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CHARACTERIZATIONS OF INFINITE-DIMENSIONAL AND NONREFLEXIVE SPACES

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Infinite-dimensional, resp. **nonreflexive spaces are characterized in terms of subsets having a finite visibility property without being starshaped.**

1. Introduction. A well-known result of Smulian [4] states that every nonreflexive normed linear space contains a decreasing sequence of nonempty closed and bounded convex sets whose intersection is empty. This result was used by V. L. Klee [1] to show that a normed linear space is nonreflexive if, and only if, it contains a decreasing sequence of closed and bounded starshaped sets whose intersection is empty. Also proved by Klee [2] is the following. Theorem [Klee]. Every infinite dimensional normed linear space contains a decreasing sequence of unbounded but linearly bounded closed convex sets whose intersection is empty. Here, a set is called linearly bounded if each straight line intersects it in a bounded set.

In the present paper other characterizations of infinite-dimensional, and of nonreflexive spaces are given which are similar in spirit and not unrelated to those mentioned above. To this end use is made of the notion of finite visibility. A set S is said to have the finite visibility property, f.v.p. for short, if for any finite $F \subset S$ there is an $x \in S$ such that the line segment $[x, y]$ is contained in S for all y in F. As customary a set S is called starshaped if an $s \in S$ exists such that the above condition is satisfied with s replacing x and S replacing F. A well-known theorem of Krasnoselski [3] implies that in a finite dimensional normed linear space X if S is closed and bounded and has f.v.p. then S is starshaped. (In fact, if $\dim X = n$, and card $S \geqq n + 1$, then the above mentioned theorem holds if the hypothesis is satisfied for all F with card $F = n + 1$.) A previous version of this paper was mainly concerned with showing that in some Banach spaces a weakly closed bounded set may have f.v.p. without being starshaped. The broader scope of the present paper is due to suggestions made by Professor Klee in a personal communication, in which he conjectured the two theorems of this paper and directed us to relevant passages in some of his works. It is indeed a pleasure to acknowledge his help.

2. Preliminary results.

LEMMA 1. *A compact subset S of a Hausdorff linear topological* *space X is starshaped if it has the finite visibility property.*

Proof. For $x \in S$, let $S_x = \{y \in S: [x, y] \subset S\}$, a closed set. The family $\{S_x: x \in S\}$ has the finite intersection property by f.v.p. so $\bigcap S_x \neq \emptyset$ by compactness, and S is starshaped.

LEMMA 2. *Let E be a closed subspace of a normed linear space X, S a closed convex linearly bounded set in E and x a point in $X \sim E$. Then $K = \mathrm{co}\,\{\{x\} \cup S\}$ is closed.*

Proof. Let $y \in \bar{K}$, $y \neq x$, and let F be the subspace spanned by x and S. Clearly $y \in F$. Thus if R is the ray emanating from x, through y, i.e. $R = \{z \in X: z = x + \alpha(y - x), \alpha \geqq 0\}$, then R is contained in F. Moreover, R cannot be parallel to E, for if parallel, then with $w \in S$, $R' = \{z \in X: z = w + \alpha(y - x), \alpha \geqq 0\}$ is contained in E and by linear boundedness there is a $w' \in R' \sim S$. But then w' and S can be separated by a hyperplane $H \subset E$, relative to E. The subspace spanned by H and x clearly determines a closed halfspace of F which contains $\{\{x\} \cup S\}$ and is disjoint from y, leading to a contradiction, since $y \in \bar{K}$. Suppose now that u is the point of intersection of R and E. It suffices to show that $u \in S$. If not, then there is an open ball B about u which is disjoint from S and $\mathrm{co}\,\{\{x\} \cup \mathrm{B}\}$ is a neighborhood of u which contains no point of the form $\lambda x + (1 - \lambda)s$ for any λ, $0 \leqq \lambda < 1$ and $s \in S$. This is impossible since $y \in \bar{K}$. Hence $y \in K$ and $K = \bar{K}$ as claimed.

LEMMA 3. *Let x be a normed linear space, E a closed subspace of X and l a line skew to E,* i.e. *l neither intersects E nor is parallel to any line of E. Let $\{C_k: k = 1, 2, \cdots\}$ be a decreasing sequence of closed convex subsets of E and $\{p_k: k = 1, 2, \cdots\}$ a sequence on l converging to some p_0. Let $K_i = \mathrm{co}\,\{\{p_i\} \cup C_i\}$ for $i \geqq 1$ and $K_0 = \mathrm{co}\,\{\{p_0\} \cup C_1\}$.*

Then $S = \bigcup\{K_i: i = 0, 1, \cdots\}$ is weakly closed. If, in addition, C_1 is linearly bounded then so is S.

Proof. To prove that S is weakly closed let $x \in X \sim S$. Then $x \notin K_0$, which is closed by Lemma 2, and convex. Thus there is a hyperplane H such that $x \in H^+$ and $K_0 \subset H^-$ where H^+ and H^- are open halfspaces determined by H. Let n_0 be such that $p_n \in H^-$ whenever $n > n_0$. Then, for such n, $K_n \subset H^-$ since $\{\{p_n\} \cup C_n\} \subset H^-$. On the other hand, as $\bigcup\{K_i: i \leqq n_0\}$ is weakly closed there is a weak neighborhood W of x which is disjoint from it. It follows that $W \cap H^+$ is a weak neighborhood of x which is disjoint from S. Hence S is weakly closed. To prove linear boundedness observe first that, as can be readily verified, in finite dimensional spaces boundedness and linear boundedness are equivalent for closed convex sets. If now l_1 is a line in X let L be the subspace spanned by $l \cup l_1$. Then $L \cap C_1$ is bounded and closed and $l_1 \cap S$ is contained in the compact set

$$\mathrm{co}\,\{\{p_k: k = 0, 1, \cdots\} \cup (C_1 \cap L)\}$$

and therefore bounded. Hence S is linearly bounded, as asserted.

LEMMA 4. *Let X be a linear space, E a subspace of X and l a line in X which is skew to E. If $p, q \in l$, $p \neq q$, and A, B are convex subsets of E then*

$$\mathrm{co}\,\{\{p\} \cup A\} \cap \mathrm{co}\,\{\{q\} \cup B\} = A \cap B\,.$$

Proof. Let $x \in \mathrm{co}\,\{\{p\} \cup A\} \cap \mathrm{co}\,\{\{q\} \cup B\}$. It suffices to show that $x \in A \cap B$. If this were not the case then $x \in [p, a) \cap [q, b)$ for some $a \in A$ and $b \in B$, with $a \neq b$. But then a, b, p, q would have to be coplanar against the assumption that l is skew to E.

LEMMA 5. *Let X be a linear space, E a subspace of X and l a line in X which is skew to E. Suppose p_i: $i = 1, 2, \cdots$ is a sequence of distinct points on l. Let $C_i \subset E$ be convex, $K_i = \mathrm{co}\,\{\{p_i\} \cup C_i\} i = 1, 2, \cdots$ and $S = \bigcup \{K_i: i = 1, 2, \cdots\}$. Then S is starshaped if, and only if, $\bigcap \{C_i: i = 1, 2, \cdots\} \neq \emptyset$ and S has f.v.p. if, and only if, $\{C_i: i = 1, 2, \cdots\}$ has the finite intersection property.*

Proof. If l' is a line such that $l' \cap (K_j \sim C_j) \neq \emptyset$ then $\mathrm{card}\,(l' \cap K_i) \leqq 1$ for any $i \neq j$. Indeed, if for some $i \neq j$ $l' \cap K_i$ contains two or more points then l' is contained in L_i, the linear span of K_i; but then $l' \cap (K_j \sim C_j) = \emptyset$ since $L_i \cap K_j \subset C_j$ by the preceding lemma. Hence $[u, p_i]$, with $u \in K_j \sim C_j$ and $i \neq j$, is not contained in S as $\mathrm{card}\,([u, p_i] \cap S) \leqq \aleph_0$. Thus $\bigcup \{[u, p_m] \subset S: m \in M\}$, where M is a set of two or more positive integers, implies that $u \in \bigcap \{C_m: m \in M\}$. It follows that for S to be starshaped it is necessary that $\bigcap \{C_i: i = 1, 2, \cdots\} \neq \emptyset$ and for it to have f.v.p. $\{C_i: i = 1, 2, \cdots\}$ has to have the finite intersection property.

For the converse note that $u \in \bigcap \{C_i: i = 1, 2, \cdots\}$ implies $S_u = S$ and if $F \subset S$ is finite then, for N sufficiently large, $F \subset \bigcup \{K_i: i = 1, 2, \cdots\}$ and this last set is contained in S_u for any $u \in \bigcap \{C_i: i = 1, 2, \cdots, N\}$.

3. Main results.

THEOREM 1. *A normed linear space is infinite-dimensional if,* *and only if, it contains a linearly bounded, weakly closed subset S which has the finite visibility property but fails to be starshaped.*

Proof. If X contains a set S with the stated properties then by the Krasnoselski theorem [3] X must be infinite-dimensional.

Assume now that X is infinite-dimensional and E is a closed subspace of X of codimension 2. By the theorem of Klee quoted in the introduction, E contains a decreasing sequence $\{C_k: k = 1, 2, \cdots\}$ of nonempty, closed, linearly bounded subsets whose intersection is empty. Let l be a line which is skew to E and $\{p_k: k = 1, 2, \cdots\}$ a sequence of distinct points on l converging to $p_0 \in l$. Let K_i, $i = 0, 1, \cdots$ and S be as in Lemma 3. Then S is weakly closed and linearly bounded by that lemma. By Lemma 4 S has f.v.p. but fails to be starshaped.

THEOREM 2. *A normed linear space X is nonreflexive if, and only if, it contains a set S which is bounded, weakly closed, has the finite visibility property but fails to be starshaped.*

Proof. If X contains a set S with the stated properties then, by Lemma 1, it fails to be reflexive.

Assume now that X is nonreflexive and, as in the construction of the proof of Theorem 1, let E be a closed subspace of X of codimension 2 and l a line skew to E. Let $\{p_k\}$ be a sequence of distinct points on l converging to $p_0 \in l$. By the Smulian theorem [3] there exists a decreasing sequence $\{C_k: k = 1, 2, \cdots\}$ of nonempty, closed and bounded convex sets in E whose intersection is empty. Let K_i, $i = 0, 1, \cdots$ and S be defined as in the proof of Theorem 1. Then the arguments used there apply again to the effect that S is weakly closed, bounded, with f.v.p. but not starshaped.

4. An example in l_1. The following is an example of a concrete subset of l_1 having all the properties of the set S of Theorem 2. Let S consist of all $x = (x_1, x_2, \cdots, x_n, \cdots) \in l_1$ such that

(i) $x_n \geqq 0$ for $n = 1, 2, \cdots$;

(ii) $\|x\| = 1$;

(iii) if $x_{2n} \neq 0$ then $x_k = 0$ for $1 \leqq k < 2n$.

To show that S has the finite visibility property let $F \subset S$ be finite and N an odd integer which is larger than the index of the first positive coordinate of each member of F. If $e_N \in S$ has 1 for its Nth coordinate then clearly $[u, e_N] \subset S$ for all $u \in F$.

To prove that S is weakly closed let $y = (y_1, y_2, \cdots, y_n, \cdots) \in l_1 \sim S$ and assume, as we may, that $\|y\| = 1$. Since $y \notin S$, there must be positive integers n, k such that $k < 2n$ and $y_k > 0$ and $y_{2n} > 0$. If $u = (u_1, \cdots, u_k, \cdots)$, $v = (v_1, \cdots, v_{2n}, \cdots) \in l_\infty$ are such that $u_k = v_{2n} = 1$ and all other coordinates $= 0$ then

$$W = \{z \in l_1 : u(z) > 0 \quad \text{and} \quad v(z) > 0\}$$

is a weak neighborhood of y which is disjoint from S. Since boundedness of S is obvious it remains to show that S is not starshaped. If now $u = (u_1, u_2, \cdots, u_k, \cdots) \in S$ and $u_k \neq 0$ then for $x = (x_1, \cdots, x_n, \cdots) \in S$ with $s_{2k} = 1$ we have $[u, x] \notin S$.

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