 < m < \sqrt{d}, \sqrt{d} - m < q < \sqrt{d} + m, q \mid (d - m^2)\})\\
&= \sum_{m=1}^{[\sqrt{d}]} c(\{q \mid \sqrt{d} - m < q < \sqrt{d} + m, q \mid d - m^2\}) \leqq \sum_{m=1}^{[\sqrt{d}]} \tau(d - m^2) ,
\end{aligned}
$$

where $\tau(n)$ denotes the number of divisors of n.

It is shown in [3] that

$$\log \tau(N) < \frac{\log 2 \log N}{\log\log N} + O\left(\frac{\log N \log\log\log N}{(\log\log N)^2}\right).$$

It follows that

$$\tau(N) < N^{\log 2/\log\log N + O(\log\log\log N/(\log\log N)^2)} .$$

Therefore, for $m = 1, 2, \cdots, [\sqrt{d}]$,

$$\tau(d - m^2) < d^{\log 2/\log\log d + O(\log\log\log d/(\log\log d)^2)} ,$$

and the theorem follows by summing this expression over the $[\sqrt{d}] < d^{1/2}$ values of m.

COROLLARY. $p(d) < d^{1/2+\log 2/\log\log d + O(\log\log\log d/(\log\log d)^2)}$.

Proof. This follows immediately from Lemma 2 and Theorem 2.

4. A lower bound on the order of $g(d)$. Theorem 2 shows that $g(d) = O(d^{1/2+\varepsilon})$ for any $\varepsilon > 0$. It will follow from Theorem 3 that $g(d) \neq o(d^{1/2})$. Thus, Theorem 2 is almost best possible. This, however, is not necessarily true of its corollary.

THEOREM 3. *There exist infinitely many positive integers d for which $g(d) > \sqrt{d}$.*

Proof. Let n be an arbitrary positive integer. Let

$$S = \{(m, q) \mid q - n \leqq m,\ n + 1 - q \leqq m,\ m \leqq n\}\ .$$

Then, for $n^2 + 1 \leqq d \leqq n^2 + 2n$, $T(d) = \{(m, q) \mid (m, q) \in S$ and $d \equiv m^2 (\mathrm{mod}\ q)\}$. Given $(m, q) \in S$, let $f(m, q)$ denote the number of integers d for which $n^2 + 1 \leqq d \leqq n^2 + 2n$ and $d \equiv m^2 (\mathrm{mod}\ q)$. Then $\sum_{d=n^2+1}^{n^2+2n} g(d) = \sum_{(m,q)\in S} f(m, q)$. However, it is easily seen that if $(m, q) \in S$, then $f(m, q) \geqq [2n/q]$. Also, note that $S = \{(m, q) \mid 1 \leqq q \leqq n,\ n + 1 - q \leqq m \leqq n\} \cup \{(m, q) \mid n + 1 \leqq q \leqq 2n,\ q - n \leqq m \leqq n\}$. If $1 \leqq q \leqq n$, then $[2n/q] > 2n/q - 1$. If $n + 1 \leqq q \leqq 2n$, then $[2n/q] = 1$. Therefore,

$$\begin{aligned}\sum_{d=n^2+1}^{n^2+2n} g(d) &= \sum_{(m,q)\in S} f(m, q) \geqq \sum_{(m,q)\in S}\left[\frac{2n}{q}\right] = \sum_{\substack{1\leqq q\leqq n\\ n+1-q\leqq m\leqq n}}\left[\frac{2n}{q}\right] + \sum_{\substack{n+1\leqq q\leqq 2n\\ q-n\leqq m\leqq n}}\left[\frac{2n}{q}\right] \\ &= \sum_{q=1}^{n} q\left[\frac{2n}{q}\right] + \sum_{q=n+1}^{2n} (2n + 1 - q)\left[\frac{2n}{q}\right] > \sum_{q=1}^{n} q\left(\frac{2n}{q} - 1\right) \\ &\quad + \sum_{q=n+1}^{2n} (2n + 1 - q) = 2n^2\ .\end{aligned}$$

It follows from this inequality that at least one of the $2n$ numbers $g(d)$ with $n^2 + 1 \leqq d \leqq n^2 + 2n$ must be greater than $(2n^2/2n) = n$. Since $n = [\sqrt{d}]$ for any such d, there is a d such that $n = [\sqrt{d}]$ and $g(d) > \sqrt{d}$. Since this is true for any positive n, the theorem follows.

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