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RADIAL AVERAGING TRANSFORMATIONS WITH VARIOUS METRICS

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In [5] one of the authors introduced the notion of a radial averaging transformation of domains in the plane, which was based on the metric given by the line element $ds^2 = (1/r^2)(dx_1^2 + dx_2^2)$ where (x_1, x_2) are the cartesian and (r, θ) are the polar coordinates. This transformation is useful in obtaining estimates for conformal capacity of condensers and for conformal radius of domains. In this paper we discuss averaging transformations in m-dimensional spaces ($m \geqq 2$), based on various metrics of the form $ds^2 = g^2(r) \sum_{i=1}^{m} (dx_i)^2$, where $g(r)$ is a positive, continuous function of $r(0 < r < \infty)$.

With the help of these transformations we are able to obtain estimates for energy integrals of the form

$$\int_{\Omega} |\nabla F|^2 g(r) r^{3-m} dx \qquad (dx = dx_1 dx_2 \cdots dx_m) .$$

These estimates can be used to compare capacities of different condensers filled with nonhomogeneous dielectric [cf. Kühnau [3] and the literature cited there]. As a further application we derive inequalities for conformal capacity and conformal radius in the plane and similar results in higher dimensional spaces. In this direction we have results for the case where $g(r) = r^{\beta}\ \beta \geqq m - 3$. They include the symmetrization results obtained by Szegö in [7]. The method presented seems to be quite general, and we believe that it might be employed also with other classes of metrics g.

1. Estimates for energy integrals. Let $g(r)$ be a positive continuous function for $0 < r < \infty$ and let $G(r)$ be a primitive of g. $(r, \theta_1, \cdots, \theta_{m-1})$ are the polar coordinates defined in the following way:

$$x_1 = r\cos\theta_1,\ x_2 = r\sin\theta_1\cos\theta_2,\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3,$$
$$\cdots\cdots$$
$$x_n = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{m-2}\sin\theta_{m-1}$$

where $0 \leqq \theta_i \leqq \pi$ for $i = 1, \cdots, m-2$ and $-\pi \leqq \theta_{m-1} \leqq \pi$. Let ρ be a fixed positive number and set:

$$\begin{cases} u = G(r) - G(\rho) & \\ v_i = \theta_i & i = 1, \cdots, m-1 . \end{cases} \tag{1.1}$$

Let Ω be an open set in R^m which does not contain the sphere $\{x; |x| \leqq \rho\}$ and the hyperplanes $\theta_i = 0\ i = 1, 2, \cdots, n-2$. Then for $F(x) \in C^1(\Omega)$ $[x = (x_1, x_2, \cdots, x_m)]$ we have

$$|\nabla F|^2 = F_r^2 + \sum_{i=1}^{m-1} r^{-2} f_i F_{\theta_i}^2 \tag{1.2}$$

where $f_i = f_i(\theta_1, \theta_2, \cdots, \theta_{i-1})$ if $i = 2, 3, \cdots, m-1$ and $f_1 = 1$.

If $d\omega = \Phi(\theta_1, \cdots, \theta_{m-2}) d\theta [d\theta = d\theta_1, \cdots, d\theta_{m-1}]$ denotes the surface element of the unit sphere, then the volume element dx is given by $dx = (r^{m-1}/g(r)) du d\omega$. Hence

$$\int_\Omega |\nabla F|^2 g(r) r^{3-m} dx = \int_{\tilde{\Omega}} \left[\tilde{F}_u^2 r^2 g^2(r) + \sum_{i=1}^{m-1} f_i \tilde{F}_{v_i}^2\right] du d\omega \tag{1.3}$$

where $\tilde{\Omega}$ is the image of Ω by (1.1), and

$$\tilde{F}(u, v_1, \cdots, v_{m-1}) = F(x_1(r, \theta), \cdots, x_n(r, \theta))$$

where $\theta = (\theta_1, \cdots, \theta_{m-1})$.

We denote

$$q^2 = r g(r) \ . \tag{1.4}$$

DEFINITION 1.1. Let D be an open set containing $\{x; |x| \leqq \rho\}$ and $D_\rho = D - \{x; |x| \leqq \rho\}$. Denote

$$l(\theta) = \int_{E\theta} g(r) dr \ , \tag{1.5}$$

where

$$E_\theta = D_\rho \cap \{x = (r, \tilde{\theta}); \tilde{\theta} = \theta\} \ .$$

we define

$$R(\theta) = G^{-1}[l(\theta) + G(\rho)] \ . \tag{1.6}$$

Clearly $R(\theta)$ does not depend on ρ. Let

$$\begin{aligned} D^* = \{x = (r, \theta); 0 \leqq r < R(\theta), 0 \leqq \theta_i \leqq \pi \ i = 1, \cdots, m-2, \\ -\pi \leqq \theta_{m-1} \leqq \pi\} \ . \end{aligned} \tag{1.7}$$

The transformation $D \to D^*$ will be called the *radial concentration with metric g*.

EXAMPLE. Suppose that D is compact and that the rays $\theta =$ constant intersect ∂D in a finite number of points $r_1(\theta) < r_2(\theta) < \cdots < r_{2\nu+1}(\theta)$. In the case where $g(r) = r^{\beta-1}$, we have

$$R(\theta) = \begin{cases} \left\{r_1^\beta + \sum_{j=1}^{2\nu} (r_{2j+1}^\beta - r_{2j}^\beta)\right\}^{1/\beta} & \text{if } \beta \neq 0 \\ r_1 \prod_{j=1}^{2\nu} \dfrac{r_{2j+1}}{r_{2j}} & \text{if } \beta = 0 \ . \end{cases} \tag{1.8}$$

REMARK. If $\tilde{D}_\rho$ is the image of D_ρ by (1.1) and $\tilde{D}_\rho^*$ is the image of D_ρ^* by (1.1), then $\tilde{D}_\rho^*$ is obtained from $\tilde{D}_\rho$ by:

$$(1.9) \quad \tilde{D}_\rho^* = \{(u, v) \mid 0 < u < l(v), 0 \leqq v_i \leqq \pi \quad \text{if} \quad i = 1, \cdots, n-2\} \\ -\pi \leqq v_{n-1} \leqq \pi$$

where $v = (v_1, \cdots, v_{n-1})$ and

$$(1.10) \qquad l(v_0) = \text{linear measure of } (\tilde{D}_\rho^* \cap \{v = v_0\}) .$$

DEFINITION 1.2. Let D be a domain which contains $|x| \leqq \rho$. Let $F(x)$ be a continuous function in R^m such that $F \equiv 0$ outside D and $F \equiv 1$ in a compact subset of D (denoted by E) such that $\{|x| \leqq \rho\} \subset E$. Suppose that in $\Omega = D - E$, $0 < F < 1$ and that on every ray through the origin F takes every value λ, $(0 < \lambda < 1)$, only a finite number of times. Let

$$(1.11) \qquad D_\lambda(F) = \{(x) \mid F(x) > \lambda\} , \qquad (0 \leqq \lambda < 1) .$$

Let D_λ^* be the g-radial concentration of D_λ and let F^* be defined as follows:

$$(1.12) \qquad F^* = \begin{cases} 1 & \text{in} \quad E^* \\ \lambda & \text{on the boundary of} \quad D_\lambda^* , \qquad 0 < \lambda < 1 \\ 0 & \text{outside} \quad D^* . \end{cases}$$

(Here E^* is defined as in Definition 1 except that in (1.7) $0 \leqq r \leqq R(\theta)$.) Then F^* will be called the *radial concentration of F, with metric g*.

The following results are proved exactly in the same way as in [5]:

(i) D^* is a starlike domain.

(ii) E^* is a compact, connected, starlike set.

(iii) If F is continuous then F^* is continuous.

(iv) If F is continuous in R^m and Lip in every compact subset of $D - E$, then F^* has the same properties with respect to $D^* - E^*$.

Also the following basic result is obtained by essentially the same method as in [5].

LEMMA 1.1. *Let D, Ω, F be as in Definition 1.2. Suppose also that $F \in C^1(\Omega)$, that $F \in C^0$ in R^m, and that on every ray $\theta =$ constant, the set of points in Ω where $(\partial F/\partial r) = 0$ is at most a finite set. Finally suppose that:*

$$p(u) = [rg(r)]_{r=r(u)} \textit{ is convex or monotone} .$$

Then we have:

$$(1.13) \qquad \int_{\Omega^*} |\nabla F^*|^2 g(r) r^{3-m} dx \leqq \int_\Omega |\nabla F|^2 g(r) r^{3-m} dx$$

where $\Omega^* = D^* - E^*, \Omega = D - E$.

We shall sketch the main ideas of the proof for the case $\Omega \subset R^2$. It is not difficult to see that the same proof works also for $\Omega \subset R^m (m > 2)$.

Proof. Consider the level line $\tilde{F}(u, v) = \lambda$ where $0 < \lambda < 1$ and $v = v_1$. Except for a finite number of λ the rays $v =$ constant intersect the level line in a finite number of points (u_j, v) with

$$u_1(\lambda, v) < u_2(\lambda, v) < \cdots < u_{2\nu+1}(\lambda, v) .$$

Actually we shall show that not only the total energy integral is diminished by radial concentration but even each of its infinitesimal parts between two level lines given by

$$\tilde{F}(u, v) = \lambda \quad \text{and} \quad \tilde{F}(u, v) = \lambda + d\lambda .$$

Introducing λ, v as new variables we find that this infinitesimal part is given by

$$\int_{v=-\pi}^{\pi} \Big(\sum_{j=1}^{2\nu+1} \Big\{ \frac{p^2(u_j)}{|\partial u_j/\partial\lambda|} + \frac{(\partial u_j/\partial v)^2}{|\partial u_j/\partial\lambda|} \Big\} \Big) d\lambda dv . \tag{1.14}$$

Because of the Schwarz inequality, (1.14) is greater than

$$\int_{v=-\pi}^{\pi} \frac{\Big[\sum_{j=1}^{2\nu+1} p(u_j)\Big]^2 + \Big[\sum_{j=1}^{2\nu+1} |\partial u_j/\partial v|\Big]^2}{\sum_{j=1}^{2\nu+1} |\partial u_j/\partial\lambda|} d\lambda dv .$$

From the monotonicity or convexity of $p(u)$ it follows [cf. 8]

$$p\Big(\sum_{j=1}^{2\nu+1} (-1)^{j+1} u_j\Big) \leqq \sum_{j=1}^{2\nu+1} p(u_j) .$$

Thus, by Minkowski's inequality we have that (1.14) is greater than

$$\int_{v=-\pi}^{\pi} \frac{\Big[p\Big(\sum_{j=1}^{2\nu+1} (-1)^{j+1} u_j\Big)\Big]^2 + \Big|\sum_{j=1}^{2\nu+1} (-1)^{j+1} \partial u_j/\partial v\Big|^2}{\sum_{j=1}^{2\nu+1} (-1)^{j+1} \partial u_j/\partial\lambda} d\lambda dv . \tag{1.15}$$

Since $(\partial u_j/\partial\lambda)$ has alternating signs, the denominator of (1.15) does not vanish. The assertion follows immediately from (1.15). We now define the radial averaging transformation with metric g in the same way as it was defined in [5] for the logarithmic measure.

DEFINITION 1.3. Let $\{D_1, \cdots, D_n\} = \mathscr{D}$ be a family of open sets in R^m, each containing the sphere $|x| \leqq \rho (\rho$ arbitrary real number).

Let $A = \{a_j\}_{j=1}^n$ where $a_j > 0$ and $\sum_{j=1}^n a_j = 1$. Let $l_j(\theta)$ be defined as in Definition 1.1 for D_j. Then set

$$l^*(\theta) = \sum_{j=1}^n a_j l_j(\theta) , \tag{1.16}$$

$$R^*(\theta) = G^{-1}[l^*(\theta) + G(\rho)] , \tag{1.17}$$

and finally define D^* as in (1.7) with $R(\theta)$ replaced by $R^*(\theta)$. We shall denote $D^* = \mathscr{R}_{g,A}(\mathscr{D})$, and call the transformation $\mathscr{D} \to D^*$ the *radial averaging transformation with metric g*.

In the special case where D_i $i = 1,2, \cdots, n$ are obtained from a fixed domain D by a combination of simple transformations such as rotations around the origin and reflections with respect to a plane through the origin, then the transformation $R_{A,g}$ becomes a symmetrization. For example, in R_3 with $g = 1$ and $a_j = 1/n$, $(j = 1, \cdots, n)$, we obtain, by rotations, the symmetrization given by:

$$R^*(\theta_1, \theta_2) = \frac{1}{n} \sum_{j=1}^n R(\theta_1 + \beta_j, \theta_2 + \gamma_j) ,$$

where β_j, γ_j are arbitrary numbers. D^* is defined as in (1.7) with R replaced by R^*.

It might be observed that the radial concentration (1.7) is a particular case of the radial averaging transformation, i.e., the case $n \equiv 1$.

DEFINITION 1.4. Let $\mathscr{D}$ and A be defined as above. Suppose that D_j is bounded. Let E_j be a compact subset of D_j, which contains $\{|x| \leqq \rho\}$. Let $\mathscr{F} = \{F_1, \cdots, F_n\}$ be a set of functions such that each F_j has the properties described in Definition 1.2 with respect to D_j and E_j. We define $D_\lambda(F_j)$ as in (1.10) and

$$D_\lambda^* = \mathscr{R}_{g,A}(D_\lambda(F_1), \cdots, D_\lambda(F_n)) .$$

Finally we define F^* as in (1.12). The transformation $\mathscr{F} \to F^*$ will be called a *radial averaging transformation on $\mathscr{F}$ with metric g*. We shall denote $F^* = \mathscr{R}_{g,A}(\mathscr{F})$.

The analogous properties to (i)-(iv) for the radial averaging transformation are verified exactly as in [5]. Also the following result is proved essentially in the same way as the parallel result in [5].

THEOREM 1.1. *Let $\mathscr{D}$, $\mathscr{F}$ be as in Definition 1.4. Suppose also that each F_j has the properties described in Lemma 1.1 with respect to D_j, E_j. Finally suppose that $p(u)$ (defined as in Lemma 1.1) is convex. Then we have:*

$$\int_{\Omega^*} |\nabla F^*|^2 \frac{g(r)}{r^{m-3}} dx \leqq \sum_{j=1}^n a_j \int_{\Omega_j} |\nabla F_j|^2 \frac{g(r)}{r^{m-3}} dx \tag{1.18}$$

where $D^* = \mathscr{R}_{g,A}(\mathscr{D}), E^* = \mathscr{R}_{g,A}(E_1, \cdots, E_n)\, \Omega^* = D^* - E^*, \Omega_j = D_j - E_j,$.

REMARKS (1) Lemma 1.1 is contained in Theorem 1.1 for the particular case $n = 1$.

(2) Szegö [7] proved (1.18) for $g(r) = r^{m-3}$ and $a_j = 1/n$ $j = 1, \cdots, n$ and for D_j and E_j $j = 1, \cdots, n$ obtained from D and E by certain rotations. He assumed further that Ω is starlike and that F has starlike level surfaces.

(3) If $g(r) = 1/r$ (i.e., the logarithmic metric) and $m = 2$, then the results obtained here coincide with the results of [5, 4]. In this case we can obtain from (1.18) inequalities for the (conformal) capacities of cylindrical condensers with a homogeneous dielectric. By the same method we can derive from (1.18) inequalities for capacities of condensers with an inhomogeneous dielectric.

2. Estimates for capacities. In this section we describe a method by which the results of Theorem 1.1, with various metrics g, can be used to derive inequalities for condensers with homogeneous dielectrics.

Let D be a bounded domain and E a compact subset of D which contains the sphere $\{|x| \leqq \rho\}$. We denote as usual $\Omega = D - E$. We assume that the boundary of Ω is sufficiently smooth so that Green's theorem may be used. Let C be the "inner boundary" of Ω, i.e., $\bar{\Omega} \cap E$.

Consider a function ω which is continuous in R^m such that $\omega \in C^1(\Omega)$, $\omega \equiv 0$ outside D and $\omega \equiv 1$ in E. Let h be defined in Ω by $h = \omega/q$, where q is a positive function of r $(0 < r < \infty)$ such that $q \in C^2(0, \infty)$. Because of the identity

$$(\text{grad}\, uv)^2 = u^2 \,\text{grad}^2 v + \text{div}\,(v^2 u \,\text{grad}\, u) - v^2 u \Delta u$$

we have

$$\int_\Omega |\nabla \omega|^2 dx = \int_\Omega |\nabla h|^2 q^2 dx - \int_\Omega h^2 q \Delta q dx - \oint \frac{\omega^2}{q} \frac{\partial q}{\partial n} dx \tag{2.1}$$

[n inner normal, ds surface element of C].

We now restrict our attention to the case where $E = \{|x| \leqq \rho\}$ in which case C is the sphere $|x| = \rho$. We also assume that ω is harmonic in Ω and that $q(r)$ is analytic for $0 < r < \infty$.

Let us apply the transformation of radial concentration with metric g, where $q^2 = rg(r)$, to D and h. We denote the resulting domain and function by D^*, h^* respectively and we set $\Omega^* = D^* - E$. (In this case $E^* = E$.) It is easily verified that

$$\int_\Omega h^2 g(r) r^{1-m} dx = \int_{\Omega^*} h^{*2} g(r) r^{1-m} dx\,. \tag{2.2}$$

Now suppose that q is chosen in such a manner that:

(2.3) (i) $q\Delta q = cg(r)r^{1-m}$ where $q^2 = r^{3-m}g(r)$ and c is an arbitrary constant.

(ii) q is positive and nondecreasing.

(iii) $p(u) = [rg(r)]_{r=r(u)}$ is convex (where $r(u) = G^{-1}[u + G(\rho)]$ see (1.1)).

Since ω is harmonic in Ω we have $0 < \omega < 1$ in Ω and since q is nondecreasing $0 < h < (1/q(\rho))$ in Ω with $h = (1/q(\rho))$ on C and $h = 0$ on the boundary of D. Furthermore since h is an analytic function of r on the intersection of any ray $\Omega =$ constant with Ω, it is clear that h satisfies all the assumptions of Lemma 1.1 (if Ω has a smooth boundary). Hence we obtain:

$$\int_{\Omega^*} |\nabla h^*|^2 q^2 dx \leqq \int_{\Omega} |\nabla h|^2 q^2 dx . \tag{2.4}$$

By (2.1), (2.2), (2.3), and (2.4) we get

$$\int_{\Omega} |\nabla \omega|^2 dx \geqq \int_{\Omega^*} |\nabla h^*|^2 q^2 dx - \int_{\Omega^*} h^{*2} q\Delta q dx - \oint_C \frac{1}{q}\frac{\partial q}{\partial n} ds . \tag{2.5}$$

But, again by (2.1), the right-hand side of (2.5) is equal to:

$$\int_{\Omega^*} |\nabla \omega^{**}|^2 dx$$

where $\omega^{**} = h^* q$; note that $\omega^{**} = 1$ on C and $\omega^{**} = 0$ on the boundary of D^*. Also, since h^* is Lip in every compact subset of Ω^*, so is ω^{**}. Hence ω^{**} is an admissible function for the variational definition of the capacity of the condenser Ω^*; if ω' is harmonic in Ω^* and $\omega' = 1$ on C and $\omega' = 0$ on the boundary of D^*, then:

$$I(\Omega^*) = \int_{\Omega^*} |\nabla \omega'|^2 dx \leqq \int_{\Omega^*} |\nabla \omega^{**}|^2 dx , \tag{2.6}$$

where $I(\Omega^*)$ is the capacity of Ω^*. (As a reference for the facts quoted here see for instance [1] and [6].) From (2.5) and (2.6) we finally obtain

$$I(\Omega^*) \leqq I(\Omega) , \tag{2.7}$$

where $I(\Omega)$ is the *m-dimensional capacity of* Ω.

To sum up this result we state

LEMMA 2.1. *Let D be a bounded domain in R^m containing the sphere $\{|x| < \rho\}$. Let $\Omega = D - \{|x| \leqq \rho\}$. Let $q(r)$ be a positive analytic function of r for $0 < r < \infty$, satisfying* (2.3). *Let D^* denote the domain obtained by radial concentration with metric g from the domain D. We assume that D^* is not the entire space R^m. Then*

$$I(\Omega^*) \leqq I(\Omega) \tag{2.8}$$

where $\Omega^* = D^* - \{|x| \leqq \rho\}$.

REMARK. In the previous discussion we assumed that the boundary of D is smooth; but the result of Lemma 2.1 is obtained for general domains D by the standard method of approximating a given domain by a sequence of domains with smooth boundary.

Using a result of Pólya-Szegö [6] on the connection between capacity and conformal radius, the following result is obtained as an immediate consequence of Lemma 2.1:

LEMMA 2.2. *Let D be a domain in the plane containing the origin and let D^* be the domain obtained from D by radial concentration with metric g. Suppose that g is analytic for $0 < r < \infty$ and satisfies* (2.3). *Denote by r_0 (resp. r_0^*) the conformal radius of D (resp. D^*) at the origin.* (*We assume that D^* is not the entire plane.*) *Then*:

$$r_0 \leqq r_0^* . \tag{2.9}$$

By the same arguments used in the proof of Lemma 2.1, one obtains the following result (based on Theorem 1.1):

THEOREM 2.1. *Let $\mathscr{D} = \{D_1, \cdots, D_n\}$ be a family of domains in R^m each of which does not contain $z = \infty$ and contains the sphere $|x| \leqq \rho$. Let q be a positive analytic function of r for $0 < r < \infty$, satisfying* (2.3). *Let $D^* = \mathscr{R}_{g,A}(\mathscr{D})$ and suppose that D^* is not the entire space. Denote*: $\Omega_j = D_j - \{|x| \leqq \rho\}\Omega^* = D^* - \{|x| \leqq \rho\}$. *Then*:

$$I(\Omega^*) \leqq \sum_{j=1}^{n} a_j I(\Omega_j) . \tag{2.10}$$

In the particular case where $g(r) = r^{m-3}$, (2.10) *holds for general condensers $\Omega_j = D_j - E_j$, $(j = 1, \cdots, n)$, where E_j is a compact subset of D_j containing the sphere $|x| \leqq \rho$.* (*In this case, $\Omega^* = D^* - E^*$ where $E^* = \mathscr{R}_{g,A}(E_1, \cdots, E_n)$.*)

The last statement of the theorem is an immediate consequence of Theorem 1.1, since in this case $q = g(r)/r^{m-3} \equiv 1$.

Again, applying the result of Pólya-Szegö [6] mentioned above (see also Hayman [1, p. 82]) we obtain from Theorem 2.1 (with $m = 2$):

THEOREM 2.2. *Let $\mathscr{D} = \{D_1, \cdots, D_n\}$ be a family of domains in the plane containing the origin and let $D^* = \mathscr{R}_{g,A}(\mathscr{D})$ where $g(r)$ is positive and analytic for $0 < r < \infty$ and satisfies* (2.3). *Denote by $r_{0,j}$ (resp. r_0^*) the conformal radius of D_j (resp. D^*) at the origin.* (*We assume that D^* is not the entire plane.*) *Then*:

$$\min(r_{0,1}, \cdots, r_{0,n}) \leqq r_0^* \, . \tag{2.11}$$

A family of functions q which satisfy (2.3) (i) is established. It depends on the constant c.

(a) Let $c > -((m-2)^2/4)$ and set $\delta = (m-2)^2 + 4c$. Then the general solution of (2.3) (i) is

$$q(r) = Ar^{(2-m+\delta)/2} + Br^{(2-m-\delta)/2} \, .$$

(b) Let $c = -((m-2)^2/4)$. Then

$$q(r) = Ar^{1-m/2} + B(\ln r)r^{1-m/2} \, .$$

(c) Let $c < -((m-2)^2/4)$. Then $\delta = ik(k > 0)$ and

$$q(r) = Ar^{k-m/2}\sin\left(\frac{k}{2}\ln r\right) + Br^{1-m/2}\cos\left(\frac{k}{2}\ln r\right) .$$

Hence the functions

$$\begin{cases} q = r^{\alpha} & \alpha \geqq 0 \\ g = r^{\gamma+m-3} & \gamma = 2\alpha \geqq 0 \end{cases} \tag{2.12}$$

satisfy (2.3). Indeed, since $\alpha \geqq 0$ (2.3) (ii) holds. An easy calculation yields $p(u) = (\gamma + m - 2)u + \rho^{\gamma+m-2}$, which is certainly convex.

Notice that for $g = r^{\beta-1}$ the domain $D^* = \mathscr{R}_{g,A}(\mathscr{D})$ in the plane is given by

$$D^* = \{(r, \theta_1); 0 \leqq r \leqq R^*(\theta_1), -\pi \leqq \theta_1 < \pi\}$$

where

$$R^*(\theta) = \begin{cases} \left\{\sum_{j=1}^{n} a_j R_j^{\beta}\right\}^{1/\beta} & \text{if} \quad \beta \neq 0 \\ \prod_{j=1}^{n} R_j^{a_j} & \text{if} \quad \beta - 0 \end{cases}$$

and R_j is defined in (1.8) with respect to D_j.

We mention also that for $g = r^{\beta-1}(\beta \geqq 0)$ in the plane, the inequality (2.11) may be replaced by:

$$r_0^* \geqq \begin{cases} \left(\sum_{j=1}^{n} a_j r_{0,j}^{\beta}\right)^{1/\beta} & \text{if} \quad \beta > 0 \\ \prod_{j=1}^{n} r_{0,j}^{a_j} & \text{if} \quad \beta = 0 \end{cases} \tag{2.11$'$}$$

The inequality for $\beta = 0$ was proved in [5]. For $\beta > 0$, (2.11)$'$ is obtained from (2.11) as follows. Set $\tilde{D}_j = (1/r_{0,j})D_j$, $(j = 1, \cdots, n)$. Then the conformal radius of $\tilde{D}_j$ at the origin is 1. Let $\tilde{D}^* = \mathscr{R}_{g,A'}(\tilde{D}_1, \cdots, \tilde{D}_n)$, where $A' = \{a_1', \cdots, a_n'\}$ and $a_k' = a_k r_{0,k}^{\beta} / \sum_{j=1}^{n} a_j r_{0,j}^{\beta}$.

If $\tilde{r}^*$ denotes conformal radius of $\tilde{D}^*$ at the origin, we have (by (2.11)) $\tilde{r}^* \geqq 1$. But $\tilde{D}^* = [(\sum_1^n a_j r_{0,j}^\beta)^{-1/\beta}]D^*$. Hence $\tilde{r}^* = r_0^* \quad (\sum_1^n a_j r_{0,j}^\beta)^{-1/\beta} \geqq 1$.

Theorem 2.2 does not hold for $g = r^{-n-1}$ with $n = 2, 3, \cdots$. This is shown by the following counterexample. Let D be the Koebe domain i.e., the entire z-plane cut along the positive real axis from $z = 1/4$ to infinity. Set $D_1 = D, D_2 = e^{i2\pi/3}D$ and $D_3 = e^{i4\pi/3}D$. Let $D^* = \mathscr{R}_{g,A}(D_1, D_2, D_3)$ with g as above and $A = \{1/3, 1/3, 1/3\}$. Then D^* is the entire z-plane, cut along the rays $\arg z = 0, (2\pi)/3, (4\pi)/3$ from $|z| = \sqrt[n]{3/4}$ to infinity. But for $n = 2, 3, \cdots$ we have $\sqrt[n]{3/4} < \sqrt[3]{1/4}$. Hence $r_0^* < 1$ while $r_{0,1} = r_{0,2} = r_{0,3} = 1$. This contradicts (2.11).

It is possible, of course, that (2.11) is valid for other families of functions g. In fact, examining the argument that proves Lemma 2.1 and Theorem 2.1 we observe that condition (2.3) is too restrictive. This condition (part (i)) guarantees that the integral

$$\int_\Omega \int h^2 q \Delta q \, dx dy$$

is preserved under our transformation. But actually we need only that this integral does not decrease.

Furthermore, even if (2.11) does not hold for a given metric g for *every* family of domains, it might hold for certain types of domains.

Finally we shall indicate an application of Theorem 1.1 concerning the harmonic radius. This notion was introduced by Hersch [2] and is defined in the following way: Let D be a domain such that the Green function $g(P, Q)$ for the Laplace operator exists. The harmonic radius R_Q is given by

$$\frac{1}{R_Q} = \lim_{P \to Q} \left(4\pi g(P, Q) - \frac{1}{|PQ|}\right)$$

where $|PQ|$ is the distance between P and Q. Following [6] we can characterize R_Q with the help of the capacity $C_\varepsilon(Q) = \int_{\Omega\varepsilon} |\nabla\omega|^2 dx$ where $\Omega_\varepsilon = D - \{P \in D \mid |PQ| < \varepsilon\}$. R_Q can be written as

$$R_Q = \lim_{\varepsilon \to 0} \left\{4\pi C_\varepsilon^{-1}(Q) - \frac{1}{\varepsilon}\right\}^{-1}. \tag{2.13}$$

LEMMA 2.3. *Let $\{D_1, D_2, \cdots, D_n\} = \mathscr{D}$ be a family of open sets in R^3 each containing the sphere $\{|x| \leqq \rho\}$, and let $R_{0,j}$ be the corresponding harmonic radius with respect to the origin. If $g = 1$ and D^* is defined by $D^* = R_{g,A}(\mathscr{D})$ [cf. Definition 1.3], then we have for the harmonic radius R_0^* of D^* at the origin*

$$\sum_{j=1}^n a_j R_{0,j} \leqq R_0^* .$$

Proof. From (2.13) we have

$$C_{\varepsilon_j}(0) = C_{\varepsilon_j} = 4\pi\Big(\frac{1}{R_{0,j} + 1/\varepsilon_j + O(\varepsilon_j)}\Big)\,, \tag{2.14}$$

and by Theorem 2.1

$$\begin{aligned}\frac{1}{R_0^* + 1/\varepsilon + O(\varepsilon)} &= I(\Omega_\varepsilon^*) \leqq \sum_{j=1}^{n} a_j I(\Omega_{j,\varepsilon_j}) \\ &= \sum_{j=1}^{n} \frac{a_j}{R_j + 1/\varepsilon_j + O(\varepsilon_j)}\end{aligned} \tag{2.15}$$

$[\Omega_{j,\varepsilon_j} = D_j - \{P \in D_j \mid |PO| < \varepsilon_j\}]$.

If we choose $\varepsilon_j = (\varepsilon R_j / \sum_{j=1}^{n} a_j R_j) = \beta R_j$. Then it follows from (2.14) that

$$\frac{\varepsilon}{\varepsilon R_0^* + 1 + O(\varepsilon^2)} \leqq \sum_{j=1}^{n} a_j \frac{\beta R_j}{\beta R_j^2 + 1 + O(\varepsilon^2)}\,. \tag{2.16}$$

The function $(\beta x/(\beta x^2 + 1))$ is concave in the interval $[0, m]$ where $m^2 \leqq 3/\beta$. Since $\varepsilon > 0$ (and therefore β) is arbitrary, it is always possible to find a number β_0 such that $(\beta_0 x/(\beta_0 x^2 + 1))$ is concave in $[0, \max_j R_j]$. Therefore

$$\begin{aligned}\sum_{j=1}^{n} a_j \frac{\beta_0 R_j}{\beta_0 R_j^2 + 1 + O(\varepsilon^2)} &\leqq \sum_{j=1}^{n} \frac{\beta_0(\sum_j a_j R_j)}{\beta 0(\sum_j a_j R_j)^2 + 1 + O(\varepsilon^2)} \\ &= \sum_{j=1}^{n} \frac{\varepsilon_0}{\beta_0(\sum_j a_j R_j) + 1 + O(1)}\end{aligned}$$

and by (2.16) we have

$$R_0^* \geqq \sum_{j=1}^{n} a_j R_j\,.$$

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