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COMPLETIONS AND CLASSICAL LOCALIZATIONS OF RIGHT NOETHERIAN RINGS

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Given a right Noetherian ring R and a prime ideal P of R, the injective hull of the right R-module R/P is a finite power of a uniquely determined indecomposable injective I_P. One forms the ring of right quotients R_P of R relative to I_P and the right ideal $M = PR_P$ of R_P generated by P. The M-adic and I_P-adic topologies are compared; they turn out to coincide on every finitely generated R_P-module when R_P is a classical quasi-local ring with maximal ideal M. This condition also implies that R satisfies the right Ore condition with respect to the multiplicative set $\mathscr{C}(P)$ introduced by Goldie, that the M-adic completion $\hat{R}_P$ of R_P is the bicommutator of I_P, and that $\hat{R}_P$ is an n by n matrix ring over a complete local ring.

Introduction. If P is a prime ideal of the commutative Noetherian ring R, then, by a theorem of Matlis [8], the completion $\hat{R}_P$ of the ring of quotients of R at P is the bicommutator of the injective hull of the R-module R/P. Recently Kuzmanovich [5] proved an analogous result for Noetherian Dedekind prime rings. Both these results are special cases of Theorem 6 below: Let P be a two-sided prime ideal of the right Noetherian ring R, and assume that the ring of right quotients R_P at P is a *classical quasi-local* ring with maximal ideal $M = PR_P$, that is, R_P/M is a simple Artinian ring and, for every right ideal E of R_P, $\bigcap_{n=1}^{\infty} E + M^n = E$. Then the bicommutator of the R-injective hull of R/P is the M-adic completion of R_P. The hypothesis of Theorem 6 is satisfied by the prime ideals of the enveloping algebra of a finitely generated nilpotent Lie algebra, by the augmentation ideal of a group ring of a finite group over a right Noetherian prime ring of characteristic zero, and by the nonidempotent prime ideals of a right and left Noetherian hereditary prime ring.

These results are consequences of Theorem 5, which states that R_P is a classical quasi-local ring with maximal ideal M if and only if R_P/M is a simple Artinian ring, and, on any finitely generated R_P-module, the M-adic topology coincides with the I_P-adic topology. Here I_P denotes the unique (up to isomorphism) P-torsionfree indecomposable injective R-module with associated prime P. By [7], Theorem 3.9, the injective hull $I_R(R/P)$ of the R-module R/P is isomorphic to a direct sum of g copies of I_P, where g is the Goldie dimension of the prime ring R/P. Thus the bicommutators of I_P and $I_R(R/P)$ are isomorphic.

Concerning terminology, we refer to [6], [7], and [8]. All rings are associative and have a unity element. Modules are right R-modules and unitary. We put

$$\mathscr{C}(P) = \{c \in R \mid \forall_{r \in R} cr \in P \Longrightarrow r \in P\} .$$

We begin by comparing topologies, generalizing the known result when R is commutative [6].

PROPOSITION 1. *If R satisfies the right Ore condition with respect to $\mathscr{C}(P)$, then on any finitely generated R_P-module the I_P-adic topology contains the M-adic topology, where $M = PR_P$.*

Proof. Let G be any finitely generated R_P-module. Take any fundamental open neighborhood GM^n of zero in the M-adic topology. We claim that GM^n is also open in the I_P-adic topology, in fact, $G/GM^n \in \mathscr{F}$, the class of all R_P-modules isomorphic to submodules of finite powers of I_P.

Since $\mathscr{F}$ is closed under module extensions, and since

$$GM^n \subseteqq GM^{n-1} \subseteqq \cdots \subseteqq GM \subseteqq G ,$$

it suffices to show that $GM^k/GM^{k-1} \in \mathscr{F}$. Put $H = GM^k$, then H is a finitely generated R_P-module. Now R_P/M is a simple Artinian ring, by [7], Theorem 5.6. Hence H/HM is isomorphic to a finite direct sum of minimal right ideals of R_P/M.

It remains to show that $R_P/M \in \mathscr{F}$. Indeed, in view of [7], Lemma 5.4, the mapping $R \xrightarrow{h} R_P \rightarrow R_P/M$ has kernel P, and so R_P/M may be regarded as an R-module extension of R/P. Actually, it is an essential extension; for, if $0 \neq [q] \in R_P/M$, then $q \notin M$, but $qc \in h(R)$, for some $c \in \mathscr{C}(P)$, and $gc \notin h(P)$, since otherwise $q = qcc^{-1} \in h(P)R_P = PR_P = M$. Thus R_P/M is isomorphic to an R-submodule of $I_R(R/P) = I_P^g$. By [7], Theorem 5.6, R_P/M is torsionfree and divisible, hence R_P/M is also isomorphic to an R_P-submodule of I_P^g, and so $R_P/M \in \mathscr{F}$.

This completes the proof. In the converse direction we have the following result. We remark that condition (1) plays an important rôle in [4], Theorem 5.3.

PROPOSITION 2. *Suppose $M = PR_P$ is a two-sided ideal of R_P. Then* (1) $\Rightarrow$ (2) $\Rightarrow$ (3):

(1) *For each right ideal E of R_P there exists a natural number n such that $E \cap M^n \subseteqq EM$.*

(2) *For each element $i \in I_P$ there exists a natural number n such that $iM^n = 0$.*

(3) *On any finitely generated R_P-module the I_P-adic topology is contained in the M-adic topology.*

Proof. Assume (1). Let $0 \neq i \in I_P$, and put $F = \{q \in R_P \mid iq = 0\}$, $E = \{q \in R_P \mid qM \subseteqq F\}$. Note that $EM \subseteqq F \subseteqq E$. Pick n so that $E \cap M^n \subseteqq EM$, then $E \cap (M^n + F) = (E \cap M^n) + F = F$. Since I_P is indecomposable, F is meet-irreducible, hence $F = E$ or $M^n + F = F$. We shall prove that $F \neq E$, hence $M^n \subseteqq F$, and so (1) implies (2).

As P is the associated prime ideal of the R-module I_P, P is the right annihilator of some nonzero R-submodule U of I_P. Putting $V = UR_P$, we see that $VM = 0$ and $V \neq 0$. Now $iR_P \cong R_P/F$, hence $0 \neq V \cap iR_P \cong G/F$, say, where $F \subseteqq G \subseteqq E, F \neq G$, hence $F \neq E$, as remained to be shown.

Assume (2), and let G be a finitely generated R_P-module. Take any fundamental open neighborhood of zero in the I_P-adic topology. By definition, this has the form $\operatorname{Ker} f$, where $f\colon G \to I_p^n$ for some positive integer n. Let $p_k\colon I_p^n \to I_P$ be the canonical projections, with $k = 1, 2, \cdots, n$, and put $G_k = p_k(f(G))$. Then G_k is a finitely generated R_P-submodule of I_P.

By assumption, there is a natural number $u(k)$ such that $G_k M^{u(k)} = 0$. Let $u = \operatorname{Max}\{u(1), \cdots, u(n)\}$, then $f(G)M^u = 0$, hence $\operatorname{Ker} f$ contains GM^u, a fundamental open neighborhood of 0 in the M-adic topology. It follows that every open set in the I_P-adic topology is also open in the M-adic topology. Thus (2) $\Rightarrow$ (3), and the proof is complete.

We know from [7], Lemma 5.2, that for each element $q \in R_P$ there exists an element $c \in \mathscr{C}(P)$ such that $qc \in h(R)$, where $h\colon R \to R_P$ canonically. This does not imply that R_P is the classical ring of quotients of $h(R)$ with denominators in $h(\mathscr{C}(P))$, unless R satisfies the right Ore condition with respect to $\mathscr{C}(P)$. (See [7], Proposition 5.5.) However, we have the following.

PROPOSITION 3. *Let P be a two-sided prime ideal of the right Noetherian ring R, and assume that $M = PR_P$ is a maximal two-sided ideal of R_P such that R_P/M is Artinian. Then, for every integer $n \geqq 1$, R_P/M^n is the classical ring of right quotients of $h(R)/(M^n \cap h(R))$, and its elements have the form $[h(r)][h(c)]^{-1}$, with $r \in R$ and $c \in \mathscr{C}(P)$.*

We could deduce this from [9], Theorem 2.4, by first proving that the ideals $h^{-1}(M^n)$ are the nth symbolic powers $P^{(n)}$ defined there in a different fashion. However, it is a bit quicker to deduce this directly from the following result by Small. (See [10], Theorem 1.)

Suppose P is the prime radical of the right Noetherian ring S, and $\mathscr{C}$ is a multiplicatively closed subset of S consisting of elements with zero right annihilators. Suppose the classical ring of right quotients of S/P has elements of the form $[s][c]^{-1}$, with $s \in S$ and $c \in$

$\mathscr{C}$. Then S satisfies the right Ore condition with respect to $\mathscr{C}$ and has a classical ring of right quotients with elements of the form sc^{-1}.

Proof. In [7], Theorem 5.6, in the proof of the implication (1) $\Rightarrow$ (2), it is shown that R_P/M is the classical ring of right quotients of R/P, and that its elements have the form $[r][c]^{-1}$, where $r \in R$, and $c \in \mathscr{C}(P)$. Since $h^{-1}(M) = P$, by [7], Lemma 5.4, the result holds for $n = 1$.

To obtain the result for $n = 2$, we shall apply Small's Theorem to the ring $S = h(R)/(M^2 \cap h(R))$. To this purpose we must show that the elements of $h(\mathscr{C}(P))$ modulo M^2 have zero right annihilators. In fact, we shall see that they have left inverse in R_P/M^2.

Take any $c \in \mathscr{C}(P)$. In view of the case $n = 1$, we have $R_P = R_Pc + M$. Hence $M = MR_P = Mc + M^2$, and so $R_P = R_Pc + Mc + M^2 = R_Pc + M^2$. By Small's Theorem, R_P/M^2 is the classical ring of right quotients of $h(R)/(M^2 \cap h(R))$, and denominators may be taken in $\mathscr{C}(P)$ modulo M^2.

Repeating the same argument, we see that $R_P = R_Pc + M^3$, and that R_P/M^3 is the classical ring of right quotients of $h(R)/(M^3 \cap h(R))$, with denominators in $\mathscr{C}(P)$ modulo M^3. Etc, etc.

In accordance with [7], we call the ring S a *classical quasi-local* ring if it is right Noetherian, it has a maximal ideal M such that S/M is Artinian, and every right ideal of S is closed in the M-adic topology. In view of the following lemma, this implies that M is the Jacobson radical of S.

LEMMA 4. *Suppose M is a primitive ideal of the ring S, and every finitely generated right ideal of S is closed in the M-adic topology. Then M is the Jacobson radical of S.*

Proof. The first assumption assures that M contains the radical. We claim the second assumption implies the converse. We shall prove that if E is any right ideal of S and $M + E = S$ then $E = S$.

Suppose $M + E = S$. Without loss in generality, we may take E to be finitely generated. Now $M = SM = M^2 + EM$, hence $M^2 + E = M^2 + EM + E = M + E = S$. Similarly $M^3 + E = S$, and so on. Hence the M-adic closure $\bigcap_{n=1}^{\infty} (E + M^n)$ of E is also S. By assumption, E is closed, hence $E = S$.

THEOREM 5. *Let P be a two-sided prime ideal of the right Noetherian ring R, and put $M = PR_P$, where R_P is the ring of right quotients of R at P. Then the following conditions are equivalent:*

(*) *R satisfies the right Ore condition with respect to $\mathscr{C}(P)$ and, for each right ideal E of R_P, there exists a natural number n*

such that $E \cap M^n \subseteq EM$.

(**) R_P *is a classical quasi-local ring with maximal ideal* M.

(***) M *is a two-sided ideal of* R_P, R_P/M *is a simple Artinian ring, and on any finitely generated right* R_P*-module the* I_P*-adic and* M*-adic topologies coincide.*

(****) M *is a two-sided ideal of* R_P, R_P/M *is simple Artinian, and for each finitely generated right ideal* E *of* R_P *there exists a natural number* n *such that* $E \cap M^n \subseteq EM$.

Proof. We shall show that $(^*) \Rightarrow (^{**}) \Rightarrow (^{***}) \Rightarrow (^{****}) \Rightarrow (^*)$.

Assume (*). In view of [7], Theorem 5.6, (**) will follow if we show that every right ideal F of R_P is closed in the M-adic topology. Now its closure is given by $E = \bigcap_{n=1}^{\infty} (F + M^n)$. Pick n so that $E \cap M^n \subseteq EM$, then

$$E \subseteq (F + M^n) \cap E = F + (M^n \cap E) \subseteq F + EM .$$

Take any $e \in E$, then $e = f + \sum_{i=1}^{k} e_i m_i$, where $f \in F$, $e_i \in E$, and $m_i \in M$. Then $[e] = \sum_{i=1}^{k} [e_i] m_i$, modulo F, hence $E/F \subseteq (E/F)M$.

It was pointed out in the discussion preceding [7], Theorem 5.6, that R_P is right Noetherian. Thus E and E/F are finitely generated R_P-modules. We may therefore invoke Nakayama's Lemma and deduce that $E/F = 0$. Thus $F = E$, and so (**) holds.

Assume (**). By Lemma 4, M is the Jacobson radical of R_P. By [7], Theorem 5.6, R satisfies the right Ore condition with respect to $\mathscr{C}(P)$. Let G be any finitely generated right R_P-module. Then, by Proposition 1, the I_P-adic topology on G contains the M-adic topology. By Proposition 2 and [4] Theorem 5.3, the converse is true. Thus (***) holds.

Assume (***). Suppse E is any finitely generated right ideal of R_P. Then EM is an open subset of E in the M-adic, hence in the I_P-adic topology. Now the I_P-adic topology on any module induces the I_P-adic topology on any submodule. Therefore, $EM = E \cap V$, where V is an open subset of R_P in the I_P-adic topology. Since R_P is a finitely generated R_P-module, V is an open set in the M-adic topology, hence $M^n \subseteq V$ for some n, and so $E \cap M^n \subseteq E \cap V = EM$. Thus (****) holds.

Assume (****). It remains to prove the right Ore condition. Given $a \in R$ and $c \in \mathscr{C}(P)$, we see from Proposition 3 that, for each positive integer n, there exist $a_n \in R$ and $c_n \in \mathscr{C}(P)$ such that $h(ac_n - ca_n) = h(u_n) \in M^n \cap h(R)$.

Let F be the right ideal generated by the u_n, then $F = u_1 R + \cdots + u_k R$, since R is right Noetherian. Taking $E = FR_P$ in the above, we see that $FR_P \cap M^n \subseteq FM$, for some n. Hence $h(u_n) = u_1 m_1 + \cdots + u_k m_k$, where the $m_i \in M$.

Pick $d \in \mathscr{C}(P)$ so that all $m_i d \in h(R)$, then $m_i d \in M \cap h(R) = h(P)$, and we may write $m_i d = h(p_i)$, where $p_i \in P$.

Put $c' = c_n d - \sum_{i=1}^k c_i p_i$ and $a' = a_n d - \sum_{i=1}^k a_i p_i$, then an easy calculation shows that $h(ac') = h(ca')$. Moreover $C' \in \mathscr{C}(P)$, since $c_n d \in \mathscr{C}(P)$ and $\sum c_i p_i \in P$. Since $ac' - ca' \in \operatorname{Ker} h$, we can find $d' \in \mathscr{C}(P)$ so that $(ac' - ca')' = 0$, hence $a(c'd') = c(a'd')$. This establishes the right Ore condition for R, and our proof is complete.

THEOREM 6. *Let P be a two-sided prime ideal of the right Noetherian ring R such that R_P is a classical quasi-local ring with maximal ideal $M = PR_P$. Then*

(a) *the M-adic completion $\hat{R}_P$ of R_P is the bicommutator of the P-torsionfree indecomposable injective R-module I_P with associated prime P,*

(b) *$\hat{R}_P$ is a ring of $n \times n$ matrices over a complete local ring $\hat{D}$ whose Jacobson radical J is finitely generated.*

Proof. (a) By Theorem 5, R satisfies the right Ore condition with respect to $\mathscr{C}(P)$. By [7], Theorem 5.6, every torsionfree R-module is P-divisible. In view of [6], Proposition 2, R_P is therefore a dense subring of the bicommutator S of I_P with respect to the finite topology, as the P-torsion theory coincides with that determined by I_P, by [7], Corollary 3.10. By [6], Corollary 1, the finite topology coincides with the I_P-adic topology on R_P, and S is the completion of R_P. By Theorem 5, the I_P-adic topology on R_P coincides with the M-adic one. Therefore S is the M-adic completion of R_P.

(b) follows immediately from (a) and [9], Corollary 2.7.

REMARK 7. By [9], Remark 3, there exists a right Noetherian ring R with a two-sided prime ideal P such that R satisfies the right Ore condition with respect to $\mathscr{C}(P)$, even though R_P is not a classical quasi-local ring with maximal ideal M. In that example R_P is not Hausdorff with respect to the M-adic topology, hence the bicommutator of I_P is not the M-adic completion of R_P.

Thus the right Ore condition does not imply the second part of (*) in Theorem 5. Conversely, Example 5.9 of [7] shows that the second part of (*) does not imply the right Ore condition.

We conclude by giving some classes of examples satisfying the condition of Theorem 5. But first we note that, in view of Theorem 3.3 of [9], each of these is also equivalent to the following, which involves only the ring R itself:

(+) For every right ideal F of R there exists a positive integer n such that $F \cap P^{(n)} \subseteq \mathrm{cl}_P(FP)$, where $P^{(n)}$ is the nth right symbolic power of P.

For notation see [9].

COROLLARY 8. *Let R be the enveloping algebra of a finitely generated nilpotent Lie algebra, and assume that P is a nonzero prime ideal of R. Then the conclusions* (a) *and* (b) *of Theorem* 6 *hold.*

Proof. In Theorem 2.6 of [9], it is shown that, if R is right and left Noetherian, $P^{(n)}$ coincides with the symbolic nth power defined by Goldie in [4]. To deduce $(+)$, we therefore refer to [2], namely to Theorem 6, Corollary 7 and Remark I.

COROLLARY 9. *Let $R = AG$ be the group ring of a finite group G over a right Noetherian prime ring A of characteristic zero, and let P be the augmentation ideal of R. Then the conclusions* (a) *and* (b) *of Theorem* 6 *hold.*

Proof. Condition $(+)$ holds by Corollary 3.7 of [9].

Actually, in this example R_P is the classical ring of right quotients of R, and $M = PR_P = 0$, because P is the P-torsion ideal of R.

COROLLARY 10. *Let R be a right and left Noetherian hereditary prime ring, and assume that P is not idempotent. Then the conclusions* (a) *and* (b) *of Theorem* 6 *hold. Furthermore, $\hat{D}$ is a complete discrete rank one valuation ring.*

Proof. By [11], R/P is a simple Artinian ring. It is known that P is an invertible ideal. By Lemma 1.1 of [3], it then follows that P has the Artin-Rees property. Now, by Corollary 2.8 of [9], $P^n = P^{(n)}$, hence condition $(+)$ holds.

It remains to show that $\hat{D}$ is a rank one valuation ring. By the remark preceding Theorem 5.6 in [7], R_P is hereditary Noetherian and quasi-local. As is well-known, this implies that $\hat{R}_P$ is hereditary Noetherian. By Morita equivalence, $\hat{D}$ is hereditary Noetherian. But it is local, hence a discrete rank one valuation ring.

For the sake of completeness, we shall show that P is invertible. Let Q be the maximal ring of right and left quotients of R and put $R.\cdot P = \{q \in Q \mid qP \subseteqq R\}$. It is known [3] that $R \subseteqq P(R.\cdot P)$ provided P is finitely generated and projective as a right R-module and "dense" in a technical sense, which means that P has zero left annihilator in R when P is a two-sided ideal. Since R is right Noetherian, right hereditary, and prime, P satisfies all three conditions.

Now $P \subseteqq (R.\cdot P)P \subseteqq R$, and P is maximal. Therefore $(R.\cdot P)P = P$ or R. Suppose the former, then $P \subseteqq P(R.\cdot P)P = P^2$, which would lead to the contradiction that P is idempotent. Therefore, $(R.\cdot P)P = R$.

Finally, consider $P\cdot .R = \{q \in Q \mid Pq \subseteqq R\}$. Then

$$P\cdot .R = (R.\cdot P)P(P\cdot .R) \subseteqq R.\cdot P .$$

By symmetry we obtain $P(P\cdot .R) = R$ and $R.\cdot P \subseteqq P\cdot .R$, and so P is invertible in Q.

For the sake of completeness, we shall also include the argument of [1] to show that P has the Artin-Rees property. Let E be any right ideal of R and put $E_k = (E \cap P^k)P^{-k}$. Since R is right Noetherian, there exists a positive integer k such that $E_k \subseteqq E_1 + \cdots + E_{k-1}$. Then $E \cap P^k = E_k P^k \subseteqq \sum_{i<k} (E \cap P^i)P^{k-i} \subseteqq EP$, and this is the Artin-Rees property.

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