PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 1, 1974

ITERATIVE TECHNIQUES FOR APPROXIMATION OF FIXED POINTS OF CERTAIN NONLINEAR MAPPINGS IN BANACH SPACES

R. L. THELE

Let D be a closed convex subset of a Banach space X, let $T: D \to D$ be *nonexpansive* (that is, $\|Tx - Ty\| \leqq \|x - y\|$ for every $x, y \in D$), and let $F_\lambda = \lambda T + (1-\lambda)I$, where $\lambda \in (0, 1)$ and I denotes the identity on D. Several authors have found conditions under which the sequences of iterates $\{T^n x\}$, or the sequences $\{F_\lambda^n x\}$, converge strongly or weakly to fixed points of T for all $x \in D$. In this paper we establish conditions under which the sequences $\{F_{1/2}^n x\}$ converge strongly to fixed points of T for all x in a neighborhood of the fixed point set of T; furthermore, our theorems hold for classes of mappings T more general than the class of nonexpansive mappings.

We complement these results by proving theorems under which local convergence of iterates entails global convergence; thus by combining our results in these two areas we obtain new theorems regarding the global convergence of iterates. Finally, we give an example of a class of mappings satisfying the various conditions of our theorems.

1. **Local and global convergence of iterates.** Let D be a convex subset of the Banach space X, and let $T: D \to D$. Adopting the terminology of Furi and Vignoli [6] we say that the sequence $\{T^n x_0\}$ of iterates of $x_0 \in D$ is *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T^n x - T^n x_0\| < \varepsilon$ for every $n = 1, 2, \cdots$ whenever $x \in D$ and $\|x - x_0\| < \delta$. We say that T has *stable iterates* if the sequence $\{T^n x\}$ of iterates of x is stable for every $x \in D$. Finally, if $x \in X$ and $B \subset X$ we define $d(x, B) = \inf\{\|x - y\|: y \in B\}$.

THEOREM 1. *Let D be a convex subset of a Banach space X and suppose that $T: D \to D$ has stable iterates. Let A be a nonempty subset of D.*

(i) *If there exists $\rho > 0$ such that $\{T^n x\}$ has a cluster point in A whenever $x \in D$ and $d(x, A) < \rho$, then $\{T^n x\}$ has a cluster point in A for every $x \in D$.*

(ii) *If there exists $\rho > 0$ such that $\{T^n x\}$ has its limit in A whenever $x \in D$ and $d(x, A) < \rho$, then $\{T^n x\}$ converges to some point of A for every $x \in D$.*

Proof. To prove the first statement, let $x \in D$ and $x_0 \in A$. For each $\lambda \in [0, 1]$ let $y_\lambda = \lambda x + (1 - \lambda)x_0$ and set $\lambda_0 = \sup\{\lambda \in [0, 1]: \{T^n y_\lambda\}$ has a cluster point in $A\}$. Let δ correspond to $\varepsilon = \rho/3$ in the definition of the stability of $\{T^n y_{\lambda_0}\}$, and choose $\lambda_1 \in [0, \lambda_0]$ such that $\| y_{\lambda_1} - y_{\lambda_0} \| < \delta$ and $\{T^n y_{\lambda_1}\}$ has a cluster point in A. If $\lambda_0 = 1$, let $\lambda_2 = \lambda_0$; if $\lambda_0 < 1$, let $\lambda_2 \in (\lambda_0, 1]$ be such that $\| y_{\lambda_2} - y_{\lambda_0} \| < \delta$. Since there exists a cluster point w in A of $\{T^n y_{\lambda_1}\}$ and a positive integer N such that $\| T^N y_{\lambda_1} - w \| < \rho/3$, we have that

$$\begin{aligned} \| T^N y_{\lambda_2} - w \| &\leqq \| T^N y_{\lambda_2} - T^N y_{\lambda_0} \| + \| T^N y_{\lambda_1} - T^N y_{\lambda_0} \| + \| T^N y_{\lambda_1} - w \| \\ &< \rho/3 + \rho/3 + \rho/3 = \rho . \end{aligned}$$

Thus $d(T^N y_{\lambda_2}, A) \leqq \| T^N y_{\lambda_2} - w \| < \rho$, entailing that $\{T^{N+n} y_{\lambda_2}\}$—and hence $\{T^n y_{\lambda_2}\}$—has a cluster point in A. If $\lambda_0 < 1$, this contradicts the definition of λ_0; thus $\lambda_0 = 1$, and since in this case $y_{\lambda_2} = x$, we have that $\{T^n x\}$ has a cluster point in A.

To prove the second statement, we let $x \in D$ and note that by our proof of the first statement $\{T^n x\}$ has a cluster point $w \in A$. Thus there exists a positive integer N such that $\| T^N x - w \| < \rho$, implying that $T^{N+n} x \to w$, whence $T^n x \to w$.

We remark that in the case of the second statement of the theorem above, A must contain a fixed point of T, since if T is continuous the limit of a sequence $\{T^n x\}$ is necessarily a fixed point. In our applications of this theorem we will assume either that A is the fixed point set of T or that A is a singleton.

COROLLARY 1. *Let D be a convex subset of a Banach space X, and let $T: D \to D$ possess stable iterates. Let x_0 be a fixed point of T for which there exists an open neighborhood U of x_0, $U \subset D$, such that T is continuously Fréchet differentiable in U and $\| T'x_0 \| < 1$. Then $T^n x \to x_0$, for every $x \in D$.*

Proof. Since T is continuously Fréchet differentiable in U and $\| T'x_0 \| < 1$, there exists a constant $k \in (0, 1)$ and an open ball $S(x_0, \rho)$ about x_0 with radius ρ, $S(x_0, \rho) \subset U$, such that if $x \in S(x_0, \rho)$ then $\| T'z \| < k$. Let $y \in S(x_0, \rho)$. Then there exists a point z in the segment from x_0 to y such that (see Fréchet [5])

$$\| Tx_0 - Ty \| \leqq \| T'z \| \, \| x_0 - y \| .$$

But $z \in S(x_0, \rho)$ so that $\| T'z \| < k$. Thus for every $y \in S(x_0, \rho)$

$$\| Tx_0 - Ty \| \leqq k \, \| x_0 - y \| .$$

By induction, $\| x_0 - T^n y \| \leqq k^n \| x_0 - y \|$ for every $n = 1, 2, 3, \cdots$. Since $k^n \to 0$, $T^n y \to x_0$ for every $y \in S(x_0, \rho)$. By part (ii) of Theorem 1, $T^n x \to x_0$ for every $x \in D$.

2. Conditions implying local convergence of iterates. The *modulus of convexity* of a Banach space X is the function $\delta\colon [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\|\colon \|x\| \leqq 1,\ \|y\| \leqq 1,\ \text{and}\ \|x - y\| \geqq \varepsilon\right\}.$$

It is well-known (cf. [9]) that δ is nondecreasing and continuous except possibly at 2. Furthermore, letting $\varepsilon_0 = \sup\{\varepsilon \in [0, 2]\colon \delta(\varepsilon) = 0\}$, X is uniformly convex if and only if $\varepsilon_0 = 0$, X is uniformly nonsquare if and only if $\varepsilon_0 < 2$, and X is strictly convex if and only if $\delta(2) = 1$.

We observe that if $x, y \in X$ satisfy the conditions

$$\|x\| \leqq d,\ \|y\| \leqq d\ ,\quad \text{and}\quad \|x - y\| \geqq \varepsilon\ ,\quad \text{then}$$
$$\left\|\frac{x + y}{2}\right\| \leqq \left(1 - \delta\left(\frac{\varepsilon}{d}\right)\right)d\ .$$

Finally, we denote by I the identity mapping on any convex subset of X.

THEOREM 2. *Let D be a convex subset of a uniformly nonsquare Banach space X. Suppose that $T\colon D \to D$ has a nonempty fixed point set A and that T satisfies the following conditions: There exist $\rho > 0$, $c > 0$, and $s \geqq 1$ with $(1 - \delta(c/s))s < 1$ such that if $x \in D$ and $d(x, A) < \rho$ then*

(i) $\|Tx - x\| \geqq cd(x, A)$, and

(ii) $\|Tx - u\| \leqq s\|x - u\|$ for every $u \in A$.

Then setting $F = 1/2(I + T)$, $d(F^n x, A) \to 0$ for every $x \in D$ for which $d(x, A) < \rho$.

Proof. We observe that if $x \notin A$ then

$$cd(x, A) \leqq \|Tx - x\| \leqq \|Tx - u\| + \|x - u\| \leqq (1 + s)\|x - u\|$$

for every $u \in A$. Thus $cd(x, A) \leqq (1 + s)d(x, A)$, so that if T is not the identity then $c \leqq 1 + s$. Therefore $c/s \leqq 1 + 1/s \leqq 2$, and moreover if $c/s = 2$, then $s = 1$ and $c = 2$.

Let $x \in D$ satisfy $0 < d(x, A) < \rho$, and for arbitrary $r > 1$ let $u_{x,r} \in A$ satisfy $\|x - u_{x,r}\| \leqq \text{minimum}\{\rho, rd(x, A)\}$. Thus $\|Tx - u_{x,r}\| \leqq s\|x - u_{x,r}\|$.

Let $d = s\|x - u_{x,r}\|$ and $\varepsilon = \|Tx - x\|$. Since $\|x - u_{x,r}\| \leqq d$, $\|Tx - u_{x,r}\| \leqq d$, and $\|(x - u_{x,r}) - (Tx - u_{x,r})\| = \varepsilon$ we obtain

$$\begin{aligned}\|Fx - u_{x,r}\| &= \frac{1}{2}\|(x - u_{x,r}) + (Tx - u_{x,r})\| \\ &\leqq (1 - \delta(\varepsilon/d))d\ .\end{aligned}$$

Now

$$\frac{\varepsilon}{d} = \frac{\| Tx - x \|}{s \| u_{x,r} - x \|} \geqq \frac{cd(x, A)}{srd(x, A)} = \frac{c}{sr} ,$$

and thus since δ is nondecreasing

$$1 - \delta(\varepsilon/d) \leqq 1 - \delta\left(\frac{c}{sr}\right) .$$

Therefore,

$$\begin{aligned} d(Fx, A) &\leqq \|Fx - u_{x,r}\| \leqq (1 - \delta(\varepsilon/d))d \leqq \left(1 - \delta\left(\frac{c}{sr}\right)\right)d \\ &= \left(1 - \delta\left(\frac{c}{sr}\right)\right)s \| x - u_{x,r} \| \leqq \left(1 - \delta\left(\frac{c}{sr}\right)\right)srd(x, A) , \end{aligned}$$

for every $r > 1$.

Let $\eta \equiv \lim_{r\to 1+} (1 - \delta(c/sr))sr$. Then $d(Fx, A) \leqq \eta d(x, A)$ whenever $d(x, A) < \rho$. If $c/s < 2$ then δ is continuous at c/s and $\eta = (1 - \delta(c/s))s < 1$. If $c/s = 2$ then $c = 2$ and $s = 1$, and since X is uniformly nonsquare, $\eta = 1 - \lim_{\varepsilon\to 2-} \delta(\varepsilon) < 1$. By induction, $d(F^n x, A) \leqq \eta^n d(x, A)$ whenever $d(x, A) < \rho$, implying that $d(F^n x, A) \to 0$ whenever $d(x, A) < \rho$.

COROLLARY 2. *If the hypotheses of Theorem 2 are satisfied and if in addition A is compact, then the sequence $\{F^n x\}$ has a cluster point in A whenever $d(x, A) < \rho$.*

Proof. Since whenever $d(x, A) < \rho$ we have $d(F^n x, A) \to 0$, we can select a sequence $\{a_n\} \subset A$ such that $\| F^n x - a_n \| \to 0$. The sequence $\{a_n\}$ has a cluster point $a \in A$ which is then a cluster point of $\{F^n x\}$.

We note two important consequences of Theorem 2:

REMARK 1. If the mapping of Theorem 2 (or Corollary 2) has a unique fixed point u then one may conclude that $F^n x \to u$ for every $x \in D$ for which $\| x - u \| < \rho$.

REMARK 2. If condition (ii) of Theorem 2 holds for $s = 1$ and if X is uniformly nonsquare then one need only verify that condition (i) holds for *some* $c \in (\varepsilon_0, 2]$.

By applying Theorem 1 to Corollary 2 we obtain:

COROLLARY 3. *If the hypotheses of Theorem 2 are satisfied, and* *if in addition A is compact and F has stable iterates, then the sequence $\{F^n x\}$ has a cluster point in A for every $x \in D$.*

REMARK 3. In a uniformly nonsquare space, for each $c \in (\varepsilon_0, 2]$ there always exists $s > 1$ such that $(1 - \delta(c/s))s < 1$.

Proof. Since $c > \varepsilon_0$, $\lim_{\varepsilon \to c-} \delta(\varepsilon) > 0$. Thus $\lim_{s \to 1+} (1 - \delta(c/s))s = 1 - \lim_{\varepsilon \to c-} \delta(\varepsilon) < 1$. Therefore, there exists $s > 1$ such that $(1 - \delta(c/s))s < 1$.

THEOREM 3. *Let D be a convex subset of a uniformly convex Banach space X. Let $T: D \to D$ possess a nonempty compact fixed point set A. Suppose that there exists a neighborhood U in D of A such that if $x \in U$ then $\| Tx - x \| \geqq cd(x, A)$ for some constant $c \in (0, 2]$, and such that T is continuously Fréchet differentiable in U with $\| T'x \| \leqq 1$ if $x \in A$. Then theree exists $\rho > 0$ such that if $x \in D$ and $d(x, A) < \rho$ then $d(F^n x, A) \to 0$.*

Proof. By the remark above there exists $s > 1$ such that $(1 - \delta(c/s))s < 1$. Let $u \in A$. Since T has a continuous Fréchet derivative in a neighborhood of u and $\| T'u \| \leqq 1$, there exists a neighborhood U_u in D of u such that if $x \in U_u$ then $\| Tx - u \| = \| Tx - Tu \| \leqq s \| x - u \|$. Letting $V = U \cap \bigcup_{u \in A} U_u$ and choosing $\rho > 0$ such that if $d(x, A) < \rho$ then $x \in V$, the hypotheses of Theorem 2 are satisfied. Therefore $d(F^n x, A) \to 0$, for each $x \in D$ with $d(x, A) < \rho$.

3. **Some examples.** Let D be a closed convex subset of a Banach space X. We consider first mappings $T: D \to D$ satisfying the condition

$$(1) \qquad \| Tx - Ty \| \leqq a \| x - y \| + b[\| x - Tx \| + \| y - Ty \|] + c[\| x - Ty \| + \| y - Tx \|]$$

where a, b, and c are nonnegative constants such that $a + 2b + 2c = 1$. In particular if $b = c = 0$, T is a nonexpansive mapping, while if $b = 1/2$, T is of a class of mappings investigated by Kannan [10]. A general fixed point theorem in uniformly convex spaces for mappings satisfying condition (1) has recently been proved by Goebel, Kirk, and Shimi in [8]. We now obtain the following application of Theorem 2 to mappings of this type:

THEOREM 4. *Let D be a nonempty, closed, bounded, and convex subset of a uniformly convex Banach space X and let $T: D \to D$ be a continuous mapping satisfying condition* (1) *above with $b \neq 0$.*

Then T has a unique fixed point u, and $F^n x \to u$, for every $x \in D$.

Proof. By the fixed point theorem of [9] T has at least one fixed point. If $Tu = u$ and $Tv = v$ and $u \neq v$, then by (1) $\|u - v\| \leqq (a + 2c)\|u - v\|$, which implies that $b = 0$, a contradiction. Thus T has a unique fixed point which we denote u.

If $x \in D$, then since $Tu = u$

$$(2)\quad \begin{aligned} \|Tx - u\| &\leqq a\|x - u\| + b\|x - Tx\| + c[\|x - u\| + \|u - Tx\| \\ &\leqq (a + b + c)\|x - u\| + (b + c)\|u - Tx\| . \end{aligned}$$

By combining terms we obtain for every $x \in D$

$$\|Tx - u\| \leqq \|x - u\| .$$

If $x \in D$ we have by inequality (2) above that

$$(1 - c)\|Tx - u\| \leqq (a + c)\|x - u\| + b\|x - Tx\| .$$

Thus

$$\begin{aligned} (1 - c)[\|x - u\| - \|x - Tx\|] &\leqq (1 - c)\|Tx - u\| \\ &\leqq (a + c)\|x - u\| + b\|x - Tx\| . \end{aligned}$$

Collecting terms we obtain

$$(1 + b - c)\|x - Tx\| \geqq (1 - a - 2c)\|x - u\| .$$

Since $1 + b - c > 0$ and $1 - a - 2c > 0$ we have for every $x \in D$

$$\|x - Tx\| \geqq \frac{1 - a - 2c}{1 + b - c}\|x - u\| .$$

The conditions of Theorem 2 are now satisfied (for $s = 1$ and for every $\rho > 0$), and thus in view of Remarks 1 and 2 above $F^n x \to u$ for every $x \in D$.

As another example we consider strongly pseudo-contractive mappings. If D is a convex subset of a Banach space X and $C \subset D$, a mapping $T: D \to D$ is said to be *strongly pseudo-contractive* relative to C [7] if for each $x \in X$ and $r > 0$ there exists a number $\alpha_r(x) < 1$ such that $\|x - y\| \leqq \alpha_r(x)\|(1 + r)(x - y) - r(Tx - Ty)\|$, for every $y \in C$. It is easily seen that if T has a fixed point $u \in C$, then u is the only fixed point of T. Conditions for the existence of fixed points for such mappings are given in [7]. The following theorem gives conditions under which strongly pseudo-contractive mappings satisfy condition (i) of Theorem 2.

THEOREM 5. *Let D be a convex subset of a Banach space X*

and let $T: D \to D$ be strongly pseudo-contractive relative to C. If T has a fixed point $u \in C$, and if for some $r > 0$ $\limsup_{x \to u} \alpha_r(x) < 1$, then there exists $c > 0$ and an open ball $S(u, \varepsilon)$ of radius ε about u such that if $x \in D \cap S(u, \varepsilon)$ then $\|x - Tx\| \geqq c\|x - u\|$.

Proof. Since $\limsup_{x \to u} \alpha_r(x) < 1$, there exists an open ball $S(u, \varepsilon)$ of radius ε about u and a constant $k \in (0, 1)$ such that if $x \in D \cap S(u, \varepsilon)$ then $\alpha_r(x) \leqq k$. Let $c = (1 - k)/(kr)$. Then $(1 - \alpha_r(x))/(\alpha_r(x)r) \geqq c$ for each $x \in D \cap S(u, \varepsilon)$. Since $Tu = u$, for each $x \in D \cap S(u, \varepsilon)$

$$\begin{aligned} \|x - u\| &\leqq \alpha_r(x)\|(1 + r)(x - u) - r(Tx - u)\| \\ &= \alpha_r(x)\|r(x - Tx) + (x - u)\| \\ &\leqq \alpha_r(x)r\|x - Tx\| + \alpha_r(x)\|x - u\|, \end{aligned}$$

yielding

$$\frac{1 - \alpha_r(x)}{\alpha_r(x)r}\|x - u\| \leqq \|x - Tx\|.$$

Thus

$$c\|x - u\| \leqq \|x - Tx\|$$

for every $x \in D \cap S(u, \varepsilon)$.

References

1. F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Zeit., **100** (1967), 201-225.
2. F. E. Browder and W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc., **72** (1966), 571-575.
3. J. B. Diaz and F. T. Metcalf, *On the structure of the set of subsequential limit points of successive approximations*, Bull. Amer. Math. Soc., **73** (1967), 516-519.
4. M. Edelstein, *A remark on a theorem of M. A. Krasnoselskii*, Amer. Math. Monthly, **73**(1966), 509-510.
5. M. Fréchet, *Sur les fonctionelles continues*, Ann. École Normale, **27** (1910), 193-216.
6. M. Furi and A. Vignoli, *A remark about some fixed point theorems*, Boll. Un. Mat. Ital., (4) **3** (1970), 197-200.
7. J. A. Gatica and W. A. Kirk, *Fixed-point theorems for lipschitzian pseudo-contractive mappings*, Proc. Amer. Math. Soc., (to appear).
8. K. Goebel, W. A. Kirk, and Tawfik N. Shimi, *A fixed point theorem in uniformly convex spaces*, Boll. Un. Mat. Ital., (to appear).
9. V. I. Gurarii, *Differential properties of the convexity moduli of Banach spaces*, Mat. Issled., **2** (1967), vyp. 1, 141-148.
10. R. Kannan, *Some results on fixed points—III*, Fund. Math., **70** (1971), 169-177.
11. M. A. Krasnoselskii, *Two remarks on the method of successive approximations*, Uspehi Mat. Nauk., (N. S.) **63** (1955), 123-127.
12. C. L. Outlaw, *Mean value iteration of nonexpansive mappings in a Banach space*, Pacific J. Math., **30** (1969), 747-750.
13. W. V. Petryshyn, *Construction of fixed points of demi-compact mappings in Hilbert spaces*, J. Math. Anal. Appl., **14** (1966), 276-284.

14. H. Schaeffer, *Über die Methode suksessiver Approximation*, Jber. Deutsch. Math. Verein., **59** (1957), 131-140.

Received December 18, 1972 and in revised form March 1, 1974. This paper is essentially a portion of the author's University of Iowa doctoral dissertation, written under the supervision of Professor W. A. Kirk.

THE UNIVERSITY OF IOWA

Current Address:
Department of Mathematics
The University of Nebraska-Lincoln
Lincoln, Nebraska, 68508