ON \((J, M, m)\)-EXTENSIONS OF BOOLEAN ALGEBRAS

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The class \(\mathcal{H}\) of all \((J, M, m)\)-extensions of a Boolean algebra \(\mathcal{A}\) can be partially ordered and always contains a maximum and a minimal element, with respect to this partial ordering. However, it need not contain a smallest element. Should \(\mathcal{H}\) contain a smallest element, then \(\mathcal{H}\) has the structure of a complete lattice. Necessary and sufficient conditions under which \(\mathcal{H}\) does contain a smallest element are derived. A Boolean algebra \(\mathcal{A}\) is constructed for each cardinal \(m\) such that the class of all \(m\)-extensions of \(\mathcal{A}\) does not contain a smallest element. One implication of this construction is that if a Boolean algebra \(\mathcal{A}\) is the Boolean product of a least countably many Boolean algebras, each of which has more than one \(m\)-extension, then the class of all \(m\)-extensions of \(\mathcal{A}\) does not contain a smallest element. The construction also has as implication that neither the class of all \((m, 0)\)-products nor the class of all \((m, n)\)-products of an indexed set \(\{\mathcal{A}_i\}_{i \in \sigma}\) of Boolean algebras need contain a smallest element.

1. Sikorski [2] has investigated the question of imbedding a given Boolean algebra \(\mathcal{A}\) into a complete or \(m\)-complete Boolean algebra \(\mathcal{B}\) and has shown that in the case where the imbedding map is not a complete isomorphism, the imbedding need not be unique up to isomorphism. He further has shown that if \(\mathcal{H}\) is the class of all \((J, M, m)\)-extensions of a Boolean algebra \(\mathcal{A}\), then \(\mathcal{H}\) has a naturally defined partial ordering on it and always contains a maximum and a minimal element. He has left as an open question whether it always contains a smallest element. La Grange [1] has given an example which implies that \(\mathcal{H}\) need not always contain a smallest element. However, the question of when does \(\mathcal{H}\) in fact contain a smallest element is of interest as it turns out that should \(\mathcal{H}\) contain a smallest element, it has the structure of a complete lattice.

In §2, necessary and sufficient conditions are given for \(\mathcal{H}\) to contain a smallest element. In addition, the principle behind La Grange's example is generalized in Proposition 2.10 to show that if \(\mathcal{A}\) is not \(m\)-representable then the class \(\mathcal{H}\) of all \((J, M, m')\)-extension of \(\mathcal{A}\), where \(J, M < \sigma\) and \(m' > M\), will not contain a smallest element.

Since the proof of this result requires that \(J\) and \(M\) have cardinality \(\leq \sigma\), it is of interest to ask if the class of all \(m\)-extensions
contain a smallest element in general, and the answer is no.

In § 3, a Boolean algebra \( \mathcal{A} \) is constructed for each cardinal \( m \) such that the class \( \mathcal{A} \) of all \( m \)-extensions of \( \mathcal{A} \) does not contain a smallest element. The construction has as implication (Theorems 3.1 and 3.2; Corollary 3.1) that for each algebra in a rather broad group of Boolean algebras, the class of all \( m \)-extensions will not contain a smallest element. In particular, this group includes all Boolean algebras which are the Boolean product of at least countably many Boolean algebras each of which has more than one \( m \)-extension.

Finally, in the last section, Sikorski's result that there is an equivalence between the class \( \mathcal{P} \) of all \((m, 0)\)-products of an indexed set \( \{\mathcal{A}_t\}_{t \in T} \) of Boolean algebras and the class of all \((J, M, m)\)-extensions of the Boolean product \( \mathcal{A}_0 \) of \( \{\mathcal{A}_t\}_{t \in T} \), for suitably defined \( J \) and \( M \), is generalized to show there is an equivalence between the class \( \mathcal{P}_n \) of all \((m, n)\)-products of \( \{\mathcal{A}_t\}_{t \in T} \) and all \((J, M, m)\)-extensions of \( \mathcal{F} \), where \( \mathcal{F} \) is the field of sets generated by a certain set \( \mathcal{I}_f \) for suitably defined \( J \) and \( M \). Then the above results imply that neither \( \mathcal{P} \) nor \( \mathcal{P}_n \) need contain a smallest element.

The notation throughout follows that of Sikorski [2].

2. Let \( n \) be the cardinality of a set of generators for the Boolean algebra \( \mathcal{A} \), let \( \mathcal{A}_{m,n} \) be a free Boolean \( m \)-algebra with a set of \( n \) free \( m \)-generators, let \( \mathcal{A}_{0,n} \) be the free Boolean algebra generated by this set of \( n \) free \( m \)-generators and let \( g \) be a homomorphism from \( \mathcal{A}_{0,n} \) to \( \mathcal{A} \). Let \( \Delta_0 \) be the kernel of this homomorphism and let \( I \) be the set of all \( m \)-ideals \( A \) in \( \mathcal{A}_{m,n} \) such that:

a. \( A \cap \mathcal{A}_{0,n} = \Delta_0 \);

b. \( A \) contains all the elements

\[
A_0 - \bigcup_{A \in \mathcal{I}_1} A, \quad \bigcup_{A \in \mathcal{I}_1} A - A_0, \\
A_0 - \bigcap_{A \in \mathcal{I}_2} A, \quad \bigcap_{A \in \mathcal{I}_2} A - A_0,
\]

where \( A_0 \in \mathcal{A}_{0,n} \) and \( \mathcal{I}_1, \mathcal{I}_2 \) are any subsets of \( \mathcal{A}_{0,n} \) of cardinality \( \leq m \) such that:

\[
g(\mathcal{I}_1) \in J, \quad g(A_0) = \bigcup_{A \in \mathcal{I}_1} g(A) \\
g(\mathcal{I}_2) \in M, \quad g(A_0) = \bigcap_{A \in \mathcal{I}_2} g(A).
\]

For each \( A \in I \) let

\[
\mathcal{A}_A = \mathcal{A}_{m,n}/A
\]

and

\[
g_A([A]_A) = g(A), \quad \text{for all } A \in \mathcal{A}_{0,n}.
\]

Set \( i_A = g_A^{-1} \). We need the following results due to Sikorski.
PROPOSITION 2.1. The ordered pair \( (i, A) \) is a \((J, M, m)\)-extension of the Boolean algebra \( A \) and if \( (i, B) \) is a \((J, M, m)\)-extension of \( A \) there is a \( A \in I \) such that \( (i, A) \) is isomorphic to \( (i, B) \). Further, if \( A, A' \in I \) then
\[
(i, A) \leq (i, A') \quad \text{if, and only if,} \quad A \supseteq A'.
\]

LEMMA 2.1. If \( S \) is a set of elements in \( A \) then the least upper bound (lub) of \( S \) exists in \( A \).

Now let \( \mathcal{A}(J, M, m) \) denote the class of all \((J, M, m)\)-extensions of \( A \).

THEOREM 2.1. Let \( \mathcal{A} \) be the class of all \((J, M, m)\)-extensions of a Boolean algebra \( A \). The following are equivalent:
1. \( \mathcal{A} \) contains a smallest element;
2. \( \mathcal{A} \) is a lattice;
3. \( \mathcal{A} \) is a complete lattice.

Proof.
1. \( \Rightarrow \) 3. It suffices to show that if \( S \) is a set of \((J, M, m)\)-extensions of \( A \) then the greatest lower bound (glb) of \( S \) exists in \( A \), which follows from noting that if \( L \) is the set of all lower bounds for the set \( S \) then \( L \neq 0 \) and by Lemma 2.1 the lub of \( L \) exists in \( A \), hence is in \( L \).

3. \( \Rightarrow \) 2. By definition.

2. \( \Rightarrow \) 1. If \( (i, B) \) is an \( m \)-completion of \( A \), \( (j, C) \in \mathcal{A} \), and \( \mathcal{A} \) a lattice, then there is an element \( (j', C') \in \mathcal{A} \) such that
\[
(j', C') \leq (j, C).
\]
Thus
\[
(j', C') \leq (i, B),
\]
so
\[
(j', C') \leq (i, B),
\]
implying
\[
(i, B) \leq (j, C).
\]
Hence \( (i, B) \) is a smallest element in \( A \).

COROLLARY 2.1. If \( J' \supseteq J \) and \( M' \supseteq M \) then the following are equivalent:
1. \( \mathcal{A}(J, M, m) \) contains a smallest element;
2. \( \mathcal{K}(J', M', m) \) is a sublattice of \( \mathcal{K}(J, M, m) \);  
3. \( \mathcal{K}(J', M', m) \) is a complete sublattice of \( \mathcal{K}(J, M, m) \).

**Proof.**

1. \( \Rightarrow \) 3. Since \( \mathcal{K}(J', M', m) \) contains a smallest element, so does \( \mathcal{K}(J, M, m) \) hence \( \mathcal{K}(J', M', m) \) and \( \mathcal{K}(J, M, m) \) are complete lattices. If \( \{ \{ i_t, \mathcal{B}_t \}_{t \in T} = S \) is a set of elements in \( \mathcal{K}(J', M', m) \), \( \{ i, \mathcal{C} \} \) is the lub of \( S \) in \( \mathcal{K}(J, M, m) \) and \( \{ i', \mathcal{C}' \} \) is the lub of \( S \) in \( \mathcal{K}(J', M', m) \), then there is an \( m \)-homomorphism \( h \) mapping \( \mathcal{C}' \) onto \( \mathcal{C} \) such that \( h i' = i \). Hence \( i \) is a \((J', M', m)\)-isomorphism. Thus \( \{ i, \mathcal{C} \} \in \mathcal{K}(J', M', m) \), implying

\[
\{ i, \mathcal{C} \} = \{ i', \mathcal{C}' \}.
\]

If \( \{ i, \mathcal{C} \} \) is the glb of \( S \) in \( \mathcal{K}(J, M, m) \) and \( \{ i', \mathcal{C}' \} \in S \), then by a similar argument, \( i \) is a \((J', M', m)\)-isomorphism, which implies \( \{ i, \mathcal{C} \} \) is the glb of \( S \) in \( \mathcal{K}(J', M', m) \).

3. \( \Rightarrow \) 2. By definition.

2. \( = \) 1. The proof is the same as that for showing 2. \( = \) 1, in Theorem 2.1.

Thus it is of particular interest to know whether \( \mathcal{K}(J, M, m) \) contains a smallest element, in general. Although, as it turns out, \( \mathcal{K}(J, M, m) \) need not contain a smallest element in general, a minimal \((J, M, m)\)-extension is always an \( m \)-completion, hence there is always a unique minimal \((J, M, m)\)-extension in \( \mathcal{K}(J, M, m) \).

**Proposition 2.2.** An \( m \)-completion \( \{ i, \mathcal{B} \} \) of the Boolean algebra \( \mathcal{A} \) is a unique minimal element in \( \mathcal{K} \).

**Proof.** That a minimal element in \( \mathcal{K} \) is an \( m \)-completion is clear.

If \( \{ i', \mathcal{B}' \} \) is another minimal element in \( \mathcal{K} \), there are \( \Delta, \Delta' \in I \) such that

\[
\{ i, \mathcal{B} \} = \{ i_\Delta, \mathcal{A}_\Delta \}
\]

and

\[
\{ i', \mathcal{B}' \} = \{ i_{\Delta'}, \mathcal{A}_{\Delta'} \}.
\]

Now \( \{ i, \mathcal{B} \} \) and \( \{ i', \mathcal{B}' \} \) minimal in \( \mathcal{K} \) imply \( \Delta \) and \( \Delta' \) are maximal \( m \)-ideals in \( I \), but if \( \Delta \) is a maximal \( m \)-ideal in \( I \) then \( g:\( \mathcal{K}, \mathcal{A} \) \) is dense in \( \mathcal{A}_\Delta \). The ideal \( \hat{\Delta}' = \langle \hat{\Delta}, A \rangle \) in \( \mathcal{A}_{\Delta, a} \) is an \( m \)-ideal and \( \Delta' \in I \), contradicting the maximality of \( \hat{\Delta} \). So \( \{ i', \mathcal{B}' \} \) is an \( m \)-completion of \( \mathcal{K} \), hence isomorphic to \( \{ i, \mathcal{B} \} \), implying
\{i', \mathcal{B}'\} = \{i, \mathcal{B}\}.

**Proposition 2.3.** If $\mathcal{A}$ is a Boolean $m$-algebra that satisfies the $m$-chain condition and

$$\bigcup_{t \in T} A_t$$

is the join of an indexed set \{A_t\}_{t \in T} in $\mathcal{A}$, then there is an indexed set \{A'_t\}_{t \in T} of disjoint elements of $\mathcal{A}$ such that

1. $\bigcup_{t \in T} A'_t = \bigcup_{t \in T} A_t$;
2. $A'_t \subseteq A_t$ for all $t \in T$.

**Proof.** Let $\mathcal{S}$ be the collection of all sets $S$ of disjoint elements in $\mathcal{A}$ such that for each $s \in S$ there is a $t \in T$ with $s \subseteq A_t$. If

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_i \subseteq \cdots$$

is a chain of sets in $\mathcal{S}$ indexed by $I$ and ordered by set theoretical inclusion, then

$$\bigcup_{i \in I} S_i = S \in \mathcal{S}.$$

By Zorn's lemma there is a maximal set in $\mathcal{S}$, say $S' = \{A_r\}_{r \in R}$, and it immediately follows that

$$\bigcup_{r \in R} A_r \neq A.$$

Now let

$$\varphi: S' \longrightarrow T$$

be a mapping such that if $A_r \in S'$ then

$$A_r \subseteq A_{\varphi(A_r)}.$$

For each $t \in T$ define

$$A'_t = \bigcup \{A_r \in S': \varphi(A_r) = t\}$$

if there is an $A_r \in S'$ such that $\varphi(A_r) = t$, otherwise define

$$A'_t = \Lambda.$$

Then

$$\{A'_t\}_{t \in T}$$

is the desired set.

**Proposition 2.4.** Let $\mathcal{A}$ be a Boolean algebra. The following are equivalent:
1. \( \mathcal{A} \) satisfies the \( m \)-chain condition:
2. for all sets \( S \) in \( \mathcal{A} \) such that \( \bigcup_{s \in S} s \) exists,

\[ \bigcup_{s \in S} s = \bigcup_{s \in S'} s \]

for some set \( S' \subseteq S \) with \( S' \leq m \); and dually for meets.

Proof.
1. \( \Rightarrow \) 2. Suppose \( \mathcal{A} \) satisfies the \( m \)-chain condition. It suffices to show that if

\[ S = \{A_t\}_{t \in T} \text{ and } V = \bigcup_{t \in T} A_t, \quad \overline{T} = m' > m, \]

then there is a set \( T' \subseteq T, \overline{T}' \leq m, \) such that

\[ \bigcup_{t \in T'} A_t = V. \]

Let \( \{i, \mathcal{B}\} \) be an \( m' \)-completion of \( \mathcal{A} \). Then \( \mathcal{B} \) satisfies the \( m \)-chain condition and

\[ V^{\mathcal{B}} = i(V^{\mathcal{A}}) = \bigcup_{t \in T'} i(A_t). \]

By Proposition 2.3, there is a set \( \{B_t\}_{t \in T} \) of disjoint elements in \( \mathcal{B} \) such that

\[ B_t \subseteq i(A_t) \quad \text{and} \quad \bigcup_{t \in T} B_t = \bigcup_{t \in T} i(A_t). \]

Since this set contains at most \( m \)-distinct elements,

\[ \bigcup_{t \in T} B_t = \bigcup_{t \in T'} B_t, \]

\( T' \subseteq T \) and \( \overline{T}' \leq m. \) Thus

\[ V^{\mathcal{B}} = \bigcup_{t \in T'} i(A_t) \]

or

\[ V^{\mathcal{A}} = \bigcup_{t \in T'} A_t. \]

2. \( \Rightarrow \) 1. Suppose \( \{A_t\}_{t \in T} \) is an \( m' \)-indexed set of disjoint elements of \( \mathcal{A}, m' > m. \) It may be assumed that \( \{A_t\}_{t \in T} \) is a maximal set of disjoint elements of \( \mathcal{A}. \) Then for some \( T' \subseteq T, \overline{T}' \leq m, \)

\[ V^{\mathcal{A}} = \bigcup_{t \in T'} A_t. \]

Since \( \overline{T}' \neq \overline{T}, \) there is a \( t \in T - T' \) such that
\[ A_{t_0} \in \{A_t\}_{t \in T} - \{A_t\}_{t \in T}, \quad \text{and} \quad A_{t_0} \neq \bigwedge_{\infty}. \]

Thus
\[ \bigcup_{t \in T} A_t \neq V_{\infty}, \]
a contradiction. Hence \( \bar{T} \subseteq m. \)

This gives, as an immediate corollary, the following result due to Sikorski [2].

**Corollary 2.2.** If \( \mathcal{A} \) is a Boolean \( m \)-algebra and satisfies the \( m \)-chain condition, it is a complete Boolean algebra.

**Proposition 2.5.** The class \( \mathcal{K}(J, M, m') \) contains a smallest element if \( \mathcal{K}(J, M, m) \) contains a smallest element, \( m' < m \).

**Proof.** Let \( \{i, \mathcal{B}\} \) be the smallest element in \( \mathcal{K}(J, M, m) \). If \( \{j, \mathcal{C}\} \in \mathcal{K}(J, M, m') \), let \( \{k, \mathcal{D}\} \) be an \( m \)-completion of \( \mathcal{C} \). Then \( \{kj, \mathcal{E}\} \in \mathcal{K}(J, M, m) \).

By the fact that \( \{i, \mathcal{B}\} \) is the smallest element in \( \mathcal{K}(J, M, m) \), there is an \( m \)-homomorphism \( h \) such that

\[ h: \mathcal{E} \longrightarrow \mathcal{B} \quad \text{and} \quad hkj = i. \]

Also \( \{i, \mathcal{B}\} \) an \( m \)-completion of \( \mathcal{A} \) implies that there is an \( m' \)-completion \( \{i, \mathcal{B}'\} \) of \( \mathcal{A} \) such that \( \mathcal{B}' \subseteq \mathcal{B} \). Thus \( hk(\mathcal{C}') \) is an \( m \)-subalgebra of \( \mathcal{B} \), hence \( \mathcal{B}' \subseteq hk(\mathcal{C}') \) and is an \( m \)-subalgebra of \( \mathcal{C} \).

Now \( kj(\mathcal{A}) \) \( m \)-generates \( k(\mathcal{C}') \) in \( \mathcal{C} \) and \( kj(\mathcal{A}) \subseteq h^{-1}(\mathcal{B}') \), hence

\[ h^{-1}(\mathcal{B}') \supseteq k(\mathcal{C}'), \]
or

\[ h(h^{-1}(\mathcal{B}')) \supseteq hk(\mathcal{C}'). \]

But

\[ h(h^{-1}(\mathcal{B}')) = \mathcal{B}', \]

thus

\[ \mathcal{B}' \supseteq hk(\mathcal{C}'), \]

so

\[ \mathcal{B}' = hk(\mathcal{C}'). \]

Since \( hkj = i, \)
\{(i, B') \leq (kj, k(C'))\}.

But a complete isomorphism implies that
\{(kj, k(C')) \cong (j, C')\},
and since isomorphic elements in \(\mathcal{A}(J, M, m)\) have been identified,
\{(i, B') = (j, C')\}.

**Lemma 2.2.** If \(\bar{J} \leq \sigma\) and \(\bar{M} \leq \sigma\) then there is a \((J, M, m)\)-isomorphism \(i\) of a Boolean algebra \(\mathcal{A}\) into the field \(\mathcal{F}\) of all subsets of a space.

**Proposition 2.6.** If the Boolean algebra \(\mathcal{A}\) is m-representable but not \(m^+\)-representable, \(m^+\) the smallest cardinal greater than \(m\), then \(\mathcal{A}(J, M, m^+)\) does not contain a smallest element if
\(\mathcal{A}_s(J, M, m^+) \neq \emptyset\).

If \(\bar{J} \leq \sigma\), \(\bar{M} \leq \sigma\) then \(\mathcal{A}_s(J, M, m^+) \neq \emptyset\).

**Proof.** Suppose \(\{j, C\} \in \mathcal{A}_s(J, M, m^+)\). Then \(\mathcal{A}\) is m-representable and if an \(m^+\)-completion \((i, B')\) of \(\mathcal{A}\) is a smallest element in \(\mathcal{A}(J, M, m^+)\), there is a surjective \(m^+\)-homomorphism
\(h: C \rightarrow B\),
which implies \(B\) is \(m^+\)-representable, hence \(\mathcal{A}\) is \(m^+\)-representable, a contradiction. Thus \(\mathcal{A}(J, M, m^+)\) does not contain a smallest element if \(\mathcal{A}_s(J, M, m^+) \neq \emptyset\).

If \(\bar{J} \leq \sigma\) and \(\bar{M} \leq \sigma\) then \(\mathcal{A}\) is \((J, M, m^+)\)-representable by Lemma 2.2, hence \(\mathcal{A}_s(J, M, m^+) \neq \emptyset\).

The next proposition is an easy generalization of Sikorski’s [2] Proposition 25.2 and will be needed for the last theorem in this section.

**Proposition 2.7.** A Boolean algebra \(\mathcal{A}\) is completely distributive, if, and only if, it is atomic.

**Corollary 2.3.** A Boolean algebra \(\mathcal{A}\) is completely distributive, if, and only if, \(\mathcal{A}\) is \(m\)-distributive, \(m = \mathcal{A}\).

The following proposition is due to Sikorski [2] and will be given without proof.

**Proposition 2.8.** If the Boolean algebra \(\mathcal{A}\) is \(m\)-distributive, then \(\mathcal{A}(J, M, m)\) contains a smallest element for arbitrary \(J\) and \(M\).
Lemma 2.3. If \( \{i, B\} \) is an \( m \)-extension of the Boolean algebra \( A \) and \( B \) is \( m \)-representable, then \( A \) is \( m \)-representable.

Proof. This follows immediately from the fact that \( A \) is \( m \)-regular in \( B \).

Now to prove the main theorem of this section.

Theorem 2.2. Let \( A \) be a Boolean algebra. Then the following are equivalent:

1. \( A \) contains a smallest element for arbitrary \( J, M \), and \( m \);
2. \( A \) is \( m \)-representable for all \( m \);
3. \( A \) is completely distributive;
4. \( A \) is atomic;
5. an \( m \)-completion of \( A \) is atomic for all \( m \);
6. an \( m \)-completion of \( A \) is in \( \mathcal{H}(J, M, m) \) for arbitrary \( J, M \), and \( m \);
7. \( \mathcal{H}(J, M, 2^m) \) contains a smallest element, where \( J = M = \emptyset \) and \( \mathcal{I} = m^* \).

Proof.

1. \( \Rightarrow \) 2. If \( A \) is \( m \)-representable but not \( m^* \)-representable, then Proposition 2.6 implies \( \mathcal{H}(J, M, m^*) \) does not contain a smallest element if \( J, M < \sigma \).

2. \( \Rightarrow \) 3. This follows from the fact that if a Boolean algebra \( A \) is \( 2^m \)-representable, it is \( m \)-distributive.

3. \( \Rightarrow \) 4. This follows from Proposition 2.7.

4. \( \Rightarrow \) 1. This follows from Proposition 2.8.

4. \( \Rightarrow \) 5. If \( \{i, B\} \) is an \( m \)-completion of \( A \) then \( i(A) \) is dense in \( B \), so \( B \) is atomic, and conversely.

2. \( \Rightarrow \) 6. This follows from noting that 2. \( \Rightarrow \) 3. and \( A \) completely distributive implies an \( m \)-completion of \( A \) is completely distributive, hence \( m \)-representable for all cardinals \( m \).

6. \( \Rightarrow \) 2. This follows from Lemma 2.3.

3. \( \Rightarrow \) 7. If \( J = M = \emptyset \) and \( \mathcal{H}(J, M, 2^m) \) contains a smallest element, then by Proposition 2.6, \( A \) is \( 2^m \)-representable, hence \( m^* \)-distributive. Since \( m^* = \mathcal{I} \), \( A \) is completely distributive, by Corollary 2.3. The converse is clear.
3. The example in § 2 of a Boolean algebra \( \mathcal{A} \) such that the class of all \((J, M, m)\)-extensions of \( \mathcal{A} \) does not contain a smallest element depends on the assumption that \( \bar{J}, \bar{M} \subseteq \sigma \). Thus it is of interest to know whether an example can be found showing that the class of all \( m \)-extensions of \( \mathcal{A} \) does not contain a smallest element, since this corresponds to the case where \( J \) and \( M \) are as large as possible. As it turns out, there are Boolean algebras \( \mathcal{A} \) such that the class of all \( m \)-extensions \( \mathcal{A} \) does not contain a smallest element.

In this section such an example will be constructed for each infinite cardinal \( m \) and several general types of Boolean algebras such that \( \mathcal{A} \) does not contain a smallest element will be given.

Throughout this section \( \mathcal{A} \) will denote the class of all \( m \)-extensions of a Boolean algebra \( \mathcal{A} \) and \( \mathcal{A}(J, M, m) \) the class of all \((J, M, m)\)-extensions.

If \( \mathcal{A} \) is a Boolean algebra and \( \{i, \mathcal{C}\} \in \mathcal{A}(J, M, m) \), let
\[
K(\mathcal{C}) = \{C \in \mathcal{C}: \text{if } i(A) \subseteq C, A \in \mathcal{A}, \text{then } A = \bigwedge_{\mathcal{A}} \},
\]
and
\[
K_p(\mathcal{C}) = \{C \in \mathcal{C}: \text{if } P = \{A \in \mathcal{A}: i(A) \supseteq C\} \text{ then } \bigcap_{A \in P} A = \bigwedge_{\mathcal{A}} \}.
\]

Note that \( K_p(\mathcal{C}) \subseteq K(\mathcal{C}) \).

**Lemma 3.1.** The set \( K_p(\mathcal{C}) \) is an ideal if and only if \( K(\mathcal{C}) \) is an ideal.

**Proof.** It follows easily that \( K_p(\mathcal{C}) \) is an ideal. If \( K(\mathcal{C}) \) is an ideal and \( \mathcal{C} \in K(\mathcal{C}) \) let
\[
P = \{A \in \mathcal{A}: i(A) \supseteq C\}.
\]
If \( A' \in \mathcal{A} \) and \( A' \subseteq A \) for all \( A \in P \), then
\[
i(A') = C \in K(\mathcal{C}).
\]
Now \( i(A') \cap C \in K(\mathcal{C}) \), hence
\[
i(A') = (i(A') - C) \cup (i(A') \cap C) \in K(\mathcal{C}),
\]
which implies \( i(A') = \bigwedge_{\mathcal{A}} \) or \( A' = \bigwedge_{\mathcal{A}} \). Thus
\[
\bigcap_{A \in P} A = \bigwedge_{\mathcal{A}},
\]
so \( C \in K_p(\mathcal{C}) \), and
\[
K_p(\mathcal{C}) = K(\mathcal{C}).
\]
Since \( K_p(\mathcal{C}) \) is an ideal, the converse is true.
Proposition 3.1. If $\mathcal{N}$ is a Boolean algebra the following are equivalent:

1. $\mathcal{N}(J, M, m)$ contains a smallest element;
2. $K(\mathscr{C}) = K_F(\mathscr{C})$ for all $\{i, \mathscr{C}\} \in \mathcal{N}(J, M, m)$;
3. $K(\mathscr{C}) = K_F(\mathscr{C})$ if $\{i, \mathscr{C}\}$ is the maximum element in $\mathcal{N}(J, M, m)$.

Proof.

1. $\Rightarrow$ 2. Suppose $\mathcal{N}(J, M, m)$ contains a smallest element $\{i, \mathscr{C}\}$, and there is an element

$$\{j, \mathscr{C}\} \in \mathcal{N}(J, M, m)$$

with the property that

$$K(\mathscr{C}) \neq K_F(\mathscr{C}).$$

Let $h$ be the unique $m$-homomorphism mapping $\mathscr{C}$ onto $\mathscr{B}$ such that $hj = i$. Let $\ker h$ be the kernel of this mapping. Then

$$K_F(\mathscr{C}) \subseteq \ker h \subseteq K(\mathscr{C}),$$

and

$$\ker h \neq K(\mathscr{C}).$$

Pick $x \in K(\mathscr{C}) - \ker h$ and let

$$\mathcal{A} = \langle x \rangle,$$

so $\mathcal{A}$ is a complete ideal. Thus

$$\{i_A, \mathscr{C}/\mathcal{A}\} \in \mathcal{N}(J, M, m),$$

where

$$i_A: \mathcal{A} \rightarrow \mathcal{C}/\mathcal{A}$$

is defined by

$$i_A(A) = [i(A)]_A.$$  

Consequently, there are unique homomorphisms $h_A$ and $h'$ mapping $\mathscr{F}$ onto $\mathcal{C}/\mathcal{A}$, $\mathcal{C}/\mathcal{A}$ onto $\mathscr{B}$, and satisfying $h_Aj = i_A$, $h'i_A = i$, respectively. Hence

$$h'h_Aj = h'i_A = i$$

and by the uniqueness of $h$,

$$h = h'h_A.$$

This implies

$$h(x) = h'h_A(x) = \bigwedge_{\mathcal{A}}.$$
a contradiction. Thus

\[ K(\mathcal{E}') = K_p(\mathcal{E}') . \]

2. \( \rightarrow \) 3. Obvious.

3. \( \rightarrow \) 1. To show that \( \mathcal{A}'(J, M, m) \) contains a smallest element, let \( \{j, \mathcal{C}\} \) be the largest element in \( \mathcal{A}'(J, M, m) \) and suppose \( \{j', \mathcal{C}'\} \in \mathcal{A}'(J, M, m) \). Let \( \{i, \mathcal{D}\} \) be an \( m \)-completion of \( \mathcal{A} \). Then there is an \( m \)-homomorphism \( h' \) mapping \( \mathcal{C} \) onto \( \mathcal{C}' \) such that \( h'j = j' \) and an \( m \)-homomorphism \( h \) mapping \( \mathcal{C} \) onto \( \mathcal{D} \) such that \( hj = i \). Thus

\[ K_p(\mathcal{E}') \subseteq \ker h \subseteq K(\mathcal{E}') , \]

which implies, by assumption, that

\[ K_p(\mathcal{E}') = \ker h = K(\mathcal{E}') , \]

so \( K_p(\mathcal{E}') \) and \( K(\mathcal{E}') \) are \( m \)-ideals in \( \mathcal{E} \). Further,

\[ h'(K_p(\mathcal{E}')) \subseteq K_p(\mathcal{E}') \subseteq K(\mathcal{E}') \subseteq h'(K(\mathcal{E}')) . \]

This implies that

\[ h'(K_p(\mathcal{E}')) = K_p(\mathcal{E}') = K(\mathcal{E}') = h'(K(\mathcal{E}')) , \]

hence \( K(\mathcal{E}') \) is an \( m \)-ideal. Let

\[ \Delta = K(\mathcal{E}') . \]

Then \( \mathcal{E}'/\Delta \) is an \( m \)-algebra and

\[ \bar{j}'(\mathcal{A}) = \{[j'(A)]_\Delta : A \in \mathcal{A}\} \]

\( m \)-generates \( \mathcal{E}'/\Delta \). Finally, \( \bar{j}'(\mathcal{A}) \) is dense in \( \mathcal{E}'/\Delta \). Thus \( \{j', \mathcal{E}'/\Delta\} \) is an \( m \)-completion of \( \mathcal{A} \), hence is equal to \( \{i, \mathcal{D}\} \), as isomorphic elements of \( \mathcal{A}'(J, M, m) \) have been identified. The \( m \)-homomorphism

\[ h_\Delta : \mathcal{E}' \longrightarrow \mathcal{E}'/\Delta \]

defined by

\[ h_\Delta(C') = [C']_\Delta \]

has the property that

\[ h_\Delta j = j_\Delta \text{ for all } A \in \mathcal{A} , \]

implying that

\[ \{i, \mathcal{E}'/\Delta\} \leq \{j', \mathcal{E}'\} . \]
Hence \(\mathcal{X}(J, M, m)\) contains a smallest element.

This, then, gives a way to construct a Boolean algebra \(\mathcal{X}\) such that \(\mathcal{X}\) does not contain a smallest element. Namely, by finding a Boolean algebra \(\mathcal{X}\) with an \(m\)-extension \(\{i, M\}\) such that \(K_p(C) \neq K(C)\). The next task is to construct such a Boolean algebra.

If \(\overline{T} = m\) and \(\mathcal{A} = \mathcal{A}_t\) for all \(t \in T\), the Boolean product of \(\{\mathcal{A}_t\}_{t \in T}\) will be called the \(m\)-fold product of \(\mathcal{A}\). Note that if \(\mathcal{A}\) is a subalgebra of the Boolean algebra \(\mathcal{A}'\), \(\mathcal{F}\) is the \(m\)-fold product of \(\mathcal{A}\) and \(\mathcal{F}'\) is the \(m\)-fold product of \(\mathcal{A}'\), then \(\mathcal{F} \subseteq \mathcal{F}'\).

**Lemma 3.2.** If \(\mathcal{A}\) is an \(m\)-regular subalgebra of the Boolean algebra \(\mathcal{A}'\) then the Boolean \(m\)-fold product \(\mathcal{F}\) of \(\mathcal{A}\) is isomorphic to an \(m\)-regular subalgebra of the Boolean \(m\)-fold product \(\mathcal{F}'\) of \(\mathcal{A}'\).

**Proof.** Since \(\mathcal{A}\) is a subalgebra of \(\mathcal{A}'\), \(\mathcal{F} \subseteq \mathcal{F}'\). Let \(\mathcal{I}(\mathcal{F}')\) be the set of all \(\mathcal{F}(A), A \in \mathcal{A}\) and \(t \in T(A \in \mathcal{A}'\) and \(t \in T\)). Then \(F \in \mathcal{I}(F \in \mathcal{F}')\) implies \(-F \in \mathcal{I}(-F \in \mathcal{F}')\) and \(\mathcal{I}(\mathcal{F}')\) are sets of generators for \(\mathcal{I}(\mathcal{F}')\). For elements \(F \in \mathcal{F}'\) of the form

\[ F = \bigcap_{t=1}^{N} F_t, \quad F_t \in \mathcal{F}, \]

define

\[ \lambda_t(F) = \left\{ \pi_t(x) : x \in \bigcap_{t=1}^{N} F_t \right\}. \]

Note that if \(F \in \mathcal{F}'\) and \(t \in T\) is such that \(\lambda_t(F) \neq \forall \mathcal{F}\), then \(\mathcal{F}(\lambda_t(F)) = F\).

In order to show \(\mathcal{F}\) is \(m\)-regular in \(\mathcal{F}'\), it suffices to prove that if \(\{F_t\}_{t \in T}\) is an \(m\)-indexed set of elements of \(\mathcal{F}\) such that

\[ \bigcap_{t \in T} F_t = \Lambda \mathcal{F} \]

then

\[ \bigcap_{t \in T} F_t = \Lambda \mathcal{F}'. \]

Now \(F_t \in \mathcal{F}\) so \(F_t\) may be rewritten as

\[ F_t = \bigcap_{p=1}^{P} \bigcup_{q=1}^{Q} F_{p,q,t}, \]

where \(P_t, Q_t\) are finite numbers and \(F_{p,q,t} \in \mathcal{F}\) for all \(p \in P_t, q \in Q_t\), and \(t \in T\). Thus
\[
\Lambda_{\mathcal{S}} = \bigcap_{t \in T} \bigcap_{p=1}^{P_t} \bigcup_{q=1}^{Q_t} E_{p,q,t} \\
= \bigcap_{s \in S} \bigcup_{q=1}^{Q_s} E_{s,q}
\]

after a suitable re-indexing, where \( S \subseteq m \) and \( E_{s,q} = E_{p,q,t} \) for suitable \( p \in P_s, t \in T \). Without loss of generality, assume that for each \( s \in S, \lambda_t(E_{s,q}) \neq \Lambda_{\mathcal{S}} \) implies \( \lambda_t(E_{s,q'}) = \bigvee_{\mathcal{S}} \) for all \( t \in T \) and \( q' \neq q \), and that \( E_{s,q} \neq \bigvee_{\mathcal{S}} \) for all \( q, 1 \leq q \leq Q_s \), and all \( s \in S \). Suppose \( F'' \in F' \) and \( F'' \subseteq F_t \) for all \( t \in T \). Then

\[
F' = \bigcup_{m=1}^{M} \bigcap_{n=1}^{N} F'_{m,n}, \quad F'_{m,n} \in \mathcal{S}'
\]

so

\[
\bigcap_{n=1}^{N} F'_{m,n} \subseteq \bigcup_{q=1}^{Q_s} E_{s,q}
\]

for \( 1 \leq m \leq M \), and all \( s \in S \). Thus to show \( F' = \Lambda_{\mathcal{S}} \), it suffices to prove that if

\[
\bigcap_{n=1}^{N} F'_{n} \subseteq \bigcup_{q=1}^{Q_s} E_{s,q}
\]

for all \( s \in S \), where \( F'_{n} \in \mathcal{S}' \), then

\[
\bigcap_{n=1}^{N} F'_{n} = \Lambda_{\mathcal{S}}.
\]

It may be assumed that for each \( n, 1 \leq n \leq N \), \( \lambda_t(F'_{n}) \neq \bigwedge_{\mathcal{S}} \) implies \( \lambda_t(F'_{n'}) = \bigvee_{\mathcal{S}} \) for all \( t \in T \) and \( n' \neq n \), and that \( F'_{n} \neq \bigvee_{\mathcal{S}} \) for all \( n, 1 \leq n \leq N \).

Now

\[
\bigcap_{n=1}^{N} F'_{n} \subseteq \bigcup_{q=1}^{Q_s} E_{s,q}
\]

implies

\[
\bigcap_{n=1}^{N} F'_{n} \cap \bigcup_{q=1}^{Q_s} -E_{s,q} = \Lambda_{\mathcal{S}},
\]

and as each \( F'_{n} \) and \( -E_{s,q} \) is of the form \( \mathcal{P}_t(A) \) for some \( A \in \mathcal{A} \) and \( t \in T \), the independence of the indexed set \( \{\mathcal{P}_t(\mathcal{A})\}_{t \in T} \) of subalgebras of \( \mathcal{F}' \) implies that for some \( n_s, 1 \leq n_s \leq N \), and some \( q_s, 1 \leq q_s \leq Q_s \),

\[
F'_{n_s} \cap -E_{s,q_s} = \Lambda_{\mathcal{S}},
\]

which implies \( F'_{n_s} \subseteq F_{s,q_s} \). This argument may be repeated for each \( s \in S \).
The set \( \{n_s: s \in S\} \) is finite so let \( \{n_s: s \in S\} = \{n_i: 1 \leq i \leq N'\} \). Let \( S_i = \{s \in S: F_{s,i} \subseteq F_{s,q}\} \). If \( t_s \in T \) is such that
\[
\lambda_{t_s}(F_{s,q}) \neq \bigvee \mathcal{A}
\]
then \( \lambda_{t_s}(F_{s,q}) \in \mathcal{A} \) and
\[
\bigcap_{s \in S_i} \lambda_{t_s}(F_{s,q}) \neq \bigwedge \mathcal{A}.
\]
Thus
\[
\bigcap_{s \in S_i} \lambda_{t_s}(F_{s,q}) \neq \bigwedge \mathcal{A},
\]
or
\[
\bigcap_{s \in S_i} \lambda_{t_s}(F_{s,q}) \neq \bigwedge \mathcal{A},
\]
hence there is an \( A_i \in \mathcal{A}, A_i \neq \bigwedge \mathcal{A} \), with
\[
A_i \subseteq \lambda_{t_s}(F_{s,q}) \text{ for all } s \in S_i.
\]
Let \( A_{t,i} \) be the set of all \( x \in \mathcal{X} \) such that \( \pi_{t,i}(x) \in A_i \). Thus \( A_{t,i} \in \mathcal{F} \) and this argument may be repeated for each \( i, 1 \leq i \leq N' \). Now
\[
\bigwedge \mathcal{A} \neq \bigcap_{i=1}^{N'} A_{t,i}
\]
and
\[
\bigcap_{i=1}^{N'} A_{t,i} \subseteq F_{s,q},
\]
for all \( s \in S \). But then
\[
\bigcap_{i=1}^{N'} A_{t,i} \subseteq \bigcap_{s \in S} \bigcup_{q=1}^{N'} F_{s,q} = \bigwedge \mathcal{A},
\]
a contradiction. Thus \( \mathcal{F} \) is \( m \)-regular in \( \mathcal{F}' \).

The next lemma assumes there is a Boolean algebra \( \mathcal{A} \) such that an \( m \)-extension is not an \( m \)-completion. Sikorski [2] cites an example due to Katetov of such a Boolean algebra for the case \( m = \sigma \). As Lemmas 3.5 and 3.6 imply, there is such an \( \mathcal{A} \) for all infinite cardinal numbers \( m \).

Assume for the moment that \( \mathcal{A} \) is a Boolean algebra such that \( \mathcal{K} \) contains more than one element and \( \{i, B\} \in \mathcal{H} \) is an \( m \)-extension that is not an \( m \)-completion. Thus there is a \( B \in \mathcal{B} \) such that \( i(A) \subseteq B, A \in \mathcal{A} \), implies \( A = \bigwedge \mathcal{A} \). Let \( \mathcal{F}' \) be the Boolean \( m \)-fold product of \( \mathcal{B}, h_0 \) an isomorphism of \( \mathcal{B} \) onto the Stone space \( \mathcal{F} \) of
the Cartesian product of $J_f$ with itself $m$ times and indexed by $T$, and
\[ B_t = \varphi, h_t(B) \text{ for all } t \in T. \]

Let
\[ B_0 = \bigcup_{t \in T} B_t, \]
where $T'$ is a fixed, but arbitrary subset of $T$ such that $T' \supseteq \sigma$, and define
\[ \mathcal{F}_0 = \langle \mathcal{F}', B_0 \rangle. \]

Since $T' \supseteq \sigma$, $\mathcal{F}_0 \neq \mathcal{F}'$.

**Lemma 3.3.** If $\mathcal{F}$ is the Boolean $m$-fold product of $\mathcal{N}$ then $\mathcal{F}$ is isomorphic to an $m$-regular subalgebra of $\mathcal{F}_0$.

**Proof.** It may be assumed, without loss of generality, that $\mathcal{A} \subseteq \mathcal{B}$. Thus $\mathcal{F} \subseteq \mathcal{F}_0$. Let $\mathcal{I}(\mathcal{F}')$ be a generating set for $\mathcal{F}(\mathcal{F}')$. Let
\[ \mathcal{I}_0 = \mathcal{I}' \cup \{B_0\}, \]
so $\mathcal{I}_0$ is a generating set for $\mathcal{F}_0$. As in the previous lemma, to prove $\mathcal{F}$ is $m$-regular in $\mathcal{F}_0$ it suffices to show that if
\[ \bigcap_{n=1}^{N} F'_n \subseteq \bigcup_{q=1}^{Q} F_{s,q} \]
for all $s \in S$, $\bar{s} \leq m$; and
\[ \bigcap_{s \in S} \bigcup_{q=1}^{Q} F_{s,q} = \bigwedge \mathcal{F}; \]
$F_{s,q} \in \mathcal{I}$ for all $s \in S$ and $1 \leq q \leq Q$, $F'_n \in \mathcal{I}_0$, $1 \leq n \leq N$; then
\[ \bigcap_{n=1}^{N} F'_n = \bigwedge \mathcal{F}_0. \]

Since $F'_n \in \mathcal{I}_0$, there is an $n$, $1 \leq n \leq N$, such that $F'_n = B_0$ or $F'_n = -B_0$, otherwise there is nothing to prove. This may be reduced to two cases:

**Case 1.**
\[ \bigcap_{n=1}^{N} F'_n \cap B_0 \subseteq \bigcup_{q=1}^{Q} F_{s,q} \]
for all $s \in S$, where $F'_n \in \mathcal{I}'$ and $F_{s,q} \in \mathcal{I}$. 

\[ \]
Case 2.

\((-B_0) \cap \bigcap_{n=1}^{N} F'_n \subseteq \bigcup_{q=1}^{Q_s} F_{s,q}\)

for all \(s \in S\), where \(F'_n \in \mathcal{F}'\) and \(F_{s,q} \in \mathcal{F}'\).

**Proof of Case 1.** If for each \(s \in S\) there is an \(n_s, 1 \leq n_s \leq N\), such that there is a \(q_s, 1 \leq q_s \leq Q_s\), with \(F'_n \subseteq F_{s,q_s}\), then

\[\bigcap_{n=1}^{N} F'_n \subseteq \bigcup_{q=1}^{Q_s} F_{s,q}\]

for all \(s \in S\), and

\[\bigcap_{n=1}^{N} F'_n \in \mathcal{F}'\]

implies

\[\bigcap_{n=1}^{N} F'_n = \Lambda_{\mathcal{F}'}\]

Thus it may be assumed there is an \(s_0\) such that

\[\bigcap_{n=1}^{N} F'_n \subseteq \bigcup_{q=1}^{Q_{s_0}} F_{s_0,q}\]

Hence for all \(n, F'_n \subseteq F_{s_0,q}\) for some \(q\), is false. If

\[\bigcap_{n=1}^{N} F'_n \cap B_0 \neq \Lambda_{\mathcal{F}'}\]

let \(x \in X\) be defined as follows. Let \(t_1, \ldots, t_n \in T\) be such that \(\lambda_{t_i}(F'_i) \neq \bigvee \mathcal{F}, 1 \leq i \leq N\). Choose an \(x \in X\) such that it satisfies the following conditions:

(a) \(\pi_i(x) \in \lambda_{t_i}(F'_i)\) if \(\lambda_{t_i}(F_{s_0,q}) = \bigvee \mathcal{F}\) for all \(q, 1 \leq q \leq Q_{s_0}\)

(b) \(\pi_i(x) \in -\lambda_{t_i}(F_{s_0,q})\) if \(\lambda_{t_i}(F_{s_0,q}) \neq \bigvee \mathcal{F}\)

for \(1 \leq i \leq N\);

(c) \(\pi_i(x) \in \eta_{t_i}(B)\) for all \(t \neq t_i, 1 \leq i \leq n\);

(c) \(\pi_i(x) \in \eta_{t_i}(B)\) for all \(t \neq t_i, 1 \leq i \leq n, 1 \leq q \leq Q_{s_0}\). Now \(x\) is well defined,

\[x \in B_0 \quad \text{and} \quad x \in \bigcap_{n=1}^{N} F'_n\]

by its definition. But \(x \not\in F_{s_0,q}\) for all \(q, 1 \leq q \leq Q_{s_0}\), hence
Proof of Case 2. If

\[ -B_0 \cap \bigcap_{n=1}^{N} F'_{n} \neq \bigwedge_{\forall x}, \]

and \( \lambda_{t_{n}}(F'_{n}) \neq \bigvee_{x}, \; t_{n} \in T \), let \( A_{n} = \varphi_{t_{n}}(-B_0), \; 1 \leq n \leq N \). Then

\[ \bigcap_{n=1}^{N} F'_{n} \cap (-B_0) = \bigcap_{n=1}^{N} (F'_{n} \cap A_{n}) \cap (-B_0) \]

and

\[ \bigcap_{n=1}^{N} (F'_{n} \cap A_{n}) \in \mathcal{F}'. \]

As before, an \( s_{0} \in S \) may be found such that

\[ \bigcap_{n=1}^{N} (F'_{n} \cap A_{n}) \subseteq \bigcup_{q=1}^{Q_{x_0}} F_{x_0,q} \].

Define \( t_{1}, \ldots, t_{N} \) as before so that \( \lambda_{t_{i}}(F'_{i} \cap A_{i}) \neq \bigvee_{x}, \; 1 \leq i \leq N \). Choose \( x \in X \) satisfying the following conditions:

(a) \( \pi_{t_{i}}(x) \in \{ \lambda_{t_{i}}(F'_{i} \cap A_{i}) \text{ if } \lambda_{t_{i}}(F'_{i},q) = \bigvee_{x}, \; 1 \leq q \leq Q_{x_0} \}
\[ \lambda_{t_{i}}(F'_{i} \cap A_{i}) - \lambda_{t_{i}}(F'_{s_0,q}) \text{ if } \lambda_{t_{i}}(F'_{s_0,q}) \neq \bigvee_{x} \]

for \( 1 \leq i \leq N \).

(b) \( \pi_{t_{q}}(x) \in -\lambda_{t_{q}}(F'_{s_0,q}) \) for each \( t_{q} \in T \) such that \( \lambda_{t_{q}}(F'_{s_0,q}) \neq \bigvee_{x}, \; 1 \leq q \leq Q_{x_0} \), and \( t_{q} \neq t_{i}, \; 1 \leq i \leq N \).

(c) \( \pi_{t}(x) \in \lambda_{t}(-B_0) \) if \( t \neq t_{i}, t_{q}; \; 1 \leq i \leq n, \; 1 \leq q \leq Q_{x_0} \).

Now \( x \) is well defined and

\[ x \in (-B_0) \cap \bigcap_{n=1}^{N} (F'_{n} \cap A_{n}) = -B_0 \cap \bigcap_{n=1}^{N} F'_{n} \],

so

\[ x \notin \bigcup_{q=1}^{Q_{x_0}} F_{x_0,q} \],

a contradiction.

Consequently, in either case

\[ \bigcap_{n=1}^{N} F'_{n} = \bigwedge_{\forall x}. \]
Lemma 3.4. If $j$ is the identity isomorphism of $\mathcal{F}$ into $\mathcal{F}_0$ and $\{i, \mathcal{C}\}$ is an $m$-completion of $\mathcal{F}_0$, then $\{ij, \mathcal{C}\}$ is an $m$-extension of $\mathcal{F}$.

Proof. All that needs to be shown is that $ij(\mathcal{F})$ $m$-generates $\mathcal{C}$. But this follows immediately from the fact that $\mathcal{A}$ $m$-generates $\mathcal{B}$ and the definition of $\mathcal{F}$ and $\mathcal{F}_0$.

Theorem 3.1. If $\mathcal{A}$ $m$-generates $\mathcal{B}$ then $\mathcal{A}(\mathcal{F})$ does not contain a smallest element.

Proof. $F \in \mathcal{F}$ and $F \supseteq B_0$ then $F = \bigvee_{\mathcal{A}}$, by definition of $B_0$. Thus if $j$ and $\{i, \mathcal{C}\}$ are defined as in Lemma 3.4, $\{ij, \mathcal{C}\}$ is an $m$-extension of $\mathcal{F}$ and $ij(B_0) \in K(\mathcal{C})$. By Proposition 3.1, $\mathcal{A}(\mathcal{F})$ does not contain a smallest element.

The results of this theorem may be generalized as follows. Let $\{\mathcal{A}_t\}_{t \in T}$ be an infinite indexed set of Boolean algebras and $\{(j_t, \mathcal{B}_t)\}_{t \in T}$ be the Boolean product of $\{\mathcal{A}_t\}_{t \in T}$. Let $T'$ be the set of all $t \in T$ such that $\mathcal{A}_t$ contains more than one element.

Theorem 3.2. The class of $m$-extensions $\mathcal{A}(\mathcal{B})$ does not contain a smallest element if $T' \geq \sigma$.

Proof. Define $\mathcal{F}'$ to be the Boolean product of $\{(j_t, \mathcal{B}_t)\}_{t \in T}$, where $\{j_t, \mathcal{B}_t\} \in \mathcal{A}(\mathcal{A}_t)$ for all $t \in T$ and $\{j_t, \mathcal{B}_t\}$ is not an $m$-completion of $\mathcal{A}$ for all $t \in T'$. For each $\mathcal{B}_t$, $t \in T'$, there is a $B_t \in \mathcal{B}_t$ such that $j_t(A) \subseteq B_t$, $A \in \mathcal{A}$, implies $A = \bigwedge_{\mathcal{A}_t}$. Let $\varphi_t$ map $\mathcal{B}_t$ into $\mathcal{B}$ and set

$$B_0 = \bigcup_{t \in T'} \varphi_t(B_t)$$

and

$$\mathcal{F}_0 = \langle \mathcal{F}', B_0 \rangle.$$

Then by an argument similar to the proofs of Lemmas 3.2, 3.3, and 3.4, and Theorem 3.1, $\mathcal{A}(\mathcal{B})$ does not contain a smallest element.

Corollary 3.1. If $\mathcal{A}_t = \mathcal{A}$ for all $t, t' \in T$ then $\mathcal{A}(\mathcal{B})$ contains a smallest element if, and only if, an $m$-extension of $\mathcal{B}$ is an $m$-completion.

Proof. If $\mathcal{A}(\mathcal{B})$ contains an $m$-extension which is not an $m$-completion, let $\mathcal{B}$ play the role of $\mathcal{A}$ in Lemmas 3.2, 3.3, and 3.4. By Theorem 3.1, $\mathcal{A}(\mathcal{F})$ does not contain a smallest element. As
the $m$-fold product $\mathcal{F}$ of $\mathcal{B}$ is isomorphic to $\mathcal{B}$, $\mathcal{K}(\mathcal{B})$ does not contain a smallest element. The converse is clear.

Now to prove the assumption on which these results are based.

**Lemma 3.5.** For each infinite cardinal number $m$ there is a Boolean algebra $\mathcal{A}$ such that an $m$-completion $\{i, \mathcal{B}\}$ of $\mathcal{A}$ contains an element $B$ with

$$B \neq \bigcup_{u \in U} \bigcap_{v \in V} A_{u,v},$$

for all $m$-indexed sets $\{A_{u,v}\}_{u \in U, v \in V}$ in $\mathcal{A}$.

**Proof.** The proof will be by constructing such an $\mathcal{A}$ for each $m$. Let $S$ be an indexing set of cardinality $m$. Let $\mathcal{D}_m$ be the Cartesian product of $S$ with itself $m$ times and indexed by $T$. Define

$$D_{t,s} = \{d \in \mathcal{D}_m : \pi_t(d) = s\}.$$  

Fix $s', s'' \in S$, $s' \neq s''$, and set $S' = S - \{s', s''\}$. Let $D = \bigcup_{t \in T} (D_{t,s'} \cup D_{t,s''})$. Thus $\overline{D} = 2^m$ and $d \in \mathcal{D}_m - D$ implies $\pi_t(d) \neq s_k$, $k = 1, 2$, for all $t \in T$.

Let

$$\mathcal{S} = \{(d) : d \in \mathcal{D}_m\} \cup \{D_{t,s} : t \in T, s \in S'\}.$$  

Let $\mathcal{A}$ be generated by $\mathcal{S}$ in $\mathcal{D}_m$ and let $\mathcal{B}$ be the $m$-field of sets $m$-generated by $\mathcal{S}$ in $\mathcal{D}_m$. Then $\mathcal{A}$ is dense in $\mathcal{B}$ and $m$-generates $\mathcal{B}$, so if $i$ is the identity map of $\mathcal{A}$ into $\mathcal{B}$, $\{i, \mathcal{B}\}$ is an $m$-completion of $\mathcal{A}$.

Let

$$B = \mathcal{D}_m - D.$$  

Suppose

$$B = \bigcup_{u \in U} \bigcap_{v \in V} A_{u,v},$$

$\{A_{u,v}\}_{u \in U, v \in V}$ an $m$-indexed set in $\mathcal{A}$. This can be written in the form

$$\bigcup_{u \in U} \bigcap_{v \in V} A_{u,v} = \bigcup_{u \in U} \bigcap_{v \in V} A_{u,v,m};$$

$A_{u,v,m}$ or $-A_{u,v,m} \in \mathcal{S}$, $\overline{M_{u,v}} < \sigma$.

Let $B' = \{d \in \mathcal{D}_m : \{d\} = A_{u,v,m}$ for some $u \in U, v \in V$, and $m \in M_{u,v}\}$. Then $\overline{B'} \leq m$, so if

$$M'_{u,v} = \{m \in M_{u,v} : A_{u,v,m}$ is not of the form $\{d\}$, $d \in \mathcal{D}_m\},$$

it follows that
It will now be shown that in fact

\[ B - \bigcup_{u \in U} \bigcap_{v \in V} \bigcup_{m \in M_{u,v}} A_{u,v,m} \leq m. \]

a contradiction. Hence it may be assumed that \( A_{u,v,m} \) is not of the form \( \{d\} \), \( d \in \mathcal{D}_m \), for all \( u \in U \), \( v \in V \), and \( m \in M_{u,v} \).

If \( A_{u,v,m} = \{d\} \), \( d \in \mathcal{D}_m \), for some \( m \in M_{u,v} \), then either

1. \( \bigcup_{m \in M_{u,v}} A_{u,v,m} = \{d\} \)
2. \( \bigcup_{m \in M_{u,v}} A_{u,v,m} = V \).

If (1) occurs, it may be assumed that \( M_{u,v} = \{1\} \) and \( A_{u,v,1} = \{d\} \). If (2) occurs, the term \( \bigcup_{m \in M_{u,v}} A_{u,v,m} \) may be dropped. Thus for all \( u \in U \), \( V \) may be written as \( V_u \cup V_u' \), where (1) \( V_u \cap V_u' = \emptyset \); (2) \( A_{u,v,m} = \{d_{u,v}\} \), \( d_{u,v} \in \mathcal{D}_m \), for all \( v \in V_u \); and (3) \( A_{u,v,m} \) is either of the form \( -D_{t,s} \) or \( D_{t,s} \), for all \( v \in V_u \). Consequently, for all \( u \in U \),

\[ \bigcap_{v \in V_u} \bigcup_{m \in M_{u,v}} A_{u,v,m} = \bigcap_{v \in V_u} \{d_{u,v}\} \bigcap \bigcup_{m \in M_{u,v}} A_{u,v,m}. \]

Let

\[ C_u = \bigcap_{v \in V_u} \bigcup_{m \in M_{u,v}} A_{u,v,m}. \]

Suppose \( U \) is the set of all ordinals \( u < \alpha \), where \( \alpha = \bar{U} \). Let \( D_i = \{d \in \mathcal{D}_m : \pi_i(d) = s_i, s_i'\} \). Now \( D_i = 2^m \) implies there is a \( d_i \in D \) such that

\[ d_i \in \bigcap_{v \in V_i} -\{d_{i,v}\}. \]

Since \( d_i \notin B \), this implies

\[ d_i \in \bigcup_{m \in M_{i,v}} A_{i,v,m}, \]

hence for some \( v_i \in V_i' \),

\[ d_i \in \bigcup_{m \in M_{i,v_i}} A_{i,v_i,m}. \]

Also, \( D_i \subseteq -D_{t,s} \) for all \( t \in T \) and \( s \in S' \), hence

\[ A_{i,v_i,m} = D_{t_i,s_i,t_i,m}. \]

for some \( t_i,m \in T \) and \( s_{t_i,m} \in S' \), for all \( m \in M_{i,v_i} \). Let \( T_i = \{t_{i,m} : m \in M_{i,v_i}\} \)
and pick $s_i \in S'$ such that $s_i \neq s_{i,m}$ for all $m \in M_{i,v_i}$. Define

$$\varphi(t) = s_i$$

for all $t \in T_i$. Let $B_1 = \emptyset$ and define $B_2 = \{d \in \mathcal{D}_m : \pi_i(d) = \varphi(t) \text{ for all } t \in T_1\}$.

Note that $B_1 \cap C_i = \emptyset$.

Suppose $i > 1$ and a finite set $T_i$ has been defined for each $i' < i$ so that $T_i \cap T_{i'} = \emptyset$ if $i', i'' < i, i' \neq i''$; $s_v \in S'$ has been chosen; $\varphi$ has been defined on each $T_i$, $i' < i$, so that $\varphi(t) = s_v$ for all $t \in T_i$; and if

$$B_i = \{d \in \mathcal{D}_m : \pi_i(d) = \varphi(t) \text{ for all } t \in \bigcup_{i < i} T_i\}$$

then

$$B_i \cap \bigcup_{i < i} C_i = \emptyset.$$

Let

$$\hat{T}_i = \bigcup_{i < i} T_i$$

and note that $\hat{T}_i < m$. Let

$$D_i = \{d \in \mathcal{D}_m : \pi_i(d) = \varphi(t) \text{ for all } t \in \hat{T}_i$$

and $\pi_i(d) = s_k, k = 1, 2$, if $t \in T - \hat{T}_i\}.$

Then $D_i \subseteq D$ and $\overline{D_i} = 2^m$, hence there is a $d_i \in D_i$ such that

$$d_i \in \bigcap_{i \in V_i} - \{d_{i,v}\}.$$

Since $d_i \in B$, this implies

$$d_i \notin \bigcap_{i \in V_i} \bigcup_{m \in M_{i,v}} A_{i,v,m},$$

hence for some $v_i \in V_i,$

$$d_i \in \bigcup_{m \in M_{i,v}} A_{i,v,m}.$$

If $B_i \cap C_i = \emptyset$ set $T_i = \emptyset$. If not, there is a $d'_i \in B_i$ such that

$$d'_i \in C_i,$$

so

$$d'_i \in \bigcup_{m \in M_{i,v}} A_{i,v,m}.$$

Note that $\pi_i(d'_i) = \pi_i(d_i)$ for all $t \in \hat{T}_i$.

It immediately follows that if
\[ d'_i \in \bigcup_{m \in M_i, \nu_i} A_{i, \nu_i, m} \]
then
\[ A_{i, \nu_i, m} = D_{t_i, m, s_i, m}, \]
where \( t_i, m \in \hat{T}_i \) and
\[ \pi_{t_i, m}(d'_i) = s_{t_i, m}, \]
for some \( m \in M_i, \nu_i \).

Let
\[ T_i = \{ t_i, m \in T - \hat{T}_i : A_{i, \nu_i, m} = D_{t_i, m, s_i, m} \text{ for some } m \in M_i, \nu_i \} \]
and pick \( s_i \in S' \) such that if \( t_i, m \in T_i \) then
\[ s_i \not= S_{t_i, m}, \]
for all \( m \in M_i, \nu_i \). Now define
\[ \varphi(t) = s_i \text{ for all } t \in T_i. \]
Thus \( T_i \cap \hat{T}_i = \emptyset \) which implies \( T_i \cap T_{i'} = \emptyset \) for all \( i' < i \). If
\[ B_{i+1} = \{ d \in D_u : \pi_t(d) = \varphi(t) \text{ for all } t \in T_i \cup \hat{T}_i \} \]
then it is clear that
\[ B_{i+1} \cap \bigcup_{i' < i} C_i = \emptyset. \]

Now let \( \hat{T} = \bigcup_{i \leq a} T_i \) and set
\[ \hat{B} = \{ d \in D_u : \pi_t(d) = \varphi(t) \text{ for all } t \in \hat{T} \}
\text{ and } \pi_t(d) \not= s', s'_i \text{ if } t \in T - \hat{T} \} . \]
Then \( \hat{B} \not= \emptyset \) and \( \hat{B} \subseteq B \). But \( \hat{B} \cap \bigcup_{u \in U} C_u = \emptyset \) which implies
\[ B - \bigcup_{u \in U} C_u \not= \emptyset. \]

If \( B' = B - \bigcup_{u \in U} C_u \) then for each \( b \in B' \),
\[ b = \bigcap_{t \in T} D_{t, s_t, b}, \]
for some \( m \)-indexed set \( \{ s_t, b \}_{t \in T} \) in \( S' \). Thus
\[ B = \bigcup_{u \in U} \bigcap_{t \in T} D_{t, s_t, b} \bigcup_{u \in U} D_{t, s_t, b}, \]
but the above construction shows that
LEMMA 3.6. If \( \{i, B\} \) is an \( m \)-completion of the Boolean algebra \( \mathcal{A} \) and there is a \( B \in B \) such that

\[
B - (\bigcup_{u \in U} \bigcap_{v \in V} A_{u,v,m} \bigcup \bigcup_{b \in B, t \in T} D_{t,s,t}) \neq \emptyset
\]

if \( B' \leq m \). Hence

\[
B - \bigcup_{u \in U} C_u > m.
\]

for all \( m \)-indexed sets \( \{A_{i,t}\}_{t \in T, s \in S} \) in \( \mathcal{A} \), then there is an \( m \)-ideal \( \Delta \) in \( B \) such that \( \{i, B\} \) is an \( m \)-extension of \( i_{\Delta} \) but not an \( m \)-completion, where \( i_{\Delta}(A) = [i(A)]_{\Delta} \) for all \( A \in \mathcal{A} \), \( B_{\Delta} = B/\Delta \) and \( j \) is the identity map of \( i_{\Delta}(\mathcal{A}) \) into \( B_{\Delta} \).

Proof. Let

\[
\Delta' = \{B' \in B : B' \subseteq B \text{ and } B' = \bigcap_{t \in T} i(A_t), \text{ for some } m \text{-indexed set } \{A_t\}_{t \in T} \text{ in } \mathcal{A}\}
\]

and let \( \Delta = \langle \Delta' \rangle_m \). Then if \( \delta \in \Delta, \delta \subseteq B \), so \( B \in \Delta \). If \( A \in \mathcal{A} \) and \( [i(A)]_{\Delta} \subseteq [B]_{\Delta} \) then \( i(A) - B \in \Delta \) so \( i(A) - B \subseteq B \) which implies \( i(A) \subseteq B \), hence \( i(A) \in \Delta \) and \( [i(A)]_{\Delta} = \Lambda_{\Delta} \), implying \( i_{\Delta}(\mathcal{A}) \) is not dense in \( B \).

It only remains to show that \( i_{\Delta}(\mathcal{A}) \) is \( m \)-regular in \( B_{\Delta} \). If

\[
\bigcap_{t \in T} [i(A_t)]_{\Delta} = \Lambda_{\Delta}
\]

then \( i(A) \subseteq i(A_t) \) for all \( t \in T \) implies \( i(A) \in \Delta \), so \( i(A) \subseteq B \). If

\[
\bigcap_{t \in T} i(A_t) \not\subseteq B,
\]

then there is an \( A \neq \Lambda_{\Delta} \) in \( \mathcal{A} \) such that

\[
i(A) \subseteq \bigcap_{t \in T} i(A_t) - B,
\]

contradicting the above statement. Thus

\[
\bigcap_{t \in T} i(A_t) \subseteq B
\]

so

\[
\bigcap_{t \in T} i(A_t) \in \Delta
\]
and

\[ \Lambda_{\mathcal{A}_d} = \left[ \bigcap_{t \in \tau} i(A_t) \right] = \bigcap_{t \in \tau} [i(A_t)]. \]

Thus if \( \mathcal{A} \) is the Boolean algebra constructed in Lemma 3.5, \( i_d(\mathcal{A}) \) is a Boolean algebra such that \( \mathcal{A}(i_d(\mathcal{A})) \) contains more than one element. Hence it is justified to assume that for each infinite cardinal \( m \) there is a Boolean algebra \( \mathcal{A} \) such that \( \mathcal{A} \) has an \( m \)-extension which is not an \( m \)-completion.

4. Let \( \{ \mathcal{A}_t \}_{t \in \tau} \) be a (fixed) indexed set of Boolean algebras. Let \( h_t \) be an isomorphism of \( \mathcal{A}_t \) onto the field \( \mathcal{F}_t \) of all open-closed subsets of the Stone space \( X_t \) of \( \mathcal{A}_t \). Let \( X \) denote the Cartesian product of all the spaces \( X_t \). Let \( \pi_t \) be the projection of \( X \) onto \( \mathcal{F}_t \) and define

\[ \phi_t : \mathcal{F}_t \rightarrow X \]

by:

if \( F \in \mathcal{F}_t \) then \( \phi_t(F) = \{ x \in X : \pi_t(x) \in F \} \).

Let \( \mathcal{F} \) be the Boolean product of \( \{ \mathcal{A}_t \}_{t \in \tau} \). Define \( h^*_t = \phi_t h_t \) and let \( \mathcal{J} \) be the set of all sets \( \bigcap_{t \in \tau} h^*_t(A_t) ; A_t \in \mathcal{A}_t, T' \subseteq T', \bar{T}' \leq n \). Define \( \mathcal{J} \) to be the field of sets generated by \( \mathcal{J} \). Let \( J \) be the set of all sets \( S \subseteq \mathcal{J} \) such that

1. \( \bar{S} \leq m \);
2. there is a \( t \in T \) such that \( S \subseteq h^*_t(\mathcal{A}_t) \);
3. the join \( \bigcup_{A \in S} A \) exists.

Let \( M' \) be the set of all sets \( S \subseteq \bar{T} \) such that

1. \( \bar{S} \leq m \);
2. there is a \( t \in T \) such that \( S \subseteq h^*_t(\mathcal{A}_t) \);
3. the meet \( \bigcap_{A \in S} A \) exists.

Let \( M'' \) be the set of all sets \( S \subseteq \bar{T} \) such that

1. \( \bar{S} \leq n \);
2. if \( A \in S \) then \( A \in h^*_t(\mathcal{A}_t) \) for some \( t \in T \);
3. if \( A, B \in S, A \neq B \), then \( A \in h^*_t(\mathcal{A}_t) \) implies \( B \in h^*_t(\mathcal{A}_t) \). Let \( M = M' \cup M'' \).

The following lemma is due to La Grange [1] and will be given without proof.

**Lemma 4.1.** If \( \{ (i_t)_{t \in \tau}, \mathcal{B} \} \in \mathcal{P}_n \) then there is one and only one \( (J, M, m) \)-isomorphism \( h \) mapping \( \mathcal{J} \) into \( \mathcal{B} \) such that

\[ hh^*_t = i_t \text{ for all } t \in T. \]
Theorem 4.1. If \{[i_t]_{t \in T}, \mathcal{B}\} \in \mathcal{P}_m$ then there is a mapping $h$ of $\widehat{\mathcal{A}}$ into $\mathcal{B}$ such that $(h, \mathcal{B})$ is a $(J, M, m)$-extension of $\widehat{\mathcal{A}}$. If $(h, \mathcal{B})$ is a $(J, M, m)$-extension of $\widehat{\mathcal{A}}$ then the ordered pair $\{(hh^*_t)_{t \in T}, \mathcal{B}\} \in \mathcal{P}_m$.

Proof. Let $h$ be the $(J, M, m)$-isomorphism from $\widehat{\mathcal{A}}$ into $\mathcal{B}$ such that $hh^*_t = i_t$ for all $t \in T$. Then $(h, \mathcal{B})$ is a $(J, M, m)$-extension of $\widehat{\mathcal{A}}$.

Conversely, if $(h, \mathcal{B})$ is a $(J, M, m)$-extension of $\widehat{\mathcal{A}}$, it follows immediately that $\{(hh^*_t)_{t \in T}, \mathcal{B}\}$ is a $(m, n)$-product of $\{(A_t)_{t \in T}\}$.

Theorem 4.2. If $\{(i_t)_{t \in T}, \mathcal{B}\}, \{(i'_t)_{t \in T}, \mathcal{B}'\}$ are two $(m, n)$-products of $\{(A_t)_{t \in T}\}$ then

\[ \{(i_t)_{t \in T}, \mathcal{B}\} \leq \{(i'_t)_{t \in T}, \mathcal{B}'\} \]

if, and only if,

\[ \{i, \mathcal{B}\} \leq \{i', \mathcal{B}'\} \]

where $\{i, \mathcal{B}\}$ and $\{i', \mathcal{B}'\}$ are the $(J, M, m)$-extensions of $\widehat{\mathcal{A}}$ induced by the $(J, M, m)$-isomorphisms $i'$ and $i$ of $\widehat{\mathcal{A}}$ into $\mathcal{B}'$ and $\mathcal{B}$, respectively, given by Lemma 4.1.

Proof. Now

\[ \{(i_t)_{t \in T}, \mathcal{B}\} \leq \{(i'_t)_{t \in T}, \mathcal{B}'\} \]

if, and only if, there is an $m$-homomorphism $h$ such that

\[ h: \mathcal{B}' \longrightarrow \mathcal{B} \]

and $hi'_t = i_t$ for all $t \in T$. Similarly,

\[ \{i, \mathcal{B}\} \leq \{i', \mathcal{B}'\} \]

if, and only if, there is an $m$-homomorphism

\[ h: \mathcal{B}' \longrightarrow \mathcal{B} \]

such that $h'i' = i$. Thus it suffices to show that $hi' = i$, if, and only if, $hi'_t = i_t$. Let $h_t^*$ be defined as above. Then $ih_t^* = i$ and $h'_ih_t^* = i'_t$, so if $hi' = i$,

\[ hi'_t = h'i'h_t^* = ih_t^* = i_t, \]

and if $hi'_t = i_t$, then

\[ hi' = h'i'h_t^*-1 = i_t h_t^*-1 = i. \]
La Grange [1] has given an example of an \((m, 0)\)-product for which \(\mathcal{P}\) does not contain a smallest element and an example of an \((m, n)\)-product for which \(\mathcal{P}_n\) does not contain a smallest element. Theorem 4.2 extends this result by showing that the question whether \(\mathcal{P}\) or \(\mathcal{P}_n\) contains a smallest element reduces to asking whether the class of all \((J, M, m)\)-extensions of \(\mathcal{A}_0\) or \(\mathcal{F}\) contains a smallest element for \(J\) and \(M\) defined appropriately in each case, where \(\mathcal{A}_0\) and \(\mathcal{F}\) are defined as above. Now the class of all \((J, M, m)\)-extensions of \(\mathcal{A}_0\) contains a smallest element only if the class of all \(m\)-extensions of \(\mathcal{A}\) contains a smallest element and Theorem 3.2 shows that the class of all \(m\)-extensions of \(\mathcal{A}\) need not contain a smallest element, which implies the same is true for \(\mathcal{P}\). Since Theorem 3.2 may be extended to Boolean algebras of the form \(\mathcal{F}\), it follows that \(\mathcal{P}_n\) need not contain a smallest element.

References


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