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LIMITS FOR MARTINGALE-LIKE SEQUENCES

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The concept of a martingale is generalized in two ways. The first generalization is shown to be equivalent to convergence in probability under certain uniform integrability restrictions. The second generalization yields a martingale convergence theorem.

1. Introduction. In what follows $\{X_n, \mathfrak{B}_n\}$ is a sequence of integrable random variables and sub-sigma fields on the probability space $(\Omega, \mathfrak{B}, P)$ such that

$$X_n \text{ is } \mathfrak{B}_n\text{-measurable}$$
$$\mathfrak{B}_n \subset \mathfrak{B}_{n+1}$$
$$\mathfrak{B} = \sigma\left(\bigcup_1^\infty \mathfrak{B}_n\right).$$

We call the sequence $\{X_n, \mathfrak{B}_n\}$ an adapted sequence. In [2] Blake defines $\{X_n, \mathfrak{B}_n\}$ as a game which becomes fairer with time provided

$$E(X_n \mid \mathfrak{B}_m) - X_m \xrightarrow{P} 0 \quad \text{as} \quad n \geqq m \longrightarrow \infty ,$$

i.e., provided, for all $\varepsilon > 0$:

$$\lim_{n > m} P(|E(X_n \mid \mathfrak{B}_m) - X_m| > \varepsilon) = 0 \quad \text{as} \quad m \longrightarrow \infty .$$

It is proven in [1] that if $\{X_n, \mathfrak{B}_n\}$ becomes fairer with time, and if there exists $Z \in L_1$ with $|X_n| \leqq Z$ for all n, then $X_n \xrightarrow{\mathscr{L}_1} X$, some $X \in \mathscr{L}_1$.

In the present paper we will show that $X_n \xrightarrow{\mathscr{L}_1} X$ under the less restrictive assumption that $\{X_n\}$ is uniformly integrable. We will further show that in the presence of uniform integrability $\{X_n, \mathfrak{B}_n\}$ becomes fairer with time if and only if $\{X_n\}$ converges in probability, i.e.,

$$E(X_n \mid \mathfrak{B}_m) - X_m \xrightarrow{P} 0 \Longleftrightarrow X_n - X_m \xrightarrow{P} 0 .$$

Finally, by using the more restrictive concept that $\{X_n, \mathfrak{B}_n\}$ is a martingale in the limit, namely,

$$\lim_{n \geqq m \to \infty} (E(X_n \mid \mathfrak{B}_m) - X_m) = 0 \quad \text{a.e.},$$

we will prove (Theorem (2)) a generalization of a standard martingale convergence theorem.

2. PROPOSITION 1. *Let the sequence $\{X_n\}$ be uniformly integrable and assume*

$$\lim_{n\to\infty}\int_A X_n \text{ exists, all } A \in \bigcup_1^\infty \mathfrak{B}_n .$$

Then there exists $X \in \mathscr{L}_1$ such that

$$\lim_{n\to\infty}\int_A X_n = \int_A X , \quad \text{all} \quad A \in \mathfrak{B} .$$

Proof. Let $A \in \mathfrak{B}$, $\delta > 0$. There exists $A_0 \in \bigcup_1^\infty \mathfrak{B}_n$ with $P(A \Delta A_0) \leqq \delta$. This, together with the augument in Neveu [3] (page 117) proves the desired result.

REMARKS. Let $\Omega = [0, 1)$ with Lebesgue measure. Let $\mathfrak{B}_n$ be the σ-field generated by the subintervals $A_{k,n} \equiv [k/2^n, (k+1)/2^n)$, $k = 0, 1, \cdots, 2^n - 1$. Set $X_n = \sum_{k=0}^{2^n-1} (-1)^k I_{A_{k,n}}$ where I_A is the indicator function of A. Then for any $A \in \bigcup \mathfrak{B}_n$ we have $\lim_{n\to\infty}\int_A X_n = 0$. Further, $\{X_n\}$ is uniformly integrable. However, $\{X_n\}$ does not converge in the $\mathscr{L}_1$-norm.

PROPOSITION 2. *Let $\{X_n\}$ be uniformly integrable and assume $\{X_n\}$ becomes fairer with time*:

$$(*) \qquad \lim_{n \geqq m \to\infty} P(|E(X_n | \mathfrak{B}_m) - X_m| > \varepsilon) = 0 .$$

Then there exists $X \in \mathscr{L}_1$ such that $X_n \xrightarrow{\mathscr{L}_1} X$.

Proof. Let $A \in \mathfrak{B}_m$, $p \geqq q \geqq m$. Then

$$\begin{aligned} \left|\int_A X_p - \int_A X_q\right| &= \left|\int_A E(X_p | \mathfrak{B}_q) - X_q\right| \\ &\leqq \int_{A(|E(X_p|\mathfrak{B}_q) - X_q| > \varepsilon)} |E(X_p | \mathfrak{B}_q) - X_q| + \varepsilon \\ &\leqq 2 \sup_k \int_{A(|E(X_p|\mathfrak{B}_q) - X_q| > \varepsilon)} |X_k| + \varepsilon . \end{aligned}$$

By uniform integrability and the assumption (*) we see that

$$\lim_{n\to\infty}\int_A X_n \quad \text{converges, all} \quad A \in \bigcup_1^\infty \mathfrak{B}_n .$$

By Proposition 1, there exists $X \in \mathscr{L}_1$ with

$$\lim_{n\to\infty}\int_A X_n = \int_A X , \quad \text{all} \quad A \in \mathfrak{B} .$$

Note that $\{E(X|\mathfrak{B}_n), \mathfrak{B}_n\}$ is a martingale and $E(X|\mathfrak{B}_n) \to X$ both in the $\mathscr{L}_1$ and the almost sure sense (Levy's Theorem). Since

$$\int |X_n - X| \leqq \int |X_n - E(X|\mathfrak{B}_n)| + \int |E(X|\mathfrak{B}_n) - X| ,$$

it will be enough to show $\int |X_n - E(X|\mathfrak{B}_n)| \to 0$. Now

$$\begin{aligned}\int |X_n - E(X|\mathfrak{B}_n)| &= \int_{(X_n \geqq E(X|\mathfrak{B}_n))} (X_n - E(X|\mathfrak{B}_n)) \\ &\quad + \int_{(X_n < E(X|\mathfrak{B}_n))} (E(X|\mathfrak{B}_n) - X_n) .\end{aligned}$$

Letting $n' \geqq n$ and setting $A = (|E(X_{n'}|\mathfrak{B}_n) - X_n| > \varepsilon)$, we have

$$\begin{aligned}\int_{(X_n \geqq E(X|\mathfrak{B}_n))} (X_n - E(X|\mathfrak{B}_n)) &\leqq \int_A |X_n| + \int_A |X_{n'}| \\ &\quad + \left| \int_{(X_n \geqq E(X_n|\mathfrak{B}_n))} (X_{n'} - X) \right| + \varepsilon \\ &\leqq 2 \sup_k \int_A |X_k| \\ &\quad + \left| \int_{(X_n \geqq E(X|\mathfrak{B}_n)} (X_{n'} - X) \right| + \varepsilon .\end{aligned}$$

By uniform integrability and condition (*), the first integral is small. Letting $n' \to \infty$, the difference in the remaining integral tends to zero. An identical analysis shows

$$\int_{(X_n < E(X|\mathfrak{B}))} (E(X|\mathfrak{B}_n) - X_n) \longrightarrow 0 .$$

REMARKS. Suppose $X_n \xrightarrow{\mathscr{L}_1} X$. Then since

$$\int_A |X_n| \leqq \int |X_n - X| + \int_A |X| ,$$

we see that $\{X_n\}$ is uniformly integrable. Further

$$\begin{aligned}P(|E(X_n|\mathfrak{B}_m) - X_m| > \varepsilon) &\leqq \frac{1}{\varepsilon} \int |E(X_n|\mathfrak{B}_m) - X_m| \\ &\leqq \frac{1}{\varepsilon} \int |X_n - X_m| ,\end{aligned}$$

so $\{X_n, \mathfrak{B}_n\}$ becomes fairer with time. It is shown (Neveu [3], page 52):

$\{X_n\}$ is Cauchy in the $\mathscr{L}_1$ norm $\Longleftrightarrow$ $\{X_n\}$ is uniformly integrable and $\{X_n\}$ is Cauchy in probability.

We tie these results together with Proposition 2 to get

THEOREM 1. *Let $\{X_n, \mathfrak{B}_n\}$ be an adapted sequence. Then the following three statements are equivalent:*

(a) *There exists $X \in \mathscr{L}_1$ and $X_n \xrightarrow{\mathscr{L}_1} X$.*

(b) *$\{X_n\}$ is uniformly integrable and $E(X_n \mid \mathfrak{B}_m) - X_m \xrightarrow{P} 0$.*

(c) *$\{X_n\}$ is uniformly integrable and $X_n - X_m \xrightarrow{P} 0$.*

COROLLARY 1. *Let the adapted sequence $\{X_n, \mathfrak{B}_n\}$ be uniformly integrable. Then*

$$E(X_n \mid \mathfrak{B}_m) - X_m \xrightarrow{p} 0 \Longleftrightarrow X_n - X_m \xrightarrow{p} 0 .$$

REMARKS. In the absence of uniform integrability we have neither implication. Consider the following two examples:

(1) Set $X_n = \sum_1^n y_k$ where $\{y_k\}$ is a sequence of independent identically distributed random variables with zero means. Set $\mathfrak{B}_n = \sigma(y_1, y_2, \cdots, y_n)$. Clearly $\{X_n, \mathfrak{B}_n\}$ is a martingale, so $E(X_n \mid \mathfrak{B}_m) - X_m \xrightarrow{P} 0$. But, if, for instance

$$y_k = \begin{cases} 1 & \text{with probability } \dfrac{1}{2} \\ -1 & \text{with probability } \dfrac{1}{2} \end{cases},$$

then

$$\begin{aligned} P(|X_n - X_m| \geqq 1) &= P\Big(\Big|\sum_1^{n-m} y_k\Big| \geqq 1\Big) \\ &= 1 - P\Big(\sum_1^{n-m} y_k = 0\Big) \sim 1 - \frac{c}{\sqrt{n-m}} \not\rightarrow 0 , \end{aligned}$$

so $X_n - X_m \overset{P}{\not\rightarrow} 0$.

(2) Let $\{y_k\}$ independent where $P(y_k = k^2) = 1/k^2$ and $P(y_k = 0) = 1 - 1/k^2$.

Then, setting $X_n = \sum_1^n y_k$ we have

$$|E(X_n \mid \mathfrak{B}_m) - X_m| = E \sum_{m+1}^{n} y_k \geqq 1$$

while

$$\begin{aligned} P(|X_n - X_m| \geqq \varepsilon) &= P\Big(\sum_{m+1}^{n} y_k \geqq \varepsilon\Big) = P\Big(\bigcup_{m+1}^{n} (y_k \geqq \varepsilon)\Big) \\ &\leqq \sum_{m+1}^{n} P(y_k \geqq \varepsilon) = \sum_{m+1}^{n} \frac{1}{k^2} \longrightarrow 0 , \end{aligned}$$

so in this case $X_n - X_m \xrightarrow{p} 0$ while $E(X_n \mid \mathfrak{B}_m) - X_m \overset{p}{\not\rightarrow} 0$.

Recall now the definition that $\{X_n, \mathfrak{B}_n\}$ be a martingale in the limit, namely:

$$(**) \qquad E(X_n | \mathfrak{B}_m) - X_m \longrightarrow 0 \text{ almost everywhere.}$$

THEOREM 2. *Let the adapted sequence $\{X_n, \mathfrak{B}_n\}$ be uniformly integrable and a martingale in the limit. Then there exists $X \in \mathscr{L}_1$ such that*

$$X_n \longrightarrow X \textit{ almost everywhere and in the } \mathscr{L}_1\textit{-norm.}$$

Proof. Clearly, $\{X_n, \mathfrak{B}_n\}$ becomes fairer with time, so from Theorem 1 there exists $X \in \mathscr{L}_1$ with $X_n \xrightarrow{\mathscr{L}_1} X$. Now, for an arbitrary subsequence $\{n'\}$,

$$|X_m - X| \leqq |X_m - E(X_{n'} | \mathfrak{B}_m)| + |E(X_{n'} - X | \mathfrak{B}_m)| + |E(X | \mathfrak{B}_m) - X| .$$

By Levy's theorem, the third term is less than $\varepsilon/3$ for large enough m. The first term is also bounded by $\varepsilon/3$ for large m, n' since $\{X_n, \mathfrak{B}_n\}$ is a martingale in the limit. We must now show that the second term is small. Note first that for an arbitrary σ-field $\mathscr{A}$ we have

$$E(X_n | \mathscr{A}) \xrightarrow{\mathscr{L}_1} E(X | \mathscr{A} .$$

Now start with the σ-field $\mathfrak{B}_1$ and note that the convergence $E(X_n | \mathfrak{B}_1) \xrightarrow{\mathscr{L}_1} E(X | \mathfrak{B}_1)$ implies the existence of subsequence $\{n_1\} \subset \{n\}$ with $E(X_{n_1} | \mathfrak{B}_1) \rightarrow E(X | \mathfrak{B}_1)$ almost everywhere. Continuing, we have $E(X_{n_1} | \mathfrak{B}_2) \xrightarrow{\mathscr{L}_1} E(X | \mathfrak{B}_2)$, and we can extract $\{n_2\} \subset \{n_1\}$ with $E(X_{n_2} | \mathfrak{B}_2) \rightarrow E(X | \mathfrak{B}_2)$ almost everywhere. Thus, there exists a subsequence $\{\bar{n}\} \subset \{n\}$ with $E(X_{\bar{n}} | \mathfrak{B}_m) \rightarrow E(X | \mathfrak{B}_m)$ a.e. for all m, namely the diagonal subsequence. Now choose $\{n'\}$ as a subsequence of $\{\bar{n}\}$, and we can bound the second term above by $\varepsilon/3$.

Applications. 1. Let $\{y_k\}$ be a sequence of independent random variables such that

$$\lim_{\substack{m \to \infty \\ n \to \infty}} \int \left| \sum_m^n y_k \right| = 0 .$$

Then $\sum_1^\infty y_k$ exists a.e. and in the $\mathscr{L}_1$-norm.

Proof. Set $S_n = \sum_1^n y_k$. Then

$$\int_A |S_n| \leqq \int_A |S_m| + \int \left| \sum_{m+1}^n y_k \right| ,$$

so it is clear that $\{S_n\}$ is uniformly integrable. Further, setting $\mathfrak{B}_n = \sigma(y_1, y_2, \cdots, y_n)$, we have

$$|E(S_n|\mathfrak{B}_m) - S_m| = \left|\int \sum_{m+1}^{n} y_k\right| \leqq \int \left|\sum_{m+1}^{n} y_k\right| ,$$

so $\{S_n, \mathfrak{B}_n\}$ is a uniformly integrable martingale in the limit.

2. Let $\{X_n, \mathfrak{B}_n\}$ be an adapted uniformly integrable sequence with $|E(X_{n+1}|\mathfrak{B}_n) - X_n| \leqq c_n$ where $\{c_n\}$ is a sequence of constants with $\sum_1^\infty c_n < \infty$. Then there exists $x \in \mathscr{L}_1$ with $X_n \to X$ almost everywhere and in the $\mathscr{L}_1$-norm.

Proof. We have

$$\begin{aligned} E(X_n|\mathfrak{B}_m) - X_m &= \sum_{m}^{n-1} E(X_{k+1} - X_k|\mathfrak{B}_m) \\ &= \sum_{m}^{n-1} E(E_{k+1} - X_k|\mathfrak{B}_k)|\mathfrak{B}_m). \end{aligned}$$

Thus

$$|E(X_n|\mathfrak{B}_m) - X_m| \leqq \sum_{m}^{n-1} c_k .$$

Editorial note. See also R. Subramanian, "On a generalization of Martingales due to Blake," Pacific J. Math., 48, No. 1, (1973), 275-278.

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