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GENERALIZED HALL PLANES OF EVEN ORDER

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The theory of generalized Hall planes of odd or zero characteristic has been developed by P. B. Kirkpatrick who has obtained a characterization of the odd order Hall planes, within the class of odd order generalized Hall planes, in terms of their homologies with affine axis. N. L. Johnson has pointed out that odd order generalized Hall planes are derivable and has characterized them in terms of their derived planes. In this paper the results of Kirkpatrick are established in the case of generalised Hall planes of characteristic two and the results of Johnson in the case of generalized Hall planes of even order. Also, a characterization of the even order Hall planes, within the class of even order generalized Hall planes, in terms of their elations with affine axis, is obtained.

1. Introduction. Let π be a projective plane, l_∞ a line of π and π_0 a Baer subplane of π such that l_∞ is a line of π_0. We call π a *generalized Hall plane* with respect to l_∞, π_0 if

(1) π is a translation plane with respect to l_∞, and

(2) π has a group of collineations which is transitive on the points of l_∞ not in π_0, and fixes every point in π_0.

We shall denote the point set $\{P \mid P \in l_\infty \text{ and } P \in \pi_0\}$ by M.

Kirkpatrick [7] has shown that any generalized Hall plane of odd order may be coordinatized by a certain type of Veblen-Wedderburn system ($V - W$ system from now on) which is a right vector space of dimension two over a subfield. A consequence of this result is that an odd order generalized Hall plane is derivable (in the sense of Ostrom [9]) and Johnson [6] has shown that odd order generalized Hall planes derive translation planes which are in semi-translation plane class 1–3a of his classification of semi-translation planes (see [4]). In § 2 we shall extend these results to include the case of even order generalized Hall planes. We shall also show that the derived planes of finite generalized Hall planes are semifield planes. Our proofs shall apply to all finite generalized Hall planes and so we shall not restrict the statement of our results to the even order case.

Section 3 is devoted to proving a result (Lemma 2) on changing of coordinate quadrangles in generalized Hall planes which will be used in § 4.

Kirkpatrick [8] has given a characterization of Hall planes of odd order which may be stated as follows: A generalized Hall plane of odd order is a Hall plane if and only if each point of M is the centre of a nontrivial involutory homology with axis in π_0. In §4 we shall establish the following analogous characterization of even order Hall planes: A generalized Hall plane of even order is a Hall plane if and only if each point of M is the centre of a nontrivial elation, with exis $\neq l_\infty$, fixing M. We note at this point that all elations in an even order translation plane are involutory.

2. Properties of generalized Hall planes. The following theorem was proved by Kirkpatrick ([7], Theorem 1) in the case of planes of odd or zero characteristic.

THEOREM 1. *If π is a generalized Hall plane with respect to l_∞, π_0 and π is coordinatized over a quadrangle* 0, *I, X, Y in π_0, such that $XY = l_\infty$, by a $V - W$ system F possessing a subsystem F_0 which coordinatizes π_0, then F_0 is a skew field and F is a right vector space over F_0.*

Proof. It is not difficult to prove that $(z\rho)\sigma = z(\rho\sigma) + \beta$ for all $z \in F\backslash F_0$ and $\rho, \sigma \in F_0$, where β depends only on ρ and σ (Kirkpatrick [7]).

Now choose $z, w \in F\backslash F_0$ such that $z + w \in F\backslash F_0$. This can be done unless $|F_0| = 2$. It follows that $((z + w)\rho)\sigma = z(\rho\sigma) + w(\rho\sigma) + \beta = z(\rho\sigma) + \beta + w(\rho\sigma) + \beta$, so $\beta = 0$ and $(z\rho)\sigma = z(\rho\sigma)$ for all $z \in F\backslash F_0$, $\rho, \sigma \in F_0$.

A similar argument establishes $z(\rho + \sigma) = z\rho + z\sigma$ for all $z \in F\backslash F_0$, $\rho, \sigma \in F_0$.

It is easy (see Kirkpatrick [7]) to prove that F_0 is a skew field and that F is a right vector space of dimension two over F_0.

COROLLARY. *π_0 is desarguesian.*

The multiplication operation in F may be described as follows if F is finite:

F is a right vector space of dimension two over a field F_0 embedded in it in the usual way, with multiplication operation

(3) $x \cdot \alpha = x\alpha$ (multiplication by a scalar) for all $x \in F$, $\alpha \in F_0$,

(4) $(z\alpha + \beta) \cdot z = zA(\alpha, \beta) + B(\alpha, \beta)$ for all $z \in F\backslash F_0$, $\alpha, \beta \in F_0$ where A and B are mappings of $F_0 \times F_0$ onto F_0 which have the properties:

(5) A and B are additive homomorphisms with $A(0, 1) = 1$ and $B(0, 1) = 0$.

(6) for any given γ and $\delta \in F_0$, the equation $(A(\alpha, \beta), B(\alpha, \beta)) = (\gamma, \delta)$ has exactly one solution (α, β) and

(7) the equation $(A(\alpha, \beta), B(\alpha, \beta)) = (\alpha\gamma, \beta\gamma + \delta)$ has exactly

one solution (α, β), given $\gamma, \delta \in F_0$; also, for this solution, $\alpha = 0$ if and only if $\delta = 0$.

For this see Kirkpatrick [7]. Conversely, it is easy to see that such a $V - W$ system coordinatizes a generalized Hall plane. We shall call such a $V - W$ system a *generalized Hall system.* From now on we shall consider only finite generalized Hall planes.

Because of (5) we can write $A(\alpha, \beta) = f(\alpha) + h(\beta)$ and $B(\alpha, \beta) = g(\alpha) + k(\beta)$ for all $\alpha, \beta \in F_0$, where f, g, h, and k are additive endomorphisms of F_0 which we shall call the *defining functions* of F. Now $\beta z = z\,h(\beta) + k(\beta)$ for all $z \in F \backslash F_0$ and $\beta \in F_0$. So, if $h(\beta) = h(\beta')$ then $(\beta - \beta')z = k(\beta - \beta')$ which implies $\beta = \beta'$. Thus h is an additive automorphism of F_0.

THEOREM 2. (i) *Finite generalized Hall planes are derivable and the derived planes are semifield planes in semi-translation class* 1–3a *of Johnson's classification.*

(ii) *Finite semifield planes in class* 1–3a *are derivable and the derived planes are generalized Hall planes.*

Proof. (i) Johnson [6] (Theorem 3.1) shows that generalized Hall planes of odd order are derivable and derive translation planes in class 1–3a. His proof rests on a theorem in Kirkpatrick [7] (Theorem 1) for odd order planes. In virtue of Theorem 1 of this paper we can say that Johnson's argument applies to the even order case as well.

It is possible to apply Theorem 11 of [9] to find a coordinate system F' in the derived plane of a generalized Hall system F defined by functions f, g, h, and k. F' has multiplication $\circ$ given by

$$\begin{aligned}(t\xi_1 + \eta_1) \circ (t\lambda_1 + \lambda_2) = {} & th^{-1}(\eta_1 h(\lambda_1) - f(h(\lambda_1)\xi_1) + h(\kappa)) \\ & + \eta_1(\lambda_2 - k(\lambda_1)) + g(h(\lambda_1)\xi_1) \\ & + kh^{-1}(\eta_1 h(\lambda_1) - f(h(\lambda_1)\xi_1)\end{aligned}$$

where $\kappa = (\lambda_2 - k(\lambda_1))\xi_1$. Because f, g, h, and k are additive endomorphisms of F_0 the multiplication $\circ$ is fully distributive and so F' is a semifield since it is necessarily a $V - W$ system, the derived system of a $V - W$ system under Theorem 11 of [9] being always a $V - W$ system.

(ii) This part is proved in Johnson [6] (Theorem 2.1).

Kirkpatrick [7] gives a class of generalized Hall planes of odd order which contains the class of Hall planes of odd order and some other planes. The following two classes (which appear in Johnson [5] in another form) contain all the finite Hall planes as well as some others of both odd and even orders. The planes are those coordinatized by the generalized Hall systems:

(i) $(z\alpha+\beta)z = z(\beta^{\theta^{-1}}+\mu^{\theta^{-1}}\alpha)+\nu\alpha^{\theta}$ where $\theta\in \mathrm{Aut}(F_0)$ and $x^{\theta}x-\mu x-\nu$ is irreducible over F_0, and

(ii) $(z\alpha+\beta)z = z(\beta+\mu\alpha)^{\theta^{-1}}+\nu\alpha^{\theta^{-1}}$ where $\theta\in \mathrm{Aut}(F_0)$ and $x^{\theta}x-\mu x-\nu$ is irreducible over F_0.

NOTE (added November 9, 1973). It has recently come to the author's attention that Theorems 1 and 2 have been proved by Vikram Jha in his thesis "On Automorphism Groups of Quasifields", University of London, 1973.

3. Change of coordinates.

Suppose $(F, +, \circ)$ is a finite $V-W$ system coordinatizing a translation plane π. Let us write $y\circ x = (y)\sigma_x$ for all $x\in F^* = F\backslash\{0\}$, $y\in F$. Then σ_x is an automorphism of $(F, +)$ and we observe that $\Gamma=\{\sigma_x \mid x\in F^*\}$ is sharply transitive on F^* and that $\sigma_x-\sigma_z$ is nonsingular if $x\neq z$, since $(F^*, \circ)$ is a loop.

LEMMA 1. *If the finite $V-W$ system $F_1=(F, +, \circ)$ coordinatizing the plane π over the quadrangle $Q\mid O, I, X, Y$ has multiplication given by $y\circ x=(y)\sigma_x$ for all $x\in F^*$ and $y\circ 0 = 0$ for all $y\in F$, then π may be coordinatized over the quadrangle $Q'\mid O, I, X, Y'=(r)$ $(r\neq 0, 1)$ by a $V-W$ system $F_2=(F, +, *)$ such that*

$$y*x=\begin{cases}(y\circ r-y)(\sigma_r-\sigma_{v(x)})^{-1}\circ v(x) & \text{for all } x\in F^*\backslash\{1-r\}\\ y-y\circ r & \text{if } x=1-r\end{cases}$$

for all $y\in F$, where $(x+r-1)\sigma_r^{-1}\circ v(x)=x$ for all $x\in F^*\backslash\{1-r\}$.

Proof. Leave the coordinates of points on the line OI unchanged. It is not difficult to show that addition is unchanged. Note that we shall use primes to distinguish point coordinates over Q' from point coordinates over Q.

Consider the line OY. The point $T_1=(0, 1-r)$ lies on OY. The line $Y'T_1$ has equation $y=x\circ r+1-r$ and since $Y'T_1\cap OI = (1, 1) = (1, 1)'$ we have $T_1=(1, 1-r)'$, whence OY has equation $y = x*(1-r)$. (The multiplication makes clear which coordinatization we are referring to.) The point $T_2=(0, \bar{y})$ lies on OY for arbitrary $\bar{y}\in F$. $Y'T_2$ has equation $y=x\circ r+\bar{y}$. $Y'T_2\cap OI = ((\bar{y})(\sigma_1-\sigma_r)^{-1}, (\bar{y})(\sigma_1-\sigma_r)^{-1})$. Hence, $T_2=((\bar{y})(\sigma_1-\sigma_r)^{-1}, \bar{y})'$. So $(\bar{y}, \bar{y}-\bar{y}\circ r)$ lies on OY for all $\bar{y}\in F$ and thus $\bar{y}*(1-r)=\bar{y}-\bar{y}\circ r$ for all $\bar{y}\in F$.

Next, consider the line l whose equation is $y=x*t$ where $t\in F^*\backslash\{1-r\}$. The line l has equation $y=x\circ m(t)$ for some $m(t)\in F$. Now $R_1=(1, t)'$ lies on l, and $Y'R_1$ has equation $y=x\circ r+1-r$, since I and $Y'=(r)$ lie on it. The point R_1 is clearly $((t+r-1)\sigma_r^{-1}, t)$, whence $m(t) = v(t)$. Now $R_2 = (\bar{y}, \bar{y} * t)'$ lies on l and $Y'R_2$ is the line $y = x \circ r + \bar{y} - \bar{y} \circ r$, since (r) and $(\bar{y}, \bar{y})$ lie on it. Thus $Y'R_2 \cap l$ has abscissa $\bar{x}$ such that $\bar{x} \circ v(t) = \bar{x} \circ r + \bar{y} - \bar{y} \circ r$, that is $\bar{x} = (\bar{y} \circ r - \bar{y})(\sigma_r - \sigma_{v(t)})^{-1}$, whence $R_2 = ((\bar{y} \circ r - \bar{y})(\sigma_r - \sigma_{v(t)})^{-1}, (\bar{y} \circ r - \bar{y})(\sigma_r - \sigma_{v(t)})^{-1} \circ v(t)) = (\bar{y}, \bar{y} * t)'$ and so $\bar{y} * t = (\bar{y} \circ r - \bar{y})(\sigma_r - \sigma_{v(t)})^{-1} \circ v(t)$ for all $\bar{y} \in F$.

Lemma 1 appears in [10] along with a number of analogous results for some other shifts of coordinates in a finite translation plane.

LEMMA 2. *Let π be a generalized Hall plane with respect to l_∞, π_0. Suppose $F_1 = (F, +, \circ)$ is a generalized Hall system (with subfield F_0 coordinatizing π_0) coordinatizing π over $Q \mid O, I, X, Y$ in π_0, $y \circ x = (y)\sigma_x$ for all $x \in F^*$ and $y \in F$ and f, g, h, and k are the defining functions of F_1. π may be coordinatized over $Q' \mid O, I, X, Y' = (\lambda)$ $(\lambda \in F_0 \backslash \{0, 1\})$ by a generalized Hall system F_2 with defining function h_λ such that*

$$h_\lambda(\eta) = M_\lambda^{-1}((1 - \lambda)\eta)(M_\lambda^{-1}(1 - \lambda))^{-1} \text{ for all } \eta \in F_0 ,$$

and where $M_\lambda(\eta) = g(\eta) + kh^{-1}(\lambda\eta) - kh^{-1}f(\eta) - \lambda h^{-1}(\lambda\eta) + \lambda h^{-1}f(\eta)$ *for all* $\eta \in F_0$.

Proof. From Lemma 1, π may be coordinatized over Q' by $F_2 = (F, +, *)$, where

$$y * x = \begin{cases} (y(\lambda - 1))(\sigma_\lambda - \sigma_{v(x)})^{-1} \circ v(x) & \text{for all } x \in F^* \backslash \{1 - \lambda\} \\ y(1 - \lambda) & \text{for } x = 1 - \lambda \end{cases}$$

and

$$((x + \lambda - 1)\lambda^{-1}) \circ v(x) = x \text{ for all } x \in F^* \backslash \{1 - \lambda\} .$$

Firstly, we consider the action of $v(x)$ for each x. This is easy to discover if $x \in F_0^* \backslash \{1 - \lambda\}$ because then $v(x) \in F_0$. (In fact, it follows that $y * x = y \circ x$ in this case.) Suppose $x \in F \backslash F_0$; then $v(x) \in F \backslash F_0$, for if $v(x) \in F_0$ then $((x + \lambda - 1)\lambda^{-1}) \circ v(x) = x$ implies $\lambda = 1$, a contradiction. Now suppose $v(x) = x\alpha_x + \beta_x$ where $\alpha_x, \beta_x \in F_0$. The equation $((x + \lambda - 1)\lambda^{-1}) \circ v(x) = x$ implies

$$((x\alpha_x + \beta_x)\alpha_x^{-1}\lambda^{-1} + (\lambda - 1)\lambda^{-1} - \beta_x\alpha_x^{-1}\lambda^{-1}) \circ (x\alpha_x + \beta_x) = x$$

and so

$$f(\rho) + h(\tau) = \rho\lambda , \tag{8}$$

and

$$g(\rho) + k(\tau) = \tau\lambda + (1 - \lambda) , \tag{9}$$

where $\rho = \alpha_x^{-1}\lambda^{-1}$ and $\tau = (\lambda - 1)\lambda^{-1} - \beta_x\alpha_x^{-1}\lambda^{-1}$.

The solution of (8) and (9) for ρ is $\rho = M_\lambda^{-1}(1 - \lambda)$ where $M_\lambda(\eta) = g(\eta) + kh^{-1}(\lambda\eta) - kh^{-1}f(\eta) - \lambda h^{-1}(\lambda\eta) + \lambda h^{-1}f(\eta)$. Note that M_λ^{-1} exists since (8) and (9) are uniquely solvable for ρ, τ by (7) and also that α_x and β_x are independent of x so we may drop the subscripts.

Next we consider the action of $(\sigma_\lambda - \sigma_{v(x)})^{-1}$: $(x\gamma + \delta)(\sigma_\lambda - \sigma_{v(x)}) = x\gamma\lambda + \delta\lambda - (x\gamma + \delta) \circ (x\alpha + \beta)$. It follows that

$$(x\xi + \eta)(\sigma_\lambda - \sigma_{v(x)})^{-1} = x\gamma + \delta ,$$

where

$$f(\alpha^{-1}\gamma) + h(\delta - \beta\alpha^{-1}\gamma) = \alpha^{-1}\gamma\lambda - \xi\alpha^{-1}$$

and

$$g(\alpha^{-1}\gamma) + k(\delta - \beta\alpha^{-1}\gamma) = (\delta - \beta\alpha^{-1}\gamma)\lambda + \beta\alpha^{-1}\xi - \eta .$$

So,

$$\begin{aligned} y * x &= (y(\lambda - 1))(\sigma_\lambda - \sigma_{v(x)})^{-1} \circ v(x) \\ &= (x\xi(\lambda - 1) + \eta(\lambda - 1))(\sigma_\lambda - \sigma_{v(x)})^{-1} \circ v(x) \end{aligned}$$

putting $y = x\xi + \eta$. Hence,

$$y * x = (x\gamma + \delta) \circ (x\alpha + \beta) ,$$

where γ and δ satisfy

$$f(\alpha^{-1}\gamma) + h(\delta - \beta\alpha^{-1}\gamma) = \alpha^{-1}\gamma\lambda - \xi\alpha^{-1}(\lambda - 1) \tag{10}$$

and

$$\begin{aligned} &g(\alpha^{-1}\gamma) + k(\delta - \beta\alpha^{-1}\gamma) \\ &\quad = (\delta - \beta\alpha^{-1}\gamma)\lambda + \beta\alpha^{-1}\xi(\lambda - 1) - \eta(\lambda - 1) . \end{aligned} \tag{11}$$

Thus $y * x = x(\gamma\lambda - \xi(\lambda - 1)) + \delta\lambda - \eta(\lambda - 1)$, where γ and δ satisfy (10) and (11). Putting $\xi = 0$ in (10) and (11), we see that $h_\lambda(\eta) = \gamma\lambda = \alpha\lambda M_\lambda^{-1}(\eta(1 - \lambda)) = M_\lambda^{-1}(\eta(1 - \lambda))(M_\lambda^{-1}(1 - \lambda))^{-1}$.

4. Characterization of Hall planes of even order. The purpose of this section is to prove the following theorem.

THEOREM 3. *A generalized Hall plane π with respect to l_∞, π_0 of even order is a Hall plane if and only if each point of $\boldsymbol{M}$ is the centre of a nontrivial elation (with axis $l \neq l_\infty$) fixing $\boldsymbol{M}$.*

Proof. If π is an even order Hall plane, Hughes [3] has shown that each point of $\boldsymbol{M}$ is the centre of a nontrivial elation (with axis $l \neq l_\infty$) fixing $\boldsymbol{M}$.

Suppose π is a generalized Hall plane of even order such that each point of M is the centre of a nontrivial elation (with axis $l \neq l_\infty$) fixing M. Let us coordinatize π over the quadrangle O, I, X, Y in π_0 (where $XY = l_\infty$) by a generalized Hall system F with subfield F_0 coordinatizing π_0 and defining functions f, g, h, and k. There is a nontrivial (Y, OY)-elation φ which fixes M because there is a nontrivial elation with centre Y and axis $l \neq l_\infty$, fixing M, and π is a translation plane with respect to l_∞.

Suppose φ takes X to (d) where $d \in F_0$. Now d belongs to the distributor of F. Thus,

$$x(d + y) = xd + xy \text{ for all } x, y \in F\,. \tag{12}$$

Putting $x = z\alpha + \beta$ and $y = z$, where $z \in F\backslash F_0$, in (12) we have

$$\begin{aligned}
&(z\alpha + \beta)(d + z) = (z\alpha + \beta)d + (z\alpha + \beta)z\\
&\qquad \Longrightarrow ((z + d)\alpha + \beta + d\alpha)(z + d) = (z\alpha + \beta)d + (z\alpha + \beta)z\\
&\qquad \Longrightarrow (z + d)(f(\alpha) + h(\beta + d\alpha)) + g(\alpha) + k(\beta + d\alpha)\\
&\quad = z\alpha d + \beta d + z(f(\alpha) + h(\beta)) + g(\alpha) + k(\beta)\,.
\end{aligned}$$

It follows readily that $h(\xi) = \xi$ for all $\xi \in F_0$. We note here that this is true whatever our original choice of quadrangle O, I, X, Y in π_0 might be (provided $XY = l_\infty$, of course).

Consider a nontrivial elation ψ fixing M with centre $P \in M$ and axis $l(\neq l_\infty) \in \pi_0$. Suppose $Q \in M\backslash\{P\}$ and $Q^\psi = R$. If we choose coordinate quadrangle such that $Q = Y$, $R = X$, $P = (1)$, and $O \in l \cap \pi_0$, then $(x, y)^\psi = (y, x)$. Suppose $(z)^\psi = (w)$ where $z \in F\backslash F_0$. Then $w \in F\backslash F_0$ and $(w)^\psi = (z)$. So $(x, xz)^\psi = (x', x'w)$, since $O^\psi = O$. Hence $x' = xz$ and we have

$$(xz)w = x \text{ for all } x \in F\,. \tag{13}$$

Now there is an automorphism ρ of F fixing F_0 pointwise and taking z to w and w to (say) v. Thus, $(xw)v = x$ for all $x \in F$. But $(w)^\psi = (z)$, so $(xw)z = x$ for all $x \in F$ and so we have $v = z$.

Suppose $w = z\lambda_z + \mu_z$ where λ_z and $\mu_z \in F_0$. The automorphism ρ taking z to w takes w to z. So $(z\lambda_z + \mu_z)\lambda_z + \mu_z = z$, whence $\lambda_z = 1$ for all $z \in F\backslash F_0$. Thus $w = z + \mu_z$. Substituting for w in (13) gives $(xz)(z + \mu_z) = x$ for all $x \in F$. So, a fortiori, $(xz)(z + \mu_z) = x$ for all $x \in F_0$ and it follows that $(z + \mu_z)(f(x) + \mu_z x + k(x)) + g(x) + k(\mu_z x + k(x)) = x$ for all $x \in F_0$. Hence, $f(x) + \mu_z x + k(x) = 0$ for all $x \in F_0$ and we see that μ_z is independent of z. Writing $\mu_z = \mu$ we have

$$f(x) + \mu x + k(x) = 0 \text{ for all } x \in F_0\,, \tag{14}$$

and

$$g(x) + kf(x) + x = 0 \text{ for all } x \in F_0 . \tag{15}$$

From Lemma 2 we see that $h_\lambda(x) = M_\lambda^{-1}((1-\lambda)x)(M_\lambda^{-1}(1-\lambda))^{-1}$, where $M_\lambda(x) = g(x) + k(\lambda x) + kf(x) + \lambda^2 x + \lambda f(x)$ for all $x \in F_0$ and $\lambda \in \bar{F}_0 = F_0 \backslash \{0, 1\}$. But $h_\lambda(x) = x$ for all $\lambda \in \bar{F}_0$ as we have seen. This means that $M_\lambda(x) = \bar{\lambda} x$ where $\bar{\lambda} = (1-\lambda)(M_\lambda^{-1}(1-\lambda))^{-1}$ for all $x \in F_0$, $\lambda \in \bar{F}_0$. So we have

$$g(x) + k(\lambda x) + kf(x) + \lambda^2 x + \lambda f(x) = \bar{\lambda} x \text{ for all } x \in F_0, \lambda \in \bar{F}_0 . \tag{16}$$

Now (14), (15), and (16) imply $k(\lambda x) + \lambda k(x) = \lambda' x$ where $\lambda' = \bar{\lambda} + 1 + \lambda^2 + \lambda\mu$. We can state this as

$$km_\lambda + m_\lambda k = m_{\lambda'} \text{ for all } \lambda \in \bar{F}_0 . \tag{17}$$

(In (17) m_α stands for the endomorphism of $(F_0, +)$ given by $m_\alpha(x) = \alpha x$.)

But k may be uniquely written

$$k = m_{\alpha_0} + m_{\alpha_1} T + \cdots + m_{\alpha_{n-1}} T^{n-1} , \tag{18}$$

where the α_i's lie in $F_0 = GF(2^n)$ and T is the automorphism of F_0 given by $T(x) = x^2$.

Substituting (18) in (17) gives

$$m_{\lambda'} + m_{\beta(1,\lambda)} T + \cdots + m_{\beta(n-1,\lambda)} T^{n-1} = 0$$

where $\beta(i, \lambda) = \alpha_i(T^i(\lambda) + \lambda)$ for all $i = 1, \cdots, n-1$ and $\lambda \in \bar{F}_0$. So $\lambda' = \beta(i, \lambda) = 0$ for all $i = 1, \cdots, n-1$ and $\lambda \in \bar{F}_0$. But $T^i(\lambda) + \lambda \neq 0$ for some $\lambda \in \bar{F}_0$ for each $i = 1, \cdots, n-1$ and thus $k = m_{\alpha_0}$. But $k(1) = 0$ and so $\alpha_0 = 0$ and $k = 0$ (the zero map). From (14) and (15) we have $f(x) = \mu x$ and $g(x) = x$. It follows that F is a Hall system ([2]) and π is a Hall plane.

Theorem 3 of this paper and the characterization of the odd order Hall planes given in [8] allow us to assert the following result:

THEOREM 4. *A finite generalized Hall plane with respect to* l_∞, π_0 *is a Hall plane if and only if each point of* $\boldsymbol{M} = l_\infty \cap \pi_0$ *is the centre of a nontrivial involutory central collineation which has axis* $\neq l_\infty$ *in* π_0 *and which fixes* $\boldsymbol{M}$.

There are other characterizations of the finite Hall planes amongst the finite generalized Hall planes. Of these we mention: A generalized Hall plane π of order $q^2 \neq 4$ is a Hall plane if and only if the kernel of π is of order q.

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REFERENCES

1. P. Dembowski, *Finite Geometries*, (*Ergebrisse der Mathematik und ihrer Grenzgebiete*, Band 44, Springer-Verlag, Berlin, Heidelberg, New York, 1968)
2. Marshall Hall, Jr. *The Theory of Groups*, The Macmillan Company, New York, 1959.
3. D. R. Hughes, *Collineation groups of nondesarguesian planes, I. The Hall Veblen-Wedderburn systems*, Amer. J. Math., **81** (1959), 921-938.
4. Norman Lloyd Johnson, *A classification of semi-translation planes*, Canad. J. Math., **21** (1969), 1372-1387.
5. ———, *Translation planes constructed from semifield planes*, Pacific J. Math., **36** (1971), 701-711.
6. ———, *A characterization of generalized Hall planes*, Bull. Austral. Math. Soc., **6** (1972), 61-67.
7. P. B. Kirkpatrick, *Generalization of Hall planes of odd order*, Bull. Austral. Math. Soc., **4** (1971), 205-209.
8. ———, *A characterization of the Hall planes of odd order*, Bull. Austral. Math. Soc., **6** (1972), 407-415.
9. T. G. Ostrom, *Semi-translation planes*, Trans. Amer. Math. Soc., **111** (1964), 1-18.
10. A. J. Rahilly, *Veblen-Wedderburn systems and translation planes*, M. Sc. Thesis, University of Melbourne, 1970.

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