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CYCLIC AMALGAMATIONS OF RESIDUALLY FINITE GROUPS

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A group G is said to be *residually finite* if the intersection of the collection of all subgroups of finite index in G is *the trivial group. This paper is concerned with the following* question. If A and B are residually finite groups, and if G is the generalized free product of A and B with a single cyclic subgroup amalgamated, then what conditions on A and B will insure that G is residually finite? The main result of this paper is that there exists a class C of residually finite *groups which contains all free groups, polycylic groups, fundamental* groups of 2-manifolds, and other common residually finite groups, and in addition C is closed under the operation of *forming* generalized free products with a single cyclic subgroup amalgamated.

A well known example of G. Higman [4] shows that the generalized free product of residually finite groups amalgamated along a single cyclic subgroup need not be residually finite. However G. Baumalag [1] has shown that such a product is residually finite if both factors of the product are free or if both factors of the product are finitely generated and torsion free nilpotent. The results here generalize these theorems of Baumslag.

In order to study generalized free products of residually finite groups, P. Stebe [6] introduced the notion of a π_c group. Let J be a subset of a group G, and let H be a subgroup of G. J is said to be *H-separable* in G if for each g in J, either $g \in H$ or there is a homomorphism α of G onto a finite group such that $\alpha(g) \notin \alpha(H)$. Let $\pi_c(G)$ denote the subset of $G \times G$ with the property that $(g, h) \in \pi_c(G)$ if and only if $\{g\}$ is $gp(h)$-separable in G. ($gp(h)$ denotes the subgroup of G generated by h.) We say that G is a π_c group if $\pi_c(G) = G \times G$.

It is not difficult to show that the most common residually finite groups are π_c groups (e.g. free groups, parafree groups, polycyclic groups, etc.). However, there are residually finite groups which are not π_c groups [2]. Such groups can be used to construct a large class of nonresidually finite groups.

Let A and B be groups with subgroups H and K respectively, and let $\alpha: H \rightarrow K$ be an isomorphism. We denote the generalized free product of A and B analgamated along H and K via the isomorphism α by $G = {}^*(A, B; H, K, \alpha)$. When we are not concerned with the amalgamating isomorphism, this notation will sometimes be shortened to read $G = *(A, B; H)$. When H is cyclic, we shall also make use of the notation $G = *(A, B; a_0 = b_0)$.

2. Some technical lemmas.

We wish here to record several lemmas that will be useful in the next section. Some of these lemmas appear elsewhere in various forms, but in any case all can be proved using standard techniques. Where necessary in this section, references for similar theorems will be provided, but explicit proofs will be omitted.

That the most common residually finite groups are π_c groups is the subject of the next two lemmas. The proof of the first part of Lemma 2.1 may be found in [6], and the second part may be proved by similar methods. The proof of Lemma 2.2 is essentially given in Theorem 1 of [6].

LEMMA 2.1. *Each finite extension of a π_c group is a π_c group, and each split extension of a finitely generated π_c group by a π_c group is a π_c group.*

LEMMA 2.2. *If G is residually a finite p-group for all primes p, and if the centralizer of each element of G is cyclic, then G is a π_c group.*

From Lemma 2.2 we conclude that free groups, parafree groups, and fundamental groups of 2-manifolds are π_c groups. Then using Lemma 2.1. we see that polycyclic groups are π_c groups. Finally a generalized free product of finite groups is a finite extension of a free group so that all such groups are π_c groups.

The proofs of some of Stebe's theorems in [6] can be altered to obtain the following useful lemma.

LEMMA 2.3. *Let $G = (A, B; H)$. If $A \cup B$ is an H-separable subset of G, and if $A \times A \cup B \times B \subset \pi_c(G)$, then G is a π_c group.*

In case H is cyclic, we obtain a more concise version of Lemma 2.3.

LEMMA 2.4. *Let $G = *(A, B; H)$. If H is cyclic, and if $A \times A \cup B \times B \subset \pi_c(G)$, then G is a π_c group.*

G. Baumslag [1] has shown that if A and B are residually finite groups, then $*(A, B; H)$ is residually finite if H is a finite group. A slight modification of the method used by Baumslag to prove the above result together with Lemma 2.3 yields the following result.

THEOREM 2.5. *If A and B are π_c groups, and if H is a finite subgroup of both A and B then $*(A, B; H)$ is a π_c group.*

3. Finite quotient groups of π_c groups. By following the construction of Higman [4], we obtain the following theorem.

THEOREM 3.1. *Let A be a residually finite group with an element a of infinite order. There is a residually finite group B with an element b of infinite order such that $G = *(A, B; a = b)$ is not residually finite.*

Proof. Let B be any residually finite group which is not a π_c group. Then there is an element b_0 of infinite order in B and an element b_1 in B such that b_1 is not $gp(b_0)$-separable. Let $G = *(A, B; a^2 = b_0)$.

Then the commutator $[a, b_1]$ is a reduced word in G and hence is not the trivial element. Let α be any homomorphism of G onto a finite group. Then $\alpha(b_1) \in gp(\alpha(b_0)) \subset gp(\alpha(a))$. It follows that $\alpha[a, b_1] = [\alpha(a), \alpha(b_1)] = 1$. Hence G is not residually finite.

Let C^* be the class of all residually finite groups A with the propterty that if B is any residually finite group then $*(A, B; H)$ is residually finite if H is cyclic. Let C denote the class of all π_c groups with the property that $*(A, B; H)$ is a π_c group whenever H is a cyclic group. According to the above theorem and Theorem 3 of [1], C^* is exactly the class of all residually finite torsion groups. In comparison, we shall show that C is considerably larger than C^*. Not only does C contain the most common π_c groups, but also C is closed under the operation of forming generalized free products with a single cyclic subgroup amalgamated.

We begin with a study of finite quotient groups of π_c groups. With each element g of a group G, we associate a set of positive integers $G(g)$ with the property that $n \in G(g)$ if and only if G has a finite quotient group in which the image of g has order n.

A subset X of $G(g)$ is said to be *cofinal* in $G(g)$ if and only if for each pair g_1, g_2 in G, either $g_1 \in gp(g_2)$, or there is a homomorphism α of G onto a finite group such that $\alpha(g_1) \notin gp(\alpha(g_2))$, and the order of $\alpha(g)$ is in X. In particular, G is a π_c group if and only if $G(1)$ is cofinal in $G(1)$. More generally, we have the following lemma.

LEMMA 3.2. *Let A and B be π_c groups, and let a_0 and b_0 be elements of infinite order in A and B respectively. Then the generalized free product $*(A, B; a_0 = b_0)$ is a π_c group if and only if $A(a_0) \cap B(b_0)$ is cofinal in both $A(a_0)$ and $B(b_0)$.*

Proof. Suppose $G = *(A, B; a_0 = b_0)$ is a π_c group. We wish to show that $A(a_0) \cap B(b_0)$ is cofinal in $A(a_0)$. Let x and y be elements of A with $x \notin gp(y)$. There is a homomorphism α of G onto a finite group such that $\alpha(x) \notin gp(\alpha(y))$. Since $\alpha(a_0) = \alpha(b_0)$, we may restrict α to A and to B to obtain $|\alpha(a_0)| \in A(a_0) \cap B(b_0)$. Thus $A(a_0) \cap B(b_0)$ is cofinal in $A(a_0)$.

Similarly, $A(a_0) \cap B(b_0)$ is cofinal in $B(b_0)$.

We now suppose that $A(a_0) \cap B(b_0)$ is cofinal in both $A(a_0)$ and $B(b_0)$. According to Lemma 2.4, we need only show that $A \times A \cup B \times B \subset \pi_c(G)$. Let $x, y \in A$ with $x \notin gp(y)$. Since $A(a_0) \cap B(b_0)$ is cofinal in $A(a_0)$, there is a homomorphism α of A onto a finite group such that $\alpha(x) \notin gp(\alpha(y))$, and $|\alpha(a_0)| \in A(a_0) \cap B(b_0)$. Let A_1 be the kernel of α. Since $|\alpha(a_0)| \in B(b_0)$, there is a normal subgroup B_1 of finite index in B such that the order of b_0 in B/B_1 is $|\alpha(a_0)|$. Observe that the isomorphism of $gp(a_0)$ onto $gp(b_0)$ defined by $a_0 \to b_0$ carries $A_1 \cap gp(a_0)$ isomorphically onto $B_1 \cap gp(b_0)$. Thus we obtain a natural homomorphism β of G onto a generalized free product of finite groups

$$G_1 = *(A/A_1, B/B_1; a_0 = b_0) .$$

Further, since $x \notin gp(y) \bmod A_1$, it follows that $\beta(x) \notin gp(\beta(y))$.

Since G_1 is a π_c group, it follows that $(x, y) \in \pi_c(G)$. Thus $A \times A \subset \pi_c(G)$. Similarly, $B \times B \subset \pi_c(G)$. An application of Lemma 2.4 completes the proof.

We say that G has *regular quotients* at g if there is a constant K_g such that $\{nKg; n = 1, 2, \cdots\} \subset G(g)$. A group G is said to have *regular quotients* if G has regular quotients at each element of infinite order in G. All π_c groups have a property approximating regular quotients. This is the subject of the next lemma.

LEMMA 3.3. *Let G be a π_c group and K any positive integer. If $x \in G$ has infinite order, then there is a homomorphism α of G onto a finite group such that K divides the order of $\alpha(x)$.*

Proof. It clearly suffices to prove Lemma 3.3 when $K = p^t$ is a power of a prime p. Since x has infinite order, $x \notin gp(x^p)$ and $x^r \notin gp(x^{p^t})$ for any r with $0 < |r| < p^t$. Thus there is a homomorphism α of G onto a finite group with the following properties.

(1) $\alpha(x) \notin gp(\alpha(x)^p)$.

(2) $\alpha(x)^r \notin gp(\alpha(x)^{p^t})$, $0 < |r| < p^t$.

Since $\alpha(x) \notin gp(\alpha(x)^p)$, it follows that $(|\alpha(x)|, p) \neq 1$. Hence p divides the order of $\alpha(x)$. Let $|\alpha(x)| = p^s Q$ wher $(p, Q) = 1$. We wish to show that $t \leqq s$.

Choose an integer $R \geqq 1$ such that $(R, p) = 1$ and $Rp^sQ > p^t$. Then $Rp^sQ = Wp^t + r$ where $|r| < p^t$. Then $(\alpha(x)^{p^t})^w = \alpha(x)^{-r}$. It follows from condition 2 above that $r = 0$. But then $RQ = Wp^{t-s}$. Since $(p, R) = (p, Q) = 1$, it follows that $t - s \leqq 0$. Hence $t \leqq s$ so that p^t divides $|\alpha(x)|$.

COROLLARY 3.3.1. *Let G be a π_c group, and let g be an element of infinite order in G. If G has regular quotients at g^k for some positive integer k, then G has regular quotients at g.*

Proof. Let $\{nL; n = 1, 2, \cdots\} \subset G(g^k)$. Let G_1 be a normal subgroup of finite index in G such that k divides the order of g in G/G_1. Suppose G_1 is of index S in G. We shall show that

$$\{nSLk; n = 1, 2, \cdots\} \subset G(g) .$$

Let G_0 be a normal subgroup of finite index in G such that g^k has order nSL in G/G_0. Let $G_n = G_0 \cap G_1$. Then g^k has order nSL in G/G_n, and k divides the order of g in G/G_n. In G/G_n,

$$|g^k| = \frac{|g|}{(|g|, k)}$$

But $(|g|, k) = k$. Thus $|g| = |g^k|k = nSLk$ in G/G_n. This completes the proof of Corollary 3.3.1.

We are now prepared to prove a theorem which will enable us to identify certain members of the class C.

THEOREM 3.4. *Let A and B be π_c groups with elements a_0 and b_0 of infinite order in A and B respectively. If A has regular quotients at a_0, then $G = *(A, B; a_0 = b_0)$ is a π_c group.*

Proof. We shall show that $A(a_0) \cap B(b_0)$ is cofinal in both $A(a_0)$ and $B(b_0)$. Let $\{Kn \mid n = 1, 2, \cdots\} \subset A(a_0)$. Let $x, y \in A$ with $x \notin gp(y)$. Let A_1 be a normal subgroup of finite index in A such that $x \notin gp(y) \bmod A_1$. Suppose a_0 has order L in A/A_1. Choose B_1 to be a normal subgroup of finite index in B such that b_0 has order KLM in B/B_1 for some positive integer M. Let A_2 be a normal subgroup of finite index in A such that a_0 has order KLM in A/A_2. Put $A_3 = A_1 \cap A_2$. Then clearly $x \notin gp(y) \bmod A_3$, and a_0 has order KLM in A/A_3. Since KLM belongs to both $A(a_0)$ and $B(b_0)$, we have shown that $A(a_0) \cap B(b_0)$ is cofinal in $A(a_0)$.

The proof that $A(a_0) \cap B(b_0)$ is cofinal in $B(b_0)$ is similar (in fact less complicated) and is omitted.

Theorem 3.4 together with Theorem 2.5 yield the following corollary.

COROLLARY 3.4.1. *If G is a group with regular quotients, then G is in the class C.*

We wish now to establish that the most common groups have regular quotients and hence belong to C. In order to prove this, we need to consider a possibly stronger property. We say G has *completely regular quotients* at an element g of infinite order in G if there is a constant K_g such that for each n, there is a characteristic subgroup H_n of finite index in G such that G has order nK_g in G/H_n.

Following closely the proof of Corollary 3.3.1, we obtain the following lemma.

LEMMA 3.5. *Let G be a finitely generated π_c group, and let g be an element of infinite order in G. If G has completely regular quotients at g^K for some positive integer K, then G has completely regular quotients at g.*

LEMMA 3.6. *Let G be a finite extension of a finitely generated π_c group A. If A has completely regular quotients, then G has regular quotients.*

Proof. Let g be an element of infinite order in G. Then $g^k \in A$ for some positive integer k. By Corollary 3.3.1., it suffices to prove that G has regular quotients at g^k.

Let L be a positive integer such that for each n, there is a characteristic subgroup A_n of finite index in A such that g^k has order nL in A/A_n. Observe that A_n is a normal subgroup of finite index in G and that g^k has order nL in G/A_n. Thus

$$\{nL; n = 1, 2, \cdots\} \subset G(g^k) .$$

This completes the proof of Lemma 3.6.

Lemma 5.14 of [1] (together with the simple observation that an element of order k in a residually finite group can be represented on a finite group so that its image has order k) shows that torsion free nilpotent groups have regular quotients. The proof in fact yields that finitely generated torsion free nilpotent groups have completely regular quotients.

It is an easy consequence then that finitely generated parafree groups have completely regular quotients. Since each generalized free product of finite groups is a finite extension of a free group, it follows that these groups also have regular quotients.

If g is an element of infinite order in a polycyclic group G, then there are integers i and k such that g^k has infinite order in $G^{(i)}/G^{(i+1)}$ ($G^{(r)}$ is the rth term of the commutator series of G). Then following Baumslag's proof of Lemma 5.14 [1], it is not difficult to show that G has completely regular quotients at g^k. It follows from Lemma 3.5 that each polycyclic group has completely regular quotients. In summary, we have the following theorem.

THEOREM 3.7. *Free groups, parafree groups, polycyclic groups and generalized free products of finite groups have regular quotients and hence belong to the class C.*

This compares favorably with Baumslag's Theorems 6 and 7 of [1]. We now proceed to show that C is in fact closed under cyclic amalgamations.

LEMMA 3.8. *Let $A \cup B$ be an H-separable subset of $G = *(A, B; H)$. Then G has regular quotients at each element of cyclic length greater than one in G.*

Proof. Let $g = a_1b_1a_2b_2 \cdots a_kb_k$ be a cyclically reduced word in G with $k \geqq 1$ and $a_i \in A - H$, $b_i \in B - H (1 \leqq i \leqq k)$. Then there is a normal subgroup N of finite index in G such that a_i, $b_i \notin H \bmod N (1 \leqq i \leqq k)$. Let $A_1 = A \cap N$ and $B_1 = B \cap N$. Since $A_1 \cap H = B_1 \cap H$, we obtain a natural homomorphism α of G onto a generalized free product of finite groups $G_1 = *(A/A_1, B/B_1; H/H \cap N)$ with the property that

$$\alpha(a_i) \in A/A_1 - H/H \cap N,\ \alpha(b_i) \in B/B_1 - H/H \cap N \quad (1 \leqq i \leqq k)\ .$$

In particular, $\alpha(g)$ has cyclic length greater than one in G_1 and hence $\alpha(g)$ has infinite order in G_1. Since G_1 has regular quotients, it follows that G has regular quotients at g. This completes the proof of Lemma 3.8.

LEMMA 3.9. *Let a_0 and b_0 denote elements of infinite order in A and B respectively and let a_1 be an arbitrary element of A. If $G_1 = *(A, B; a_0 = b_0)$ is a π_c group, then $G_2 = *(A, B; a_1a_0a_1^{-1} = b_0)$ is also a π_c group.*

The proof of Lemma 3.9 is fairly straightforward and is omitted.

THEOREM 3.10. *If A and B belong to the class C and if H is a subgroup of A and B that is either finite or cyclic, then $G = *(A, B; H)$ is a member of C.*

Proof. We consider only the case that H is infinite cyclic so that $G = {}^*(A, B; a_0 = a_0)$. (The case that H is finite can be handled in a similar fashion.) Let D be any π_c group, and let g_0 and d_0 denote elements of G and D respectively such that $gp(g_0)$ is isomorphic to $gp(d_0)$. We wish to show that $K = {}^*(G, D; g_0 = d_0)$ is a π_c group. By Theorem 2.5, we may assume that $gp(g_0)$ is infinite. Also making use of Lemma 3.9, we assume that g_0 is a cyclically reduced word in G.

If g_0 has length greater than one, then G has regular quotients at g_0, and we may apply Theorem 3.4 to obtain our result.

It remains only to consider the case that g_0 has length one (with no loss of generality we assume that $g_0 \in B$). But then applying the definition of the class C twice we obtain that

$$\begin{aligned} K &= {}^*({}^*(A, B; a_0 = b_0), D; g_0 = d_0) \\ &= {}^*(A, {}^*(B, D; g_0 = d_0; a_0 = b_0) \end{aligned}$$

is a π_c group.

There are several interesting questions concerning the class C which the author has been unable to answer.

Question 1. *Is there a π_c group not in class C?*

Question 2. *If G is in C, does G have regular quotients?*

The author strongly suspects that both questions 1 and 2 have an affirmative answer, and that all that is required is a suitably general example of a π_c group without regular quotients. Note however, that Lemma 3.3 indicates that some care will be required in constructing such an example (if it exists).

In any case, a theorem analogous to Theorem 3.10 can be established for groups with regular quotients.

THEOREM 3.11. *If A and B have regular quotients, and if H is a subgroup of A and B such that H is either finite or cyclic, then ${}^*(A, B; H)$ also has regular quotients.*

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