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A MEASURE OF CONVEXITY FOR COMPACT SETS

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For a subset M of d-dimensional real vector space R^d let

$$c(M) = \inf\{\lambda \geqq 0 | M + \lambda \operatorname{conv} M \text{ is convex}),$$

where $\operatorname{conv} M$ is the convex hull of M and $+$ denotes vector addition of sets. Among the compact subsets of R^d, the convex sets are characterized by the equality $c(M) = 0$. It is proved that $c(M) \leqq d$ for arbitrary subsets of R^d, with equality if and only if M consists of $d + 1$ affinely independent points. If M is either unbounded or connected, then $c(M) \leqq d - 1$; the bound $d - 1$ is best possible in either case.

For subsets M_1, M_2 of d-dimensional real vector space R^d, the Minkowski (or vector) sum is defined by

$$M_1 + M_2 = \{x_1 + x_2 | x_i \in M_i, i = 1, 2\}.$$

Minkowski addition plays an essential role in the theory of convex bodies, due to the fact that the sum of convex sets is always convex. On the other hand, the sum $M_1 + M_2$ may be convex without M_1, M_2 being convex; for instance, if M is the boundary of a convex body K, then $M + M = K + K$. Moreover, it is easy to see that the sum of an arbitrary subset $M \subset R^d$ and a suitable multiple of its convex hull is always convex. This leads us to the definition below. In the following, the abbreviations cl, int, rel int, bd, aff, conv, dim denote, respectively, closure, interior, relative interior, boundary, affine hull, convex hull, dimension.

For a subset $M \subset R^d$, define

$$c(M) = \inf\{\lambda \geqq 0 | M + \lambda \operatorname{conv} M \text{ is convex}\}$$

(here $\lambda A = \{\lambda x | x \in A\}$). The empty set $\emptyset$ is considered as convex, hence $c(\emptyset) = 0$. Clearly $M + \lambda \operatorname{conv} M$ is convex for all $\lambda > c(M)$. If we write

$$M_\lambda = (1 + \lambda)^{-1}(M + \lambda \operatorname{conv} M),$$

then $M_\lambda \subseteq \operatorname{conv} M \subseteq \operatorname{conv} M_\lambda$, hence we may also write

$$c(M) = \inf\{\lambda \geqq 0 \mid M_\lambda = \operatorname{conv} M\}.$$

Our main object is to prove the following theorem.

THEOREM. *For every set $M \subset R^d$,*

$$0 \leqq c(M) \leqq d.$$

The equality sign on the left holds if M is convex, and for bounded M it holds only if cl *M is convex. The equality sign on the right holds if and only if M consists of $d+1$ affinely independent points.*

Restricted to the family of compact sets, the functional c might, therefore, serve as a "measure of convexity" (or rather "measure of non-convexity", since it is minimal, instead of maximal, for convex sets; but this is immaterial). The term "measure of convexity" is chosen in reminiscence of the "measures of symmetry" for convex bodies, as defined by Grünbaum [**2**, p. 234]. The functional c has certain properties analogous to those which Grünbaum proposes to consider for measures of symmetry, for instance

$$c(TM) = c(M)$$

for every $M \subset R^d$ and every nonsingular affine transformation T of R^d. Furthermore, it can be shown that (compare Grünbaum [**2**, p. 243])

$$c(M_1 + M_2) \leqq \max\{c(M_1), c(M_2)\}$$

for $M_1, M_2 \subset R^d$. However, in contrast to the situation studied by Grünbaum, the following should be pointed out. If $\|\,\|$ is a Euclidean norm on R^d, and if the set of nonempty, compact subsets of R^d is endowed with the Hausdorff metric defined by

$$\rho(M_1, M_2) = \max\left\{\sup_{x\in M_1}\inf_{y\in M_2}\|x-y\|,\ \sup_{x\in M_2}\inf_{y\in M_1}\|x-y\|\right\},$$

then c is not continuous, even if restricted to the compact sets with interior points. For instance, take a triangle $T \subset R^2$ and replace one of its edges by the two segments which join the endpoints of the edge to an interior point of T. The resulting nonconvex quadrangle Q, which can be chosen arbitrarily close to T in the Hausdorff metric, has $c(Q) = 1$, whereas $c(T) = 0$.

Let us now proceed to the proof of the theorem. We split it into a series of simple propositions, thereby proving some additional results.

First let M consist of $d+1$ affinely independent points; without loss of generality we may assume that $M = \{e_1, \cdots, e_{d+1}\}$, where

$$\langle e_i, e_j \rangle = \begin{cases} 1, & \text{if} \quad i = j, \\ -1/d, & \text{if} \quad i \neq j; \end{cases}$$

here $\langle\, , \rangle$ denotes a scalar product on R^d. Writing $T_i = \{x \in \operatorname{conv} M \mid \langle x, e_i \rangle \geqq 0\}$ for $i = 1, \cdots, d+1$ we easily see that

$$e_i + d \operatorname{conv} M = (1+d)T_i,$$

hence

$$M + d \operatorname{conv} M = \bigcup_{i=1}^{d+1} (1+d)T_i = (1+d) \operatorname{conv} M,$$

which shows that $c(M) \leqq d$. If, on the other hand, $M + \lambda \operatorname{conv} M$ is convex for some $\lambda \geqq 0$, then $0 \in M + \lambda \operatorname{conv} M$, hence $0 \in e_i + \lambda \operatorname{conv} M$ for suitable $i \in \{1, \cdots, d+1\}$, from which we deduce that $\lambda \geqq d$. This shows that $c(M) = d$.

(1) PROPOSITION. *$M_d = \operatorname{conv} M$ for every set $M \subset R^d$, hence $c(M) \leqq d$.*

The following simple proof has been communicated to me by Dr. Wolfgang Weil. It is clear that $M_d \subseteq \operatorname{conv} M$. *Let $x \in (1+d)\operatorname{conv} M$.* By Carathéodory's theorem, there exists a set $Y \subseteq M$ of affinely independent points such that $x \in (1+d)\operatorname{conv} Y$. Since $c(Y) = \dim \operatorname{aff} Y \leqq d$, we have $Y + d \operatorname{conv} Y = (1+d)\operatorname{conv} Y$, hence

$$x \in (1+d)\operatorname{conv} Y = Y + d \operatorname{conv} Y \subseteq M + d \operatorname{conv} M;$$

thus conv $M \subseteq M_d$, which proves (1).

(2) PROPOSITION. *$M_{d-1} \supseteq (\operatorname{bd} \operatorname{conv} M) \cap (\operatorname{conv} M)$ for $M \subset R^d$.*

Proof. Let $x \in (\operatorname{bd} \operatorname{conv} M) \cap (\operatorname{conv} M)$. The point x lies in a supporting hyperplane H of the convex set conv M, hence

$$x \in H \cap \operatorname{conv} M = \operatorname{conv}(H \cap M) = (H \cap M)_{d-1} \subseteq M_{d-1},$$

where we have applied (1) to a suitable R^{d-1}.

(3) PROPOSITION. cl $M_{d-1} \supseteq$ bd conv M *for* $M \subset R^d$.

Proof. Let $x \in$ bd conv M. Because of (2) we may assume that $x \notin$ conv M. Let U be a convex neighborhood of x; then U contains a point $y \in$ conv M. By Carathéodory's theorem, y is contained in the relative interior of a simplex S whose vertices belong to M. If dim $S < d$, put $H =$ aff S and $z = y$; if dim $S = d$, let $z \in \operatorname{conv}\{x, y\} \cap$ bd S (z exists because of $x \notin S$) and let H be the affine hull of a proper face of S containing z. Then $z \in \operatorname{conv}(H \cap M) = (H \cap M)_{d-1} \subseteq M_{d-1}$. Since $z \in U$ and U was arbitrary, we arrive at $x \in$ cl M_{d-1}, which proves (3).

Clearly the index $d - 1$ in Proposition (2) or (3) cannot be replaced by a smaller one, as is shown by a set consisting of $d + 1$ affinely independent points.

(4) PROPOSITION. *If* $M \subset R^d$ *and* conv $M \subseteq$ cl M_λ *for some* $\lambda \geqq d - 1$, *then* $c(M) \leqq \lambda$.

Proof. Let $\epsilon > 0$. Let $x \in \operatorname{conv} M$. If $x \in$ bd conv M, then (2) shows that $x \in M_{d-1} \subseteq M_{\lambda+\epsilon}$. Suppose, therefore, that $x \in$ int conv M. The set $x + \epsilon(1+\lambda)^{-1}(-\operatorname{conv} M + x)$ is a neighborhood of x, hence, because of $x \in \operatorname{cl} M_\lambda$, it contains a point $y \in M_\lambda$. But then we have

$$x \in (1+\lambda)(1+\lambda+\epsilon)^{-1}y + \epsilon(1+\lambda+\epsilon)^{-1}\operatorname{conv} M$$

$$\subseteq (1+\lambda)(1+\lambda+\epsilon)^{-1}(1+\lambda)^{-1}(M + \lambda \operatorname{conv} M) + \epsilon(1+\lambda+\epsilon)^{-1}\operatorname{conv} M$$

$$= M_{\lambda+\epsilon}.$$

We have proved that $\operatorname{conv} M \subseteq M_{\lambda+\epsilon}$, hence $c(M) \leqq \lambda + \epsilon$. As $\epsilon > 0$ was arbitrary, the assertion (4) follows.

(5) PROPOSITION. $c(M) \leqq d - 1$ *for every unbounded set* $M \subset R^d$.

Proof. Let $M \subset R^d$ be unbounded. We have to distinguish two cases:

First case. conv M contains a line L. Without loss of generality we may assume that $0 \in L$. Let E be a subspace of R^d complementary to L, and let π denote projection on to E in the direction of L. Then (e.g., Grünbaum [**3**, p. 24])

$$\operatorname{cl}\operatorname{conv} M = (\pi \operatorname{cl}\operatorname{conv} M) + L.$$

Let $\lambda > d - 1$, and let $x \in \operatorname{conv} M$. Then

$$\pi x \in \pi \operatorname{conv} M = \operatorname{conv} \pi M = (\pi M)_\lambda$$

by (1) (applied to E), hence there exists a point $m \in M$ such that

$$(1+\lambda)\pi x \in \pi m + \lambda \operatorname{conv} \pi M.$$

This yields

$$\begin{aligned}(1+\lambda)x &\in (1+\lambda)(\pi x + L) \subseteq \pi m + \lambda \operatorname{conv} \pi M + L\\ &= m + \lambda \operatorname{conv} \pi M + L \qquad (\text{since } \pi m + L = m + L)\\ &\subseteq \operatorname{cl} M + \lambda(\pi \operatorname{cl} \operatorname{conv} M + L)\\ &= \operatorname{cl} M + \lambda \operatorname{cl} \operatorname{conv} M \subseteq (1+\lambda) \operatorname{cl} M_\lambda.\end{aligned}$$

We have proved that $\operatorname{conv} M \subseteq \operatorname{cl} M_\lambda$, hence (4) implies $c(M) \leqq \lambda$. As $\lambda > d-1$ was arbitrary, we deduce $c(M) \leqq d-1$.

Second case. conv M does not contain a line. Let $\lambda > d-1$, and let $x \in \operatorname{conv} M$. If $x \in \operatorname{bd} \operatorname{conv} M$, then $x \in \operatorname{cl} M_{d-1}$ by (3). Suppose, therefore, that $x \in \operatorname{int} \operatorname{conv} M$. Since conv M is unbounded, there exists a direction of infinity, that is, a vector $u \neq 0$ such that $z + \alpha u \in \operatorname{cl} \operatorname{conv} M$ for all $z \in \operatorname{cl} \operatorname{conv} M$ and all $\alpha \geqq 0$ (e.g., Grünbaum [**3**, p. 23]). Since conv M does not contain a line, the halfline $\{x - \alpha u \mid \alpha \geqq 0\}$ contains a (unique) point $y \in \operatorname{bd} \operatorname{conv} M$. By (3) we have $y \in \operatorname{cl} M_{d-1}$, hence any given neighborhood U of y contains a point $z \in M_{d-1}$. The halfline $L_z = \{z + \alpha u \mid \alpha \geqq 0\}$ is contained in cl conv M, hence

$$\begin{aligned}L_z &\subset z + (\lambda - d + 1)(1+\lambda)^{-1}(\operatorname{cl} \operatorname{conv} M - z)\\ &\subseteq \operatorname{cl}[d(1+\lambda)^{-1} M_{d-1} + (\lambda - d + 1)(1+\lambda)^{-1} \operatorname{conv} M]\\ &= \operatorname{cl} M_\lambda.\end{aligned}$$

Since the neighborhood U may be chosen arbitrarily small, we see that $x \in \operatorname{cl} M_\lambda$. We have proved that $\operatorname{conv} M \subseteq \operatorname{cl} M_\lambda$, hence (4) yields $c(M) \leqq \lambda$. As $\lambda > d-1$ was arbitrary, Proposition (5) is proved.

Clearly, the bound $d-1$ in (5) cannot be replaced by a smaller number: If M consists of d parallel lines, or halflines, and aff $M = R^d$, then $c(M) = d-1$.

(6) PROPOSITION. *For bounded $M \subset R^d$, $c(\operatorname{cl} M) \leqq c(M)$.*

Proof. It is well known that $\operatorname{cl} A + \operatorname{cl} B \subseteq \operatorname{cl}(A+B)$ and $\operatorname{conv} \operatorname{cl} A \subseteq \operatorname{cl} \operatorname{conv} A$ for arbitrary $A, B \subset R^d$, and that these relations hold with the equality sign if A is bounded. We deduce that $(\mathrm{cl}\, M)_\lambda \subseteq \mathrm{cl}\, M_\lambda$ in general, and that $(\mathrm{cl}\, M)_\lambda = \mathrm{cl}\, M_\lambda$ for bounded M. Now if $\lambda > c(M)$, then $M_\lambda = \mathrm{conv}\, M$ and hence $(\mathrm{cl}\, M)_\lambda = \mathrm{cl}\, M_\lambda = \mathrm{cl}\,\mathrm{conv}\, M = \mathrm{conv}\,\mathrm{cl}\, M$, so that $c(\mathrm{cl}\, M) \leqq \lambda$. The assertion follows.

The example of a triangle from which the relative interior of an edge has been omitted, shows that $c(\mathrm{cl}\, M) < c(M)$ is possible. However, the following holds true.

(7) PROPOSITION. *If* $c(M) > d-1$, *then* $c(\mathrm{cl}\, M) = c(M)$.

Proof. Let $M \subset R^d$ be a set with $c(M) > d-1$. If $c(\mathrm{cl}\, M) < c(M)$, we may choose $\lambda > d-1$ with $c(\mathrm{cl}\, M) < \lambda < c(M)$. Then we have

$$\mathrm{conv}\, M \subseteq \mathrm{conv}\,\mathrm{cl}\, M = (\mathrm{cl}\, M)_\lambda \subseteq \mathrm{cl}\, M_\lambda,$$

and (4) gives $c(M) \leqq \lambda$, a contradiction.

(8) PROPOSITION. *If* $M \subset R^d$ *is bounded and* $c(M) = 0$, *then* $\mathrm{cl}\, M$ *is convex.*

Proof. Let $c(M) = 0$. First suppose that M is compact. Let $\lambda > 0$, and let $x \in \mathrm{conv}\, M$. Then $x \in M_\lambda$, hence $x = (1+\lambda)^{-1}(m_\lambda + \lambda y)$ with suitable $m_\lambda \in M$ and $y \in \mathrm{conv}\, M$, which implies

$$m_\lambda \in x + \lambda(-\mathrm{conv}\, M + x).$$

Since $\lambda > 0$ may be chosen arbitrarily small and since $\mathrm{conv}\, M$ is bounded, we see that $x \in \mathrm{cl}\, M$; hence $M = \mathrm{conv}\, M$. If M is bounded, but not necessarily closed, we have $c(\mathrm{cl}\, M) \leqq c(M) = 0$, hence $\mathrm{cl}\, M$ is convex.

It is now clear that in order to complete the proof of the theorem, it only remains to prove the following.

(9) PROPOSITION. *If* $M \subset R^d$ *is compact and* $c(M) = d$, *then* M *consists of* $d+1$ *affinely independent points.*

Proof. Let M be compact and such that $c(M) = d$. By (1) (applied to R^{d-1}) M cannot be contained in a hyperplane, hence $\mathrm{conv}\, M$ has interior points. Write

$$R_\lambda = \mathrm{conv}\, M \setminus M_\lambda.$$

We assert that

(10) $$\operatorname{cl} R_\lambda \subset \operatorname{int} \operatorname{conv} M \qquad \text{for} \quad \lambda > d-1.$$

For the proof let $x \in \operatorname{bd} \operatorname{conv} M$. By (2) we have $x \in M_{d-1}$. If $\alpha = (\lambda - d + 1)(1+\lambda)^{-1}$, then

$$x + \alpha(\operatorname{conv} M - x) \subseteq (1-\alpha)M_{d-1} + \alpha \operatorname{conv} M = M_\lambda,$$

hence

$$R_\lambda \subseteq \operatorname{conv} M \setminus \bigcup_{x \in \operatorname{bd} \operatorname{conv} M} (x + \alpha(\operatorname{conv} M - x)]$$

which proves (10) because of $\alpha > 0$.

If $\mu > \lambda$, then

$$\begin{aligned} M_\mu &= (1+\mu)^{-1}M + (\mu-\lambda)(1+\mu)^{-1}(1+\lambda)^{-1}\operatorname{conv} M + \lambda(1+\lambda)^{-1}\operatorname{conv} M \\ &\supseteq (+\mu)^{-1}M + (\mu-\lambda)(1+\mu)^{-1}(1+\lambda)^{-1}M + \lambda(1+\lambda)^{-1}\operatorname{conv} M \\ &= M_\lambda; \end{aligned}$$

hence $\lambda < \mu$ implies $R_\lambda \supseteq R_\mu$. Since the sets cl R_λ are compact and nonempty for $\lambda < d$, there exists a point

$$z \in \bigcap_{0<\lambda<d} \operatorname{cl} \mathrm{R}_\lambda.$$

By (10), $z \in \operatorname{int} \operatorname{conv} M$; but

(11) $$z \notin \operatorname{int}(1+d)^{-1}(m + d \operatorname{conv} M) \qquad \text{for } m \in M,$$

since otherwise for sufficiently large $\lambda < d$,

$$z \in \operatorname{int}(1+\lambda)^{-1}(m + \lambda \operatorname{conv} M) \subset \operatorname{int} M_\lambda,$$

which implies $z \notin \operatorname{cl} R_\lambda$, a contradiction.

By Carathéodory's theorem there exists an affinely independent set $Y \subseteq M$ such that $z \in \operatorname{conv} Y$, and some subset $Y' \subseteq Y$ satisfies $z \in$ rel int conv Y'. If $\dim \operatorname{aff} Y' < d$, then every point of rel int conv Y' is contained in rel int $(1+d)^{-1}(y + d \operatorname{conv} Y')$ for suitable $y \in Y'$. Since $z \in$ int conv M, we must have rel int conv Y, $\subset$int conv M, from which we deduce that $z \in \operatorname{int}\ (1+d)^{-1}(y + d \operatorname{conv} M)$, which contradicts (11). Hence Y is the set of vertices of a d-simplex S.

The only interior point of S which is not contained in

$$\bigcup_{y \in Y} \operatorname{int}(1+d)^{-1}(y+dS)$$

is the centroid of S. Hence, according to (11), z is the centroid of the simplex S. If M contains a point $m \notin Y$, we can replace an appropriate point of Y by m to obtain an affinely independent set $\bar{Y} \subseteq M$ which also satisfies $z \in \operatorname{conv} \bar{Y}$. By the argument above it follows that $\bar{Y}$ is the set of vertices of a d-simplex, of which z is the centroid. Since Y and $\bar{Y}$ differ in precisely one point, this is impossible. Hence $M = Y$, which completes the proof.

REMARK. There are many closed unbounded sets $M \subset R^d$ which satisfy $c(M) = 0$, but are not convex; for instance, in R^1 the set of all integer points, or in the plane a parabola. Hence it is only for compact sets M that $c(M)$ measures, in some sense, the nonconvexity of M.

REMARK. Since Proposition (1) is an immediate consequence of Carathéodory's theorem, it is clear that an improvement of this theorem for special sets may yield a corresponding improvement of (1). In particular, the following holds true (for the required variant of Carathéodory's theorem see Danzer, Grünbaum and Klee [**1**, p. 117] and the references given there).

(12) PROPOSITION. *Suppose the set $M \subset R^d$ is the union of at most d connected sets or is compact and the union of at most d convexly connected sets; then $c(M) \leqq d-1$.*

The bound $d-1$ cannot be replaced by a smaller number, even if one assumes that M is compact and connected: The union of all those edges of a d-simplex which contain a specified vertex of the simplex provides a counterexample.

REMARK. For convex bodies $K \subset R^d$ we could also study the derived functional b defined by $b(K) = c(\operatorname{bd} K)$. A moment's reflection shows that this is a well-known functional, namely the so-called Minkowski measure of symmetry (Grünbaum [**2**, p. 246]). Hence we have $d^{-1} \leqq c(\operatorname{bd} K) \leqq 1$, with equality on the left if and only if K is a simplex, and equality on the right if and only if K has a centre of symmetry.

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