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CONTENT OF THE FRUSTUM OF A SIMPLEX

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In the Euclidean space of n dimensions, R^n, the $(n-1)$-dimensional content of the portion of an $(n-1)$-dimensional simplex contained in a semispace is evaluated. Also, in R^n, the content of the portion of an n-dimensional simplex contained in a semispace is evaluated.

More precisely, the following theorems are proved.

Set up a Cartesian coordinate system in R^n and refer to a general point in the n-space by $(y_1, y_2, \cdots, y_n)$. Let S_n, S_{n-1} and H be defined as follows:

$$S_n\colon \{(y_1, y_2, \cdots, y_n) \mid y_i \geqq 0,\ i = 1, \cdots, n,\ \sum y_i \leqq 1\}$$
$$S_{n-1}\colon \{(y_1, y_2, \cdots, y_n) \mid y_i \geqq 0,\ i = 1, \cdots, n,\ \sum y_i = 1\}$$

and

$$H\colon \{(y_1, y_2, \cdots, y_n) \mid \sum a_i y_i \leqq z\}\ .$$

Let $[f(x) \mid x = x_1, x_2, \cdots, x_{r+1}]$ denote the rth divided difference of $f(x)$ with arguments for x as $x_1, x_2, \cdots, x_{r+1}$. Define $x_- = x$ if $x < 0$ and $x_- = 0$ if $x \geqq 0$.

THEOREM 1. *The content of the frustum $S_n \cap H$ expressed as a ratio of the content of S_n, $C[S_n]$, say, $C[S_n] = (n!)^{-1}$, is given by*

$$\frac{C[S_n \cap H]}{C[S_n]} = [\{(x - z)_-\}^n \mid x = a_0, a_1, a_2, \cdots, a_n]$$

where a_0 is defined by $a_0 = 0$.

THEOREM 2. *The $(n-1)$-content of the frustum $S_{n-1} \cap H$ expressed as a ratio of $C[S_{n-1}] = \sqrt{n}/(n-1)!$ is given by*

$$\frac{C[S_{n-1} \cap H]}{C[S_{n-1}]} = [\{(x - z)_-\}^{n-1} \mid x = a_1, a_2, \cdots, a_n]\ .$$

An algorithm suitable for automatic computation of the divided differences occurring in the above theorems is discussed.

The result of Theorem 1 has applications (see Ali, 1969) to the statistical problem of the distribution of linear combination of ordered observations arising from a population uniformly distributed over [0, 1] while the result of Theorem 2 may find application in linear programming and allocation theory.

G. Varsi [7] has considered the problem in Theorem 2 and by means of a successive dissection technique, he arrives at an algorithm suitable for automatic computation. It is shown that the formula of the present paper leads to the algorithm proposed by Varsi.

The evaluation of the divided differences occurring in the above theorems is discussed in §3. For numerical computation of these divided differences, an algorithm suitable for automatic computation is discussed in §4.

The particular choice of S_n and S_{n-1} in the above theorems does not involve any loss of generality as shown below.

Consider in R^n an n-simplex T_n whose vertices are V_i for $i = 1, 2, \cdots, n+1$. Let the co-ordinates of V_i referred to an n-dimensional cartesian co-ordinate system with origin at V_{n+1} be denoted by $x_{i,1}, x_{i,2}, \cdots, x_{i,n})$ for $i = 1, 2, \cdots, n$. Let σ_n denote the semispace given by σ_n: $\{(x_1, x_2, \cdots, x_n) \mid \sum c_i x_i \leqq z\}$.

The frustum is defined by $T_n \cap \sigma_n$ and let $C[T_n \cap \sigma_n]$ denote its content.

Define the $n \times n$ matrix V in double suffix notation as $V = (x_{i,j})$. Let $X' = (x_1, x_2, \cdots, x_n)$ and $Y' = (y_1, y_2, \cdots, y_n)$. Then it is easily checked that the linear transformation from X to Y given by $X = V'Y$ transforms T_n to the simplex S_n as defined in Theorem 1 and σ_n is transformed to H given by H: $\{(y_1, \cdots, y_n) \mid \sum a_i y_i \leqq z\}$. Therefore, it follows that $C[T_n \cap \sigma_n] = \| V \| C[S_n \cap H]$, with $\sum c_j x_{i,j} = a_i$.

Likewise, in R^n let T_{n-1} denote an $(n-1)$-simplex. With origin not on the $(n-1)$-flat passing through T_{n-1}, refer to the n vertices V_i for $i = 1, \cdots, n$ with co-ordinates as before. Let σ_n be defined as before. Proceeding in an analogous manner as in the former case it is seen that

$$C[T_{n-1} \cap \sigma_n] = \| V \| C[S_{n-1} \cap H]$$

where a_i is defined exactly as in the former case.

2. Divided difference.

For convenience we state some standard results on divided differences.

The rth divided difference of a function $f(x)$ with arguments $x = x_0, x_1, \cdots, x_r$ is defined as:

$$(1) \qquad \begin{aligned}[] [f(x) \mid x = x_0, x_1, \cdots, x_r] &= \sum_{i=0}^{r} f(x_i) / \prod_{\substack{j=0 \\ j \neq i}}^{r} (x_i - x_j) \\ &= |A| / |B| \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{r-1} & f(x_0) \\ 1 & x_1 & x_1^2 & \cdots & x_1^{r-1} & f(x_1) \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & x_r & x_r^2 & \cdots & x_r^{r-1} & f(x_r) \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{r} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{r} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & x_r & x_r^2 & \cdots & x_r^{r} \end{bmatrix}$$

when $x_0, x_1, \cdots, x_r$ are distinct.

Finally, we state the following well-known result: (see Steffensen, [6, p. 19]). For integral r,

$$(2) \qquad [x^{n+r} | x = a_0, a_1, \cdots, a_n] = \begin{cases} 0 & \text{if } -n \leqq r < 0 \\ 1 & \text{if } r = 0 \\ \sum' a_0^{r_0} a_1^{r_1} \cdots a_n^{r_n} & \text{for } r > 0 \\ (r_0 + r_1 + \cdots + r_n = r) & \end{cases}$$

where $\sum'$ denotes the summation over all the distinct products with nonnegative integral exponents whose sum is r.

For definitions of divided differences of $f(x)$ with coincident arguments the reader is referred to, for example, Hildebrand [3, p. 40], Steffensen [6, p. 20] and Isaacson and Keller [4, p. 254].

3. Divided difference of $\{(x-z)_-\}^r$. Consider the rth divided difference of $\{(x-z)_-\}^r$ with possibly coincident arguments $a_0, a_1, \cdots, a_r$ for x. We rule out the trivial case when $z = a_0 = a_1 = \cdots = a_r = 0$. Suppose $a_0, a_1, \cdots, a_r$ are relabelled as $b_1, \cdots, b_s$, ($b_i \neq b_j$ for $i \neq j$) where b_ν is repeated $p_\nu + 1$ times $p_\nu \geqq 0, \nu = 1, 2, \cdots, s$, so that $p_1 + p_2 + \cdots + p_s + s = r + 1$. Taking appropriate limits of (1) (cf. Isaacson and Keller, [4, p. 254]) we obtain

$$\begin{aligned} &[\{(x-z)_-\}^r | x = a_0, a_1, \cdots, a_r] \\ &= \frac{1}{\prod_{\nu=1}^{s} p_\nu !} \left[\prod_{\nu=1}^{s} \frac{\partial^{p_\nu}}{\partial_{b_\nu}^{p_\nu}} \right] [\{(x-z)_-\}^r | x = b_1, \cdots, b_s] \end{aligned}$$

where the divided difference on the right is given by (1).

Another alternative form of (3) is obtained by taking appropriate limit of $|A|/|B|$ in (1), for which we refer to Ali (1969). For example:

$$[\{(x-z)_-\}^3 \,|\, x = c, c, c, d]$$
$$= \begin{vmatrix} 1 & c & c^2 & \{(c-z)_-\}^3 \\ 0 & 1 & 2c & 3\{(c-z)_-\}^2 \\ 0 & 0 & 2 & 6(c-z)_- \\ 1 & d & d^2 & \{(d-z)_-\}^3 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & c & c^2 & c^3 \\ 0 & 1 & 2c & 3c^2 \\ 0 & 0 & 2 & 6c \\ 1 & d & d^2 & d^3 \end{vmatrix} .$$

The following two special cases of coincident arguments are of interest.

(i) Decompose $a_0, a_1, \cdots, a_r$ into disjoint sets

$$S_1: \{a_\nu \,|\, a_\nu - z < 0\} \quad \text{and} \quad S_1^*: \{a_\nu \,|\, a_\nu - z \geqq 0\} .$$

Let the a_ν belonging to S_1 be renamed as $\alpha_1, \alpha_2, \cdots, \alpha_J$ while those belonging to S_1^* be renamed as $\beta_1, \beta_2, \cdots, \beta_K$ so that $J + K = r + 1$. If $\alpha_1, \cdots, \alpha_J$ are distinct (whether $\beta_1, \cdots, \beta_K$ are distinct or not) we have

$$[\{(x-z)_-\}^r \,|\, x = a_0, a_1, \cdots, a_r]$$
$$= \sum_{\nu=1}^{J} (\alpha_\nu - z)^r \Big/ \sum_{\substack{j=1 \\ j\neq\nu}}^{J} (\alpha_\nu - \alpha_j) \sum_{k=1}^{K} (\alpha_\nu - \beta_k) .$$

Likewise, if $a_0, a_1, \cdots, a_r$ are decomposed into $S_2: \{a_\nu \,|\, a_\nu - z \leqq 0\}$ and $S_2^*: \{a_\nu \,|\, a_\nu - z > 0\}$ and the a_ν belonging to S_2 are relabelled as $\alpha_1, \cdots, \alpha_J$ while those belonging to S_2^* are distinct, say $\beta_1, \cdots, \beta_K$ then

$$[\{(x-z)_-\}^r \,|\, x = a_0, a_1, \cdots, a_r]$$
$$= 1 - \sum_{\nu=1}^{K} (\beta_\nu - z)^r \Big/ \prod_{j=1}^{J} (\beta_\nu - \alpha_j) \prod_{\substack{k=1 \\ k\neq\nu}}^{K} (\beta_\nu - \beta_k) .$$

The last step follows from the fact that

$$[(x-z)^r \,|\, x = a_0, a_1, \cdots, a_r] \equiv 1 .$$

4. Computation of divided differences of $\{(x-z)_-\}^r$. Consider the rth divided difference of $\{(x-z)_-\}^r$ with arguments for $x = a_0, a_1, \cdots, a_r$. Let as before the set of a_ν satisfying $a_\nu - z < 0$ be relabelled as $\alpha_1, \cdots, \alpha_J$ while the remaining a_ν satisfying $a_\nu - z \geqq 0$ be relabelled as $\beta_1, \cdots, \beta_K$, so that $K + J = r + 1$.
Define

$$A_{\lambda\mu} = [\{(x-z)_-\}^{\lambda+\mu-1} \,|\, x = \alpha_1, \cdots, \alpha_\lambda, \beta_1, \cdots, \beta_\mu] .$$

Further let

$$X_\lambda = \alpha_\lambda - z \quad \text{for} \quad \lambda = 1, \cdots, J$$

and

$$Y_\mu = \beta_\mu - z \quad \text{for} \quad \mu = 1, \cdots, K .$$

Then

$$A_{\lambda\mu} = [(X_-)^{\lambda+\mu-1} | X = X_1, \cdots, X_\lambda, Y_1, \cdots, Y_\mu] .$$

Further by the use of (1) (temporarily assuming that $\alpha_1, \cdots, \alpha_\lambda, \beta_1, \cdots, \beta_\mu$ are distinct) the following recurrence relation is easily verified.

$$A_{\lambda\mu} = \frac{Y_\mu A_{(\lambda-1)\mu} - X_\lambda A_{\lambda(\mu-1)}}{Y_\mu - X_\lambda} .$$

It is readily checked from (1) with $[f(x) | x = a] = f(a)$ that $A_{\lambda 0} = 1$ for $\lambda = 1, 2, \cdots, J$ and $A_{0\mu} = 0$ for $\mu = 1, 2, \cdots, K$. Define $A_{00} = 1$. The recurrence formula then gives $A_{11} = (X_1)/(X_1 - Y_1)$ as it should be.

The above recurrence formula sets up an algorithm to compute successive values of $A_{\lambda\mu}$. This algorithm was proposed by Varsi (from geometrical considerations) and is suitable for automatic computation. We note that

$$[\{(x - z)_-\}^r | x = a_0, a_1, \cdots, a_r] = A_{JK} .$$

The Algorithm of Varsi.

Compute $u_j = a_j - z$ for $j = 0, 1, 2, \cdots, r$. Label the u_j which are nonnegative as $Y_1, \cdots, Y_K$ and the remaining u_j as $X_1, \cdots, X_J$ so that $K + J = r + 1$.

The following notations are computational rather than mathematical notations.

Step 1. Set $A_0 = 1, A_1 = A_2 = \cdots = A_K = 0$.

Step 2. For each value of h repeat step 3 for $h = 1, 2, \cdots, J$.

Step 3.

$$A_k \longleftarrow \frac{Y_k A_k - X_h A_{k-1}}{Y_k - X_h} \quad \text{for} \quad k = 1, 2, \cdots, K .$$

(The expression on the right hand side is computed and stored in location A_k.) Then the quantity in A_K after the above set of operations is the value of $[\{(x - z)_-\}^r | x = a_0, a_1, \cdots, a_r]$.

It is to be noted that the above algorithm does not result in any indeterminacy for coincident values of $a_0, a_1, \cdots, a_r$ since $Y_\mu > X_\lambda$ for all $\lambda = 1, \cdots, J$, and $\mu = 1, \cdots, K$.

5. **Proof of the theorems.** Consider the simplex L_n defined by

$$(4) \qquad L_n: \{(x_1, x_2, \cdots, x_n) \mid \sum x_j \leqq L \quad \text{and} \quad x_j \geqq 0, j = i, \cdots, n\}$$

and the semispace H defined by

$$(5) \qquad H: \{(x_1, x_2, \cdots, x_n) \mid a_1x_1 + \cdots + a_nx_n \leqq z\} .$$

Temporarily assume that $a_0, a_1, \cdots, a_n$ are distinct, where $a_0 = 0$. This restriction will be removed later.
Let

$$(6) \qquad F(z) = \frac{C[L_n \cap H]}{C[L_n]} = n!\,L^{-n}\int_{L_n \cap H} dx_1 \cdots dx_n .$$

It is easily shown that $F(z)$ is a distribution function and that $0 \leqq F(z) \leqq 1$.

Let the characteristic function (Fourier-Stiltjes transform) of $F(z)$ be $\phi(t)$, (see Loeve [5, p. 184]) where $\phi(t)$ is defined by

$$\phi(t) = \int_{-\infty}^{+\infty} e^{itz} dF(z) .$$

It is easily seen that

$$\phi(t) = \frac{n!}{L^n}\int_{L_n} e^{it(a_1x_1 + a_2x_2 + \cdots + a_nx_n)} \cdot dx_1 dx_2 \cdots dx_n .$$

Let $x_i = Ly_i$ for $i = 1, \cdots, n$.
Then we have

$$\phi(t) = n!\int_{S_n} e^{itL(a_1y_1 + \cdots + a_ny_n)} \cdot dy_1 dy_2 \cdots dy_n$$

where S_n: $y_1 + y_2 + \cdots + y_n \leqq 1$ and $y_i \geqq 0, i = 1, 2, \cdots, n$.

Straightforward computation shows that for integral values of $r_1, r_2, \cdots, r_n$;

$$n!\int_{S_n} y_1^{r_1} y_2^{r_2} \cdots y_n^{r_n} dy_1 dy_2 \cdots dy_n = \frac{r_1!\,r_2! \cdots r_n!\,n!}{(n + r_1 + r_2 + \cdots + r_n)!}$$

so that by an easy computation we have

$$\begin{aligned} \mu'_r &= \int_{-\infty}^{+\infty} z^r dF(z) = L^r \int_{S_n} (a_1y_1 + a_2y_2 + \cdots + a_ny_n)^r dy_i \\ &= \frac{n!\,r!\,L^r}{(n + r)!} \sum{}' a_0^{r_0} a_1^{r_1} \cdots a_n^{r_n} \end{aligned}$$

where $\sum'$ is the sum of distinct products of nonnegative exponents whose sum is r, with $a_0 = 0$.

Hence from (2) we obtain

$$\mu'_r = \frac{n!r!L^r}{(n+r)!}[x^{n+r} | x = a_0, a_1, \cdots, a_n]$$
$$= \frac{n!L^r}{(n+r)!}\left[\frac{d^r x^{n+r}}{dx^r}\right]_{x=\zeta} \quad \text{(cf. Steffensen p. 23)}$$

where ξ is a number between the smallest and the largest of the numbers $a_0, a_1, \cdots, a_n$. Hence $|\mu'_r| \leqq M^r L^r$, where M denotes the largest value of the numbers $|a_0|, |a_1|, \cdots, |a_n|$, and for some $c > 0$,

$$\left|\sum_{r=1}^{\infty} \frac{\mu'_r c^r}{r!}\right| \leqq \sum_{r=0}^{\infty} \frac{L^r |Mc|^r}{r!} = e^{|Mc|L}$$

which is finite for all values of c. Therefore, the series $\sum_{r=1}^{\infty} (\mu'_r c^r / r!)$ is absolutely convergent for all finite values of $c > 0$. Hence from a well-known theorem of Cramer [2], (for a proof see, for example, Wilks [8, p. 125]) we have

$$\phi(t) = \sum_{r=0}^{\infty} \frac{\mu'_r}{r!}(it)^r$$
$$= n! \sum_{r=0}^{\infty} \frac{(iLt)^r}{(n+r)!}[x^{n+r} | x = a_0, a_1, \cdots, a_n]$$
$$= n! \sum_{s=0}^{\infty} \frac{(iLt)^{s-n}}{s!}[x^s | x = a_0, a_1, \cdots, a_n]$$

since $[x^s | x = a_0, a_1, \cdots, a_n] = 0$ for $s < n$, from (2).

Hence, we have

$$\phi(t) = n!(iLt)^{-n} \sum_{s=0}^{\infty} \sum_{\nu=0}^{n} \frac{(iLta_\nu)^s}{s!} \Bigg/ \left[\prod_{\substack{j=0 \\ j \neq \nu}}^{n} (a_\nu - a_j)\right]$$
$$= n!(iLt)^{-n} \sum_{\nu=0}^{n} \frac{1}{\prod_{\substack{j=0 \\ j \neq \nu}}^{n} (a_\nu - a_j)} \sum_{s=0}^{\infty} \frac{(iLta_\nu)^s}{s!}$$
$$= n!(iLt)^{-n} \sum_{\nu=0}^{n} e^{iLta_\nu} \Bigg/ \prod_{\substack{j=0 \\ j \neq \nu}}^{n} (a_\nu - a_j) .$$

By the inversion formula (Loeve, [5, p. 186]) we obtain

$$\frac{d}{dz}F(z) = \frac{n!}{2\pi}\int_{-\infty}^{+\infty} (iLt)^{-n} \left[\sum_{\nu=0}^{n} e^{it(La_\nu - z)} \Bigg/ \prod_{\substack{j=0 \\ j \neq \nu}}^{n} (a_\nu - a_j)\right] dt .$$

The above integral is analytic everywhere and the range of integration may be changed to the contour Γ consisting of the real axis from $-\infty$ to $-c$, the small semicircle with radius c with center at the origin and the real axis from c to ∞.

Now by the use of

$$\frac{1}{2\pi i^n}\int_\Gamma z^{-n}e^{i\alpha z}dz = -(\alpha_-)^{n-1}/(n-1)!$$

we have

$$\left(\frac{d}{dz}\right)F(z) = -nL^{-n}\sum_{\nu=0}^{n}\{(La_\nu - z)_-\}^{n-1}/\prod_{\substack{j=0\\ j\neq\nu}}^{n}(a_\nu - a_j)$$

and therefore integration over z yields

$$\begin{aligned} F(z) &= \sum_{\nu=0}^{n}\left\{\left(\left(a_\nu - \frac{z}{L}\right)_-\right)^n \Big/ \prod_{\substack{j=0\\ j\neq\nu}}^{n}(a_\nu - a_j)\right\} + K \\ &= \left[\left(\left(x - \frac{z}{L}\right)_-\right)^n \mid x = a_0, a_1, a_2, \cdots, a_n\right] + K . \end{aligned}$$

It is easily verified that: for $z/L < \min(a_0, a_1, \cdots, a_n)$, $L_n \cap H = \varnothing$, and hence $F(z) = 0$; since in this case

$$\left[\left\{\left(x - \frac{z}{L}\right)_-\right\}^n \middle| x = a_0, a_1, \cdots, a_n\right] = 0 ,$$

we immediately have $K = 0$, so that,

$$F(z) = \left[\left(\left(x - \frac{z}{L}\right)_-\right)^n \middle| x = a_0, a_1, \cdots, a_n\right] .$$

Hence substituting $L = 1$, Theorem 1 is proved for the case when a_0, $a_1, \cdots, a_n$ are distinct.

The distance of the $(n-1)$ flat $\sum x_i = L$ from the origin is $L/\sqrt{n}$. Consider the simplexes L_n, $(L + \delta L)_n$ as defined in (3) and the semispace H as in (4). The elementary volume $C[(L + \delta L)_n] - C[L_n]$ divided by $\delta L/\sqrt{n}$ by letting $\delta L \to 0$ gives the $(n-1)$-dimensional content of the portion of the simplex $\sum x_i = L$, $x_i \geqq 0$ contained in H. From (6) this volume is equal to $\sqrt{n}(d/dL)C[L_n \cap H]$

$$\begin{aligned} &= \sqrt{n}\left(\frac{d}{dL}\right)C(L_n)F(z) = \sqrt{n}\frac{d}{dL}\frac{L^n}{n!}F(z) \\ &= \sqrt{n}\frac{d}{n!\,dL}([(Lx - z)_-^n \mid x = a_0, a_1, \cdots, a_n]) . \end{aligned}$$

A simple calculation shows that the last expression is equal to

$$\frac{\sqrt{n}}{(n-1)!}[(Lx - z)_-^{n-1} \mid x = a_1, a_2, \cdots, a_n] .$$

Hence, setting $L = 1$, we finally obtain

$$C[S_{n-1} \cap H] = \frac{\sqrt{n}}{(n-1)!}[(x - z)_-)^{n-1} \mid x = a_1, a_2, \cdots, a_n]$$

so that

$$\frac{C[S_{n-1} \cap H]}{C[S_{n-1}]} = [((x - z)_-)^{n-1} | x = a_1, a_2, \cdots, a_n] .$$

Hence, Theorem 2 is established when the a_ν are distinct.

The continuity theorem for the characteristic function along with the definition of divided differences for k coincident argument show that (with the definition of the divided difference for coincident arguments) both Theorem 1 and Theorem 2 are also true for the case of coincident arguments with expressions for divided differences as given in §3. In particular, the algorithm discussed in §4, is not only suitable for numerical computation, but also can be applied in all cases since there is no indeterminacy for coincident arguments.

6. Asymptotic case. A sequence of real numbers $(c_{1n}, c_{2n}, \cdots, c_{nn})$ will be said to obey Condition C if the following is satisfied:

Condition C:

$$\lim_{n\to\infty} \max_{1 \le j \le n} (c_{jn} - \bar{c}_n)^2 \Big/ \sum_{\nu=1}^{n} (c_{\nu n} - \bar{c}_n)^2 = 0 ,$$

where $\bar{c}_n = (c_{1n} + c_{2n} + \cdots + c_{nn})/n$.

THEOREM. *If the sequence $(c_{1n}, c_{2n}, \cdots, c_{nn})$ satisfies Condition C,*

$$\lim_{n\to\infty} [\{(x - z)_-\}^{n-1} | x = c_{1n}, \cdots, c_{nn}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du$$

where $z = \bar{c}_n + [\sum_{i=1}^{n} (c_{in} - \bar{c}_n)^2 / n(n+1)]^{1/2} t$.

Before proving this general result we state the following result obtained from statistical considerations by Ali [1]:

LEMMA.

$$\lim_{n\to\infty} [\{(x - z)_-\}^{n} | x = a_0, a_1, \cdots, a_n] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du$$

where $a_0 = 0$, *and* $\bar{a}_n = (a_0 + a_1 + \cdots + a_n)/(n+1)$ *and* $z = \bar{a}_n + [\sum_{i=0}^{n} (a_i - \bar{a}_n)^2 / (n+1)(n+2)]^{1/2} \cdot t$ *provided the sequence* $a_0, a_1, \cdots, a_n$ *satisfies Condition C.*

Let us now consider $[\{(x - z)_-\}^{n-1} | x = a_1, \cdots, a_n]$. Write $c_i = a_i - a_1$, $i = 1, \cdots, n$; so that $c_1 = 0$. It is readily checked that if the sequence $(a_1, \cdots, a_n)$ obeys Condition C so does the sequence $(c_1 = 0, c_2, \cdots, c_n)$. Straightforward application of the above Lemma shows that

$$\lim_{n\to\infty} [\{(x - z)_-\}^{n-1} | x = a_1, \cdots, a_n] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du$$

where $\bar{a}_n = (a_1 + \cdots + a_n)/n$, and $z = \bar{a}_n + [\sum (a_i - \bar{a}_n)^2/n(n+1)]^{1/2} t$. This proves the theorem.

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