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INTEGRAL REPRESENTATIONS OF WEAKLY COMPACT OPERATORS

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Let X be a completely regular space and E, F locally convex spaces. Denote by $C_{rc} = C_{rc}(X, E)$ the space of all continuous functions f from X into E for which $f(X)$ is relatively compact. Uniformly continuous weakly compact operators from C_{rc} into F are repreesented by integrals with respect to $\mathcal{L}(E, F)$ valued measures on the algebra generated by the zero sets. Necessary and sufficient conditions for an operator to be continuous, with respect to certain topologies, are obtained. A sufficient condition for extending a measure to all Baire sets is given.

Introduction. In [5] D. Lewis represented weakly compact operators from the space $C(S)$ of all scalar-valued continuous functions on a compact space into a locally convex space. The representation was given by means of integrals with respect to vector-valued measures on the Borel field. In [1] Bartle, Dunford, and Schwartz gave a similar representation for operators from $C(S)$ into a Banach space. Also Grothendieck [2] noted that the family of all weakly compact operators fron $C(S)$ into a locally convex space E corresponds exactly to the family of E-valued measures on the Baire algebra. In this paper we will give integral representations for weakly compact operators from C_{rc} into F by means of integrals with respect to $\mathcal{L}(E, F)$ valued measures on the algebra generated by the zero sets. Necessary and sufficient conditions for an operator to be continuous with respect to certain locally convex topologies are given. Also a result is obtained on the extension of measures to all Baire sets.

1. Definitions and preliminaries.

Let X be a completely regular Hausdorff space and let $B = B(X)$ denote the algebra of subsets of X generated by the zero sets. By $Ba = Ba(X)$ and $Bo = Bo(X)$ we will denote the σ-algebras of Baire and Borel sets respectively. Let $M(X)$ denote the space of all bounded finitely-additive regular (with respect to the zero sets) measures on B (see Varadarajan [8]). The spaces of all σ-additive and all τ-additive members of $M(X)$ will be denoted by $M_\sigma(X)$ and $M_\tau(X)$ respectively. The set $M_\sigma(Ba)$ is the space of all real-valued Baire measures while $M_\tau(Bo)$ denotes the space of all bounded real-valued regular Borel measures m with the property that $m(G\alpha) \to 0$ for every net $\{G_\alpha\}$ of closed sets which decreases to the empty set.

Let E be a real locally convex Hausdorff space. For p a continuous seminorm on E, we define $M_p(B, E')$ as the set of all E'-valued (E' is the dual of E) finitely-additive measures m on B with the following two properties:

(1) For every $s \in E$, the function ms, from B into the reals $R, G \to m(G)s$, is in $M(X)$.

(2) $\|m\|_p = m_p(X) < \infty$, where for G in B the $m_p(G)$ is defined to be the supremum of all $|\Sigma m(G_i)s_i|$ for all finite B-partitions $\{G_i\}$ of G, i.e., $G_i \in B$, and all finite collections $s_i \in B_p = \{s \in E: p(s) \leqq 1\}$. The set $M_{\sigma,p}(B, E')$ consists of those m in $M_p(B, E')$ for which $ms \in M_\sigma(X)$ for all s in E. The spaces $M_{\tau,p}(B, E')$, $M_{\sigma,p}(Ba, E')$, and $M_{\tau,p}(Bo, E')$ are defined similarly. As shown in **[3]** if m is in any one of the spaces $M_p(B, E')$, $M_{\sigma,p}(B, E')$, $M_{\tau,p}(B, E')$, $M_{\sigma,p}(Ba, E')$, $M_{\tau,p}(Bo, E')$, then m_p belongs to $M(X)$, $M_\sigma(X)$, $M_\tau(X)$, $M_\sigma(Ba)$, $M_\tau(Bo)$ respectively. Every $m \in M_{\sigma,p}(B, E')$ $[m \in M_{\tau,p}(B, E')]$ has a unique extension μ to a member of $M_{\sigma,p}(Ba, E')$ [to a member of $M_{\tau,p}(Bo, E')$]. Moreover, the restriction of μ_p to B coincides with m_p. Let $\{p : p \in I\}$ be a generating family of continuous seminorms on E which is directed, i.e., given p_1, p_2 in I there exists $p \in I$ with $p \geqq p_1, p_2$. Let $M(B, E') = \cup\{M_p(B, E') \cdot p \in I\}$ with analogous definitions for $M_\sigma(B, E')$, $M_\tau(B, E')$, $M_\sigma(Ba, E')$ and $M_\tau(Bo, E')$.

Denote by $C_{rc} = C_{rc}(X, E)$ the space of all continuous functions f from X into E for which $f(X)$ is relatively compact. Every f in C_{rc} has a unique continuous extension $\hat{f}$ to all of the Stone Cěch compactification βX. By $C^b(X)$ we denote the space of all bounded continuous real-valued functions on X. Let Ω and Ω_1 be, respectively, the class of all compact and all zero sets in βX disjoint from X. Let $Q \in \Omega$. We define β_Q to be the locally convex topology generated by the family of seminorms $f \to \|fg\|_p = \sup\{p(f(x)g(x)): x \in X\}$ where $p \in I$ and $g \in B_Q = \{h \in C^b: \hat{h}(x) = 0 \text{ for } x \text{ in } Q\}$. The topologies β and β_1 on C_{rc} are defined to be the inductive limits of the topologies β_Q as Q ranges over Ω and Ω_1 respectively. For a fixed $p, \beta_{p,Q}$ is the locally convex topology on C_{rc} generated by the seminorms $f \to \|gf\|_p$, $g \in B_Q$. As shown in **[3]**, $\beta_{p,Q}$ is the finest locally convex topology on C_{rc} which agrees with $\beta_{p,Q}$ on p-bounded sets. Let β_p and $\beta_{1,p}$ denote the inductive limits of the topologies $\beta_{p,Q}$ as Q ranges over Ω and Ω_1 respectively. The topologies β' and β'_1 are the projective limits of the topologies β_p and $\beta_{1,p}$, respectively, as p ranges over I. If u is the uniform topology, then $\beta' \leqq \beta \leqq \beta_1 \leqq u$ and $\beta'_1 \leqq \beta_1$.

For G in B and $m \in M_p(B, E')$ we define $\int_G f dm = \lim \Sigma\, m(G_i) f(x_i)$ where the limit is taken over the directed set of all finite B-partitions $\{G_i\}$ of G and $x_i \in G_i$. The map $f \to T_m(f) = \int_X fdm$ is a uniformly continuous linear functional on C_{rc}. Moreover, $\|m\|_p = \sup\{|T_m(f)|: \|f\|_p \leqq 1\}$. The mapping $m \to T_m$ is a one-to-one linear map from $M(B, E')$ into $(C_{rc}, u)'$. The space $M_\sigma(B, E')$ is the dual space of each of the topologies β_1 and β_1' while $M_\tau(B, E')$ is the dual space of each of the topologies β and β'. Given any $m \in M_p(B, E')$ there exists a unique $\hat{m}$ in $M_{\tau,p}(Bo(\beta X), E')$ such that $\int_X fdm = \int_{\beta X} \hat{f} d\hat{m}$ for all f in C_{rc}. As shown in [3], m is σ-additive iff $\hat{m}_p(Z) = 0$ for all Z in Ω_1. Similarly m is τ-additive iff $\hat{m}_p(Q) = 0$ for all Q in Ω. Moreover, if m is σ-additive or τ-additive, then $\hat{m}(Q) = m(Q \cap X)$ and $\hat{m}_p(Q) = \hat{m}_p(Q \cap X)$ for all Q in $B(\beta X)$.

Let now F be another real locally convex Hausdorff space and let $\{q: q \in J\}$ be a generating directed family of continuous seminorms on F. Let $\mathscr{L}(E, F)$ denote the space of all continuous operators from E into F. We define $M(B, \mathscr{L}(E, F))$ to be the space of all finitely-additive $\mathscr{L}(E, F)$ valued measures m on B with the following two properties:

(1) For each $x' \in F'$ the set function $x'm: B \to E'$, $(x'm)(G)s = x'(m(G)s)$, $s \in E$, is in $M(B, E')$.

(2) Given $q \in J$ there exists p in I such that for all x' in the polar B_q^0 of B_q in F' the $x'm$ is in $M_p(B, E')$ and $\|m\|_{p,q} = m_{p,q}(X) < \infty$ where for Q in $B, m_{p,q}(Q) = \sup\{(x'm)_p(Q): x' \in B_q^0\}$. We define $M_\sigma(B, \mathscr{L}(E, F))$, $M_\tau(B, \mathscr{L}(E, F))$, $M_\sigma(Ba, \mathscr{L}(E, F))$ and $M_\tau(Bo, \mathscr{L}(E, F))$ analogously. Let $m \in M(B, \mathscr{L}(E, F))$ and f a function from X into E. We say that f is m-integrable over G in B if

(i) For each $x' \in F'$, the integral $\int_G fd(x'm)$ exists

(ii) there exists a vector in F denoted by $\int_G fdm$ such that for all $x' \in F'$ we have $x'\left(\int_G fdm\right) = \int_G fd(x'm)$.

Since F is a locally convex Hausdorff space, the $\int_G fdm$ is unique whenever it exists. If f is m-integrable over all G in B, we say that f is m-integrable.

2. Continuous linear operators from C_{rc} into F. Let $E, F, \{p: p \in I\}, \{q: q \in J\}$ be as in paragraph 1. Recall that a linear operator T from a topological vector space A into another B is weakly compact if it maps bounded subsets of A into weakly relatively compact subsets of B. We need the following lemma due to Grothendieck [2].

LEMMA 1. *Let T be an operator from a topological vector space A into another B and let T' and T'' denote, respectively, the transpose and the second transpose of T. The following are equivalent:*

(1) *T is weakly compact*

(2) *T'' maps A'' into B*

(3) *If B' is equipped with the Mackey topology $m(B',B)$ and A' with the strong topology $\beta(A',A)$, then T' is continuous.*

LEMMA 2. *Let f_0 be in C_{rc} and G in B. Define ϕ on $M(B,E')$ by $\phi(m)=\int_G f_0 dm$. Then ϕ belongs to the $(C_{rc},u)''$.*

Proof. Let $A=\{f\in C_{rc}:\|f\|_p\leqq\|f_0\|_p$ for all p in $I\}$. Then A is u-bounded and hence the polar A^0 in $(C_{rc},u)'$ is a strong neighborhood of zero. We will finish the proof by showing that ϕ is bounded on A^0. To this end consider an arbitrary m in A^0. Let $\epsilon>0$ be given. There exists a B-partition $G_1,G_2,\cdots,G_n$ of G and $x_i\in G_i$ such that $\left|\int_G f_0 dm\right|\leqq|\Sigma m(G_i)s_i|+\epsilon$, $s_i=f_0(x_i)$. By the regularity of ms_i we can find zero sets $Z_i\subset G_i$ such that $|\Sigma m(G_i)s_i|\leqq|\Sigma m(Z_i)s_i|+\epsilon$. Again by the regularity of $|ms_i|$ ($|ms_i|$ is the absolute variation of ms_i) we can find pairwise disjoint cozero sets $U_1,\cdots,U_n$, $Z_i\subset U_i$ such that $\Sigma|ms_i|(U_i-Z_i)<\epsilon$. For each i choose $h_i\in C^b$, with $0\leqq h_i\leqq 1$, such that $h_i=1$ on Z_i and $h_i=0$ on $X-U_i$. Set $h=\Sigma h_i s_i$. Then $h\in A$ and hence $\left|\int_X hdm\right|\leqq 1$. But

$$\left|\int_X hdm\right|\geqq\left|\sum\int_{Z_i} h_i s_i dm\right|-\left|\sum\int_{U_i-Z_i} h_i d_i ms_i)\right|$$

$$\geqq\left|\sum m(Z_i)s_i\right|-\epsilon\geqq\left|\int f_0\,dm\right|-3\,\epsilon.$$

Since $\epsilon>0$ was arbitrary we conclude that $\left|\int_G f_0\,dm\right|\leqq 1$ and this completes the proof.

THEOREM 3. *If T is a continuous weakly compact operator from (C_{rc},u) into F, then there exists a unique $m\in M(B,\mathcal{L}(E,F))$ such that:*

(1) *Every f in C_{rc} is m-integrable and $\int_X fdm=T(f)$*

(2) *If $p\in I$ and $q\in J$ are such that $\|T\|_{p,q}=\sup\{q(T(f))$: $\|f\|_p\leqq 1\}\leqq\infty$, then $\|m\|_{p,q}=\|T\|_{p,q}$.*

(3) *For every $x'\in F'$, we have $T'x'=x'm$*

(4) *For every bounded set S in E the set $V_{m,S} = \{\Sigma m(G_i)s_i : \{G_i\}$ is a finite B-partition of X, $s_i \in S\}$ is weakly relatively compact.*
Conversely, if $m \in M(B, \mathcal{L}(E,F))$ is such that

(5) *holds, then every f in C_{rc} is m-integrable and the operator $T(f) = \int_X fdm$ is u-continuous and weakly compact.*

Proof. Suppose that T is u-continuous and weakly compact. By Lemma 1, T'' maps $(C_{rc}, u)''$ into F. If $f \in C_{rc}$ and G in B, the function $f\chi_G$ (χ_G is the characteristic function of G) defines an element of $(C_{rc}, u)''$ by $\langle \mu, f\chi_G \rangle = \int_G fd\mu$, $\mu \in M(B, E') = (C_{rc}, u)'$. Thus we may consider $f\chi_G$ as an element of $(C_{rc}, u)''$. Define $m(G): E \to F$ by $m(G)s = T''(\chi_G s)$, G in B. It is easy to see that $m(G) \in \mathcal{L}(E, F)$. In this way we define a map $m: B \to \mathcal{L}(E,F)$ which is clearly finitely additive. If $x' \in F'$ and s in E, then $(x'm)(G)s = x'(T''(\chi_G s)) = \langle T'x', \chi_G s\rangle = T'x'(G)s$. Thus $x'm = T'x' \in M(B, E')$. Let $q \in J$. Since T is u-continuous there exists $p \in I$ such that $\|T\|_{p,q} < \infty$. Let $x' \in B_q^0$. Then for f in C_{rc} with $\|f\|_p \leqq 1$ we have $|\langle f, x'm\rangle| = |\langle f, T'x'\rangle| \leqq |\langle Tf, x'\rangle| \leqq \|T\|_{p,q}$. Thus $\|x'm\|_p \leqq \|T\|_{p,q}$ which proves that $\|m\|_{p,q} \leqq \|T\|_{p,q}$ and so m is in $M(B, \mathcal{L}(E,F))$. Let G be in B and $f \in C_{rc}$. For $x' \in F'$ we have $x'(T''(\chi_G f)) = \langle T'x', \chi_G f\rangle = \langle x'm, \chi_G f\rangle = \int_G fd(x'm)$. This shows that $\int_G fdm = T''(\chi_G f) \in F$. Taking $G = X$ we get $\int_X fdm = T''(f) = T(f)$. For $f \in C_{rc}$ with $\|f\|_p \leqq 1$ and $x' \in B_q^0$ we have $|x'(T(f))| = |\int fd(x'm)| \leqq \|x'm\|_p \leqq \|m\|_{p,q}$. This proves that $\|T\|_{p,q} \leqq \|m\|_{p,q}$. For the uniqueness of m, suppose m_1 is another element in $M(B, \mathcal{L}(E,F))$ such that $\int_X fdm_1 = T(f)$ for all $f \in C_{rc}$. Then for $x' \in F'$ we have $\int_X fd(x'm) = \int_X fd(x'm_1)$ for all f in C_{rc}. This implies that $x'm = x'm_1$ and hence $m = m_1$ since F is a locally convex Hausdorff space. Finally, let S be a bounded subset of E and $W = V_{m,S}$. Let $A = \{f \in C_{rc} : f(X) \subset S\}$. Then A is u-bounded and therefore $T(A)$ is weakly relatively compact. We will finish the proof of (4) by showing that E is contained in the weak closure of $T(A)$. Let $G_1, \cdots, G_n$ be a B-partition of X and $s_1, \cdots, s_n$ in S. Let $x'_1, \cdots, x'_N \in F'$. There exist $q \in J$ and $M > 0$ such that $x'_i \in MB_q^0$. Let $p \in I$ be such that $\|T\|_{p,q} < \infty$. Since S is bounded, $d = \sup\{p(s): s \in E\} < \infty$. By the regularity of $(x'_j m)_p$ we can find zero sets $Z_1, \cdots, Z_n$ with $\Sigma_{i=1}^n (x'_j m)_p (G_i - Z_i) < \epsilon/2d$ (where $\epsilon > 0$ is arbitrary) for $j = 1, \cdots, N$. Next, again by regularity, we can find pairwise disjoint cozero sets $U_1, \cdots, U_n$ with $Z_i \subset U_i$ such that for each $j, 1 \leqq j \leqq N$, we have $\Sigma_{i=1}^n (x'_j m)_p (U_i - Z_i) < \epsilon/2d$. For each i between 1 and n we pick a function $h_i \in C^b$ mith $0 \leqq h_i \leqq 1$, such that $h_i = 1$ on Z_i and $h_i = 0$ on the complement of U_i. The function $h = \Sigma_1^n h_i s_i$ is in A and hence $T(h) \in T(A$. Moreover

$$\left| x'_j (T(h) - \sum m(G_i)s_i \right|$$

$$= \left| x'_j \left(\sum m(Z_i)s_i - \sum m(G_i)s_i + \sum \int_{U_i - Z_i} h_i s_i dm \right) \right| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This shows that $\Sigma m(G_i)s_i$ is in the weak closure of $T(A)$ and the proof of (4) is complete. Conversely, suppose that $m \in M(B, \mathscr{L}(E,F))$ satisfies (4). Let $G \in B$ and $f \in C_{rc}$. Denote by D_G the set of all $\alpha = \{G_1, \cdots, G_n; x_1, \cdots, x_n\}$ where $\{G_i\}$ is a B-partition of G and $x_i \in G_i$. For α, γ in D_G we write $\alpha \geqq \gamma$ if the B-partition of G for α is a refinement of the one in γ. Then D_G becomes a directed set.

For $\alpha = \{G_1, \cdots, G_n; x_1, \cdots, x_n\}$ in D_G we define $z_\alpha = \Sigma m(G_i)f(x_i)$. By (4) the net $\{z_\alpha\}$ is contained in a weakly compact set. Hence there exists a subnet which converges weakly to a vector z in F. But for each $x' \in F'$ we have $x'(z_\alpha) \to \int_G fd(x'm)$. Thus $x'(z) = \int_G fd(x'm)$ which shows that $\int_G fdm = z$. Define $T: C_{rc} \to F$, $T(f) = \int_X fdm$. Then T is u-continuous and weakly compact. For the continuity, let $q \in J$. Choose $p \in I$ such that $\|m\|_{p,q} < \infty$. If $x' \in B_q^0$ and $\|f\|_p \leqq 1$, we have $|x'(T(f))| = \left| \int_X fd(x'm) \leqq \|x'm\|_p \leqq \right.$ $\|m\|_{p,q}$. It follows that $\|T\|_{p,q} \leqq \|m\|_{p,q}$ and the continuity of T is established. To prove the weak compactness consider an arbitrary bounded set A in C_{rc} and let S denote the convex circled hull of $\cup \{f(X): f \in A\}$. Then S is bounded in E. Let $W = V_{m,S}$. Clearly W is convex and circled. By hypothesis W is also weakly relatively compact. It follows that the polar W^0 of W in F' is a $m(F',F)$ neighborhood of zero. We will show that $T'(W^0) \subset A^0$. Let $x' \in W^0$ and $f \in A$. If $G_1, \cdots, G_n$ is a B-partition of X and $x_i \in G_i$, then $|x'(\Sigma_1^n m(G_i)f(x_i)| \leqq 1$. This implies that $|x'(\int fdm)| \leqq 1$. Thus $|\langle T'x', f\rangle| = |\langle x', T(f)\rangle| \leqq 1$ which proves that $T'x' \in A^0$. Now the result follows Lemma 1.

By the preceding theorem, given a continuous weakly compact operator T from C_{rc} into F there exists $m \in M(B, \mathscr{L}(E,F))$ which represents T. Since the operator $\hat{T}: C(\beta X, E) \to F$, $\hat{T}(\hat{f}) = T(f)$, is also weakly compact and since the dual of $C(\beta X, E)$ (with the uniform topology) is $M_\tau(B_0(\beta X), E')$ we can find, using an argument analogous to that of Theorem 2, an $\hat{m} \in M_\tau(B_0(\beta X), \mathscr{L}(E,F))$ representing $\hat{m}$. The next theorem gives necessary and sufficient conditions on m and $\hat{m}$ so that T is β_1' continuous.

THEOREM 4. *Let T be a u-continuous and weakly compact operator from C_{rc} into F and let m and $\hat{m}$ be as above. The following are equivalent:*

(1) *T is β_1' continuous*

(2) *Given $q \in J$ there exists p in I with $\|T\|_{p,q} < \infty$ such that $m_{p,q}(Z_n) \to 0$ whenever $\{Z_n\}$ is a sequence of zero sets decreasing to the empty set.*

(3) *Given $q \in J$ there exists $p \in I$ with $\|T\|_{p,q} < \infty$ such that for each Z in Ω_1 we have $\inf\{\hat{m}_{p,q}(V)$: V cozero set, $V \supset Z\} = 0$.*

Proof. $(1 \Rightarrow 3)$. Since T is β_1'-continuous there exists $p \in I$ such that $T^{-1}(B_q)$ is a $\beta_{1,P}$ neighborhood of zero. Let now Z be in Ω_1. Then there exists $g \in C^b(X)$ with $\hat{g}(Z) = 0$ such that $W = \{f \in C_{rc} : \|gf\|_p \leqq 1\} \subset T^{-1}(B_q)$. Let $\epsilon > 0$ be given and set $V = \{x \in \beta X : |\hat{g}(x)| < \epsilon\}$. Then V is a cozero set containing Z. For a given $\delta > 0$ there exist $x' \in B_q^0$, a $B_0(\beta X)$ partition $G_1, \cdots, G_N$ of V and s_i in E with $p(s_i) \leqq 1$ such that $|\Sigma\, x'\hat{m}(G_i)s_i| > \hat{m}_{p,q}(V) - \delta$. Next we choose compact sets $Z_i \subset G_i$ and pairwise disjoint open sets 0_i with $Z_i \subset 0_i \subset V$ such that $|\Sigma\, x'\hat{m}(G_i)s_i - \Sigma\, x'\hat{m}(Z_i)s_i| < \delta$ and $\Sigma\,(x'\hat{m})_p\,(0_i - Z_i) < \delta$. For each $i, 1 \leqq i \leqq n$, we pick $h_i \in C^b(X)$ with $0 \leqq h_i \leqq 1$, $\hat{h}_i = 1$ on Z_i and $\hat{h}_i = 0$ in the complement of 0_i in βX. Set $h = \Sigma\, h_i s_i$. Then $1/\epsilon h \in W$ and so $q(Th)) \leqq \epsilon$. Thus

$$\hat{m}_{p,q}(V) < \delta + \Sigma\,|x'\hat{m}(G_i)s_i| \leqq \delta + \delta + \Sigma \left| \int_{0_i - Z_i} \hat{h}_i s_i d(x'\hat{m}) \right|$$
$$+ |x'(T(h))| \leqq 3\delta + \epsilon.$$

Since $\delta > 0$ was arbitrary we conclude that $\hat{m}_{p,q}(V) \leqq \epsilon$. $(3 \Rightarrow 2)$. Let $x' \in F'$. If $x' \in MB_q^0$ for some $q \in J$ and if $p \in I$ is as in (2), then from the fact that $(x'\hat{m})_p(Z) = 0$ for all Z in Ω_1 and from $\int_X fd(x'm) = \int_{\beta X} \hat{f}d(x'\hat{m})$, which holds for all f in C_{rc}, it follows that $x'm$ is σ-additive and hence $(x'\hat{m})_p(A) = (x'm)_p(A \cap X)$ for each A in $B(\beta X)$. Let now $\{Z_n\}$ be a sequence of zero sets in X which decreases to the empty set. For each n there exists a zero set F_n in βX such that $F_n \cap X = Z_n$. Let $\epsilon > 0$ be given. By (3) there exists a cozero set V in

βX containing $\cap F_n$ such that $\hat{m}_{p,q}(V) < \epsilon$. Since $(\cap F_n) \cap (\beta x - V) = \emptyset$ there exists N such that $F_1 \cap \cdots \cap F_N \subset V$.

Now it follows that for $n \geqq N$ we have

$$m_{p,q}(Z_n) \leqq m_{p,q}(Z_N) = \hat{m}_{p,q}(F_1 \cap \cdots \cap F_N) < \epsilon.$$

$(2 \Rightarrow 1)$. Let $q \in J$ and choose $p \in I$ satisfying (2). For $x' \in B_q^0$ and $Z_n \downarrow \emptyset$ we have $(x'm)_p\,(Z_n) \leqq m_{p,q}(Z_n) \to 0$ which implies that $x'm$ is σ-additive and so $(x'\hat{m})_p\,(A) = (x'm)_p(A \cap X)$ for each A in $B(\beta X)$. Let Z be in Ω_1. There exists $h \geqq 0$ in C^b such that $Z = \{x \in \beta X: \hat{h}(x) = 0\}$. For each n set $F_n = \{x \in \beta X: \hat{h}(x) \leqq 1/n\}$. Then $Z_n = F_n \cap X$ is a zero set in X and $Z_n \downarrow \emptyset$. Given $r > 0$ there exists n such that $m_{p,q}(Z_n) < 1/2r$. Choose $g \in C^b$, $0 \leqq g \leqq 1$ with $\hat{g} = 0$ on Z and $\hat{g} = 1$ on the complement of V in βX, where $V = \{x \in \beta X: \hat{h}(x) < 1/n\}$. Let now $f \in C_{rc}$ with $\|f\|_p \leqq r$ and $\|fg\|_p \leqq \delta = 1/2\|m\|_{p,q}$. If

$$x' \in B_q^0,\ |x'\textstyle\int f dm| = |x'\int \hat{f} d\hat{m} \leqq \left| x' \displaystyle\int_V \hat{f} dm \right|$$

$$+ \left| x' \int_{\beta X - V} \hat{g}\hat{f} d\hat{m} \right| \leqq r \cdot 1/2r + \delta \|m\|_{p,q} \leqq 1.$$

This shows that $q(\int f dm) \leqq 1$. Thus $\{f \in C_{rc}: \|f\|_p \leqq r,\ \|fg\|_p \leqq \delta\} \subset T^{-1}(B_q)$ and $s_0\, T^{-1}(B_q)$ is a $\beta_{p,Z}$ neighborhood of zero. Since this is true for all Z in Ω_1 it follows that $T^{-1}(B_q)$ is a $\beta_{1,p}$ neighborhood of zero which proves that T is β_1' continuous.

We have an analogous theorem for β' with a similar proof.

THEOREM 5. *Let T, m and $\hat{m}$ be as in Theorem 3. The following are equivalent*:

(1) *T is β'-continuous*

(2) *Given $g \in J$ there exists $p \in I$, $\|T\|_{p,q} < \infty$ such that for each G in Ω we have* $\inf\{\hat{m}_{p,q}(V)$: *V cozero set, $G \subset V\} = 0$.*

(3) *Given $g \in J$ there exists $p \in I$ with $\|T\|_{p,q} < \infty$ that $m_{p,q}(Z_\alpha) \to 0$ for each net $\{Z_\alpha\}$ of zero sets in X which decreases to the empty set.*

THEOREM 6. *Suppose T is a linear operator from C_{rc} into F which is β_1-continuous and that every weakly closed bounded subset of F is weakly sequentially complete. Then there exists $m \in M(B, \mathscr{L}(E, F))$, with respect to which each f in C_{rc} is integrable, such that $T(f) = \int f dm$ for all f in C_{rc}. Moreover, if T is β_1' continuous, given $q \in J$ there exists $p \in I$ with $\|m\|_{p,q} < \infty$ such that $m_{p,q}(Z_n) \to 0$ whenever $\{Z_n\}$ is a sequence of zero sets which decreases to the empty set.*

Proof. Since T is β_1-continuous, T' maps F' into the space $M_\sigma(B, E') = (C_{rc}, \beta_1)'$. Let Z be a zero set in X. There exists $g \in C^b$ such that $Z = \{x : g(x) = 0\}$. For each n let

$$V_n = \{x \in X : |g(x)| < 1/n\}.$$

Choose f_n in $C^b, 0 \leqq f_n \leqq 1$ with $f_n = 1$ on Z and $f_n = 0$ on $X - V_n$. Then $f_n \to X_Z$ pointwise. An arbitrary element of $B(X)$ can be written as a finite disjoint union of sets of the form $Z - F$ where $F \subset Z$ and F, Z are zero sets. It follows that for G in $B(X)$ there exists a bounded sequence $\{f_n\}$ in C^b which converges pointwise to χ_G. For μ in $M_G(B, E')$ and $s \in E$ we have $\langle \mu, f_n s\rangle = \int f_n d(\mu s) \to \int \chi_G d(\mu s) = \langle \mu, \chi_G s\rangle$. Thus $f_n s \to \chi_G s$ in the $\sigma((C_{rc}, \beta_1)'', M_\sigma(B, E'))$ topology and hence $T''(f_n s) \to T''(\chi_G s)$ in the $\sigma(F'', F')$ sense. But $T''(f_n s) = T(f_n s)$ and the set $\{T(f_n s) : n = 1, 2, \cdots\}$ is $\sigma(F, F')$ bounded. Also the sequence $\{T''(f_n s)\}$ is weakly Cauchy. By hypothesis there exists $a \in F$ that $T''(f_n s) \to a$ in the $\sigma(F, F')$ topology. This implies that $T''(\chi_G s) = a \in F$. Define $m(G)s = T''(\chi_G s)$. It is easy to see that $m \in M(B, \mathscr{L}(E, F))$ and that $T(f) = \int f dm$ for all f in C_{rc}. Assume next that T is β_1'-continuous. Let $\hat{T} : C(\beta X, E) \to F$, $\hat{T}(\hat{f}) = T(f)$. As in the case of T we can find $\bar{m} \in M(B(\beta X), \mathscr{L}(E, F))$ such that $\hat{T}(\hat{f}) = \int \hat{f} d\bar{m}$ for all f in C_{rc}. Now to complete the proof we use an argument similar to that of Theorem 4.

If $m \in M_\sigma(Ba, \mathscr{L}(E, F))$, then the restriction of m to B is in $M_\sigma(B, \mathscr{L}(E, F))$. The following result is a partial converse.

THEOREM 7. *Let $m \in M_\sigma(B, \mathscr{L}(E, F))$ be such that for any $s \in E$ the set $(ms)(B)$ is weakly relatively compact in F. Then there exists a unique $\bar{m}$ in $M_\sigma(Ba, \mathscr{L}(E, F))$ whose restriction to B coincides with m. Moreover, if $\|m\|_{p,q} < \infty$, then $\|\bar{m}\|_{p,q} = \|m\|_{p,q}$.*

Proof. Let $G \in Ba$ and set $W = \{Z : Z \subset G, Z \text{ a zero set}\}$. If we order W by inclusion, it becomes a directed set. For $s \in E$, $\{m(Z)s : Z \in W\}$ is a net in F. By hypothesis there exists a subnet which converges weakly to some a in F. For $x' \in F'$, $x'm$ is σ-additive and thus has a unique extension to a member $\mu_{x'}$ of $M_\sigma(Ba, E')$. Moreover $x'm(Z)s \to \mu_{x'}(G)s$. Thus $x'(a) = \mu_{x'}(G)s$. Define $\bar{m}(G)s = x'(a)$. Then $x'\bar{m} = \mu_{x'} \in M_\sigma(Ba, E')$. Furthermore $\bar{m}(G) \in \mathscr{L}(E, F)$. Indeed if $\|m\|_{p,q} < \infty$, then for $x' \in B_q^0$ we have $|x'\bar{m}(G)s| = |\mu_{x'}(G)s| \leqq p(s)\|\mu_{x'}\|_p = p(s)\|x'm\|_p \leqq p(s)\|m\|_{p,q}$. Thus $q(\bar{m}(G)s) \leqq p(s)\|m\|_{p,q}$ which proves that $\bar{m}(G) \in \mathscr{L}(E, F)$. Also $\|x'\bar{m}\|_p = \|\mu_{x'}\| = \|x'm\|_p$ implies that $\|\bar{m}\|_{p,q} = \|m\|_{p,q}$. Finally suppose λ is another extension. Then for each x' in F' both $x'\lambda$ and $x'\bar{m}$ are extensions of $x'm$ and hence they are equal. This implies that $\lambda = \bar{m}$.

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