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ON THE EXISTENCE OF STRONG LIFTINGS IN SECOND COUNTABLE TOPOLOGICAL SPACES

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Let X be a second countable topological space, $\mathfrak{A}$ a σ-field of subsets of X containing all open sets and μ a finite positive measure on $\mathfrak{A}$, such that $(X, \mathfrak{A}, \mu)$ is a complete measure space and $\mu(U) > 0$ for every nonempty open $U \subset X$.

Then there exists a lifting $\phi : \mathfrak{A} \to \mathfrak{A}$ which satisfies $U \subset \phi(U)$ for every open subset $U \subset X$.

Basic notations and definitions. Throughout this paper $\mathbf{N}$ denotes the nonnegative integers and $\mathbf{R}_+$ the nonnegative real numbers. Moreover

X is a second countable topological space ($\neq \emptyset$),
$\mathfrak{A}$ is a σ-field of subsets of X containing all open sets,
$\mu : \mathfrak{A} \to \mathbf{R}_+$ is a countable additive measure, satisfying $\mu(U) > 0$ for every nonempty open subset $U \subset X$.

For $A, B \in \mathfrak{A}$ we denote by $A \subsetneq B$ the fact that $\mu(A \setminus B) = 0$ and write $A \sim B$ if $A \subsetneq B$ and $B \subsetneq A$.

A subset $\mathcal{F} \subset \mathfrak{A}$ is called an $\cap$*-system* iff $\emptyset \in \mathcal{F}$, $X \in \mathcal{F}$ and $A \cap B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$.

For an $\cap$-system $\mathcal{F} \subset \mathfrak{A}$ a mapping $\delta : \mathcal{F} \to \mathfrak{A}$ is a *partial* (lower) *density* iff it satisfies the following conditions:

(i) $A \sim B$ implies $\delta(A) = \delta(B)$
(ii) $A \sim \delta(A)$
(iii) $\delta(\emptyset) = \emptyset$ and $\delta(X) = X$
(iv) $\delta(A \cap B) = \delta(A) \cap \delta(B)$ for every $A, B \in \mathcal{F}$.

A mapping $\delta : \mathfrak{A} \to \mathfrak{A}$ with the above properties is called a (lower) *density* and if moreover

(v) $\delta(X \setminus A) = X \setminus \delta(A)$ for every $A \in \mathfrak{A}$

is fulfilled δ is a *lifting*.

A lifting or density $\delta : \mathfrak{A} \to \mathfrak{A}$ is called *strong* iff in addition

(vi) $U \subset \delta(U)$ for every open $U \subset X$.

Let us restate one of the fundamental properties of partial densities. For an $\cap$-system $\mathcal{F} \subset \mathfrak{A}$, a partial density $\delta : \mathcal{F} \to \mathfrak{A}$, and $A, B \in \mathcal{F}$ with $A \subsetneq B$ we have $\delta(A) \subset \delta(B)$. In particular δ preserves inclusions.

Existence of Strong Densities. The following two lemmas will provide us with the basic tools for the proof of the main theorem.

LEMMA 1. *Let $\mathcal{F} \subset \mathfrak{A}$ be an $\cap$-system, $\delta : \mathcal{F} \to \mathfrak{A}$ a partial density, and $F_0 \in \mathfrak{A}$ arbitrary. Suppose $A_0 \in \mathfrak{A}$ is such that $A_0 \sim F_0$ and for F'', F', F in $\mathcal{F}$,*

$$F'' \subsetsim F_0 \quad \textit{implies} \quad \delta(F'') \subset A_0,$$

and

$$F_0 \cap F \subsetsim F' \quad \textit{implies} \quad A_0 \cap \delta(F) \subset \delta(F').$$

Then $\mathcal{F}_0 = \mathcal{F} \cup \{F \cap F_0 : F \in \mathcal{F}\}$ is an $\cap$-system containing $\mathcal{F} \cup \{F_0\}$ and the mapping δ_0: $\mathcal{F}_0 \to \mathfrak{A}$ defined by

$$\delta_0(F) = \begin{cases} \delta(F), & \text{if } F \in \mathcal{F} \\ \delta(F') \cap A_o & \text{if } F = F' \cap F_0 \text{ with } F' \in \mathcal{F} \end{cases}$$

is a partial density which extends δ.

Proof. It is obvious that $\mathcal{F}_0$ is an $\cap$-system containing $\mathcal{F} \cup \{F_0\}$. To show that δ_0 is well-defined let F, F' be elements of $\mathcal{F}$.

(1) Assume $F \sim F' \cap F_0$. This implies

$$F \subsetsim F_0, \quad F \subsetsim F' \text{ and } F' \cap F_0 \subsetsim F.$$

From our assumptions concerning A_0 we may therefore conclude

$$(*) \qquad \delta(F) \subset A_0 \text{ and } A_0 \cap \delta(F') \subset \delta(F).$$

Since δ is a partial density $F \subsetsim F'$ implies $\delta(F) \subset \delta(F')$. This inclusion combined with (*) gives us $\delta(F) \subset A_0 \cap \delta(F') \subset \delta(F)$. Hence the equality $\delta(F) = A_0 \cap \delta(F')$ is established.

(2) Assume $F \cap F_0 \sim F' \cap F_0$. Then we have

$$F \cap F_0 \subsetsim F' \text{ and } F' \cap F_0 \subsetsim F.$$

Again it follows from our assumptions concerning A_0 that

$$A_0 \cap \delta(F) \subset \delta(F') \text{ and } A_0 \cap \delta(F') \subset \delta(F)$$

and hence that

$$A_0 \cap \delta(F') = A_0 \cap \delta(F).$$

(1) and (2) together prove that δ_0 is well-defined and that for $F, G \in \mathcal{F}_0$ the fact $F \sim G$ implies $\delta_0(F) = \delta_0(G)$. According to its definition δ_0 obviously satisfies the other conditions for a partial density. It is also clear that δ_0 extends δ.

LEMMA 2. *Let $\mathcal{F} \subset \mathfrak{A}$ be a countable $\cap$-system, $F_0 \in \mathfrak{A}$ arbitrary, $\mathcal{F}_0 = \mathcal{F} \cup \{F \cap F_0 : F \in \mathcal{F}\}$ and $\delta : \mathcal{F} \to \mathfrak{A}$ a partial density. Then there exists a partial density $\delta_0 : \mathcal{F}_0 \to \mathfrak{A}$ which extends δ.*

Proof. According to Lemma 1 the assertion of Lemma 2 is true if we can prove the existence of a set $A_0 \in \mathfrak{A}$ such that $F_0 \sim A_0$ and for F, F', F'' in $\mathcal{F}$,

$$F'' \subsetneq F_0 \text{ implies } \delta(F'') \subset A_0,$$

and

$$F_0 \cap F \subset F' \text{ implies } A_0 \cap \delta(F) \subset \delta(F').$$

As an easy calculation shows $A_0 \in \mathfrak{A}$ fulfills the last two conditions if and only if

$$\delta(F'') \subset A_0 \subset \delta(F') \cup C\delta(F)$$

whenever F, F', $F'' \in \mathcal{F}$ satisfy $F'' \subsetneq F_0 \subsetneq F' \cup CF$.

To establish the existence of such an A_0 let

$$\mathcal{G} := \{\delta(F'') : F'' \subset \mathcal{F} \text{ and } F'' \subsetneq F_0\}$$

and

$$\mathcal{H} := \{\delta(F') \cup C\delta(F) : F, F' \in \mathcal{F} \text{ and } F_0 \subsetneq F' \cup CF\}$$

Since $\mathcal{F}$ is countable the sets $\mathcal{G}$ and $\mathcal{H}$ are also countable.
Therefore $A := \cup \mathcal{G}$ and $B := \cap \mathcal{H}$ are in $\mathfrak{A}$.
Property (ii) of partial densities implies

$$G \subsetneq F_0 \text{ and } F_0 \subsetneq H$$

for every $G \in \mathcal{G}$ and $H \in \mathcal{H}$. Due to the countable additivity of μ we therefore get:

$$A \subsetneq F_0 \text{ and } F_0 \subsetneq B. \tag{†}$$

Let $G \in \mathcal{G}$ and $H \in \mathcal{H}$ be arbitrary. Then there are $F, F', F'' \in \mathcal{F}$ with $F'' \subsetneq F_0 \subsetneq F' \cup CF$, $G = \delta(F'')$, and $H = \delta(F') \cup C\delta(F)$. From $F'' \subsetneq F' \cup CF$ we conclude

$$F'' \cap F \subsetneq F' \text{ and hence } \delta(F'') \cap \delta(F) \subset \delta(F')$$

which implies $G = \delta(F'') \subset \delta(F') \cup C\delta(F) = H$. This gives us

$$(\ddagger) \qquad A \subset B.$$

It is a consequence of (†) and (‡) that

$$A_0 := (F_0 \cup A) \cap B \sim F_0$$

and

$$G \subset A_0 \subset H$$

for all $G \in \mathcal{G}$, $H \in \mathcal{H}$.
Hence the statement of Lemma 2 is proved.

PROPOSITION 1. *Let $\mathcal{B}$ be a countable base of the topology of X and $\mathcal{F}$ the field generated by $\mathcal{B}$. Then there is a partial density $\delta: \mathcal{F} \to \mathfrak{A}$ satisfying $U \subset \delta(U)$ for every $U \in \mathcal{B}$.*

Proof. Without loss of generality we may assume that $\mathcal{B}$ is stable under finite unions and intersections and contains $\varnothing$ and X.

For $B \in \mathcal{B}$ let $\mathcal{B}(B) := \{A \in \mathcal{B} : A \sim B\}$ and define $\phi : \mathcal{B} \to \mathfrak{A}$ by $\phi(B) = \cup \mathcal{B}(B)$. Since $\cup \mathcal{B}(B)$ is open for every $B \in \mathcal{B}$ and since $\mathfrak{A}$ contains all open sets ϕ is a well-defined map from $\mathcal{B}$ to $\mathfrak{A}$. It is an immediate consequence of the definition of ϕ that for $A, B \in \mathcal{B}$ the equivalence $A \sim B$ implies $\phi(A) = \phi(B)$. Since $\mathcal{B}(B)$ is countable we have $B \sim \phi(B)$.

According to our assumptions every nonempty open set has strictly positive measure. Therefore $\mathcal{B}(\varnothing) = \{\varnothing\}$ and hence $\phi(\varnothing) = \varnothing$. Since $X \in \mathcal{B}(X)$ the condition $\phi(X) = X$ is fulfilled.

ϕ also preserves finite intersections. To show this let $A, B \in \mathcal{B}$ be arbitrary. Because $\mathcal{B}$ is stable under finite intersections we have

$$\{A' \cap B' : A' \in \mathcal{B}(A),\ B' \in \mathcal{B}(B)\} \subset \mathcal{B}(A \cap B)$$

and consequently

$$\begin{aligned} \phi(A) \cap \phi(B) &= (\cup \mathcal{B}(A)) \cap (\cup \mathcal{B}(B)) \\ &= \cup\{A' \cap B' : A' \in \mathcal{B}(A),\ B' \in \mathcal{B}(B)\} \\ &\subset \cup \mathcal{B}(A \cap B) = \phi(A \cap B). \end{aligned}$$

To prove the inverse inclusion let $C \in \mathcal{B}(A \cap B)$ be arbitrary. Since $\mathcal{B}$ is stable under finite unions we have $A \cup C \in \mathcal{B}(A)$ and hence $C \subset \phi(A)$. For the same reasons $C \subset \phi(B)$ is true and therefore the inclusion

$$\phi(A \cap B) \subset \phi(A) \cap \phi(B)$$

holds.

Using Lemma 2, the fact that $\mathscr{F}$ is countable, and the fact that $\mathscr{B}$ is an $\cap$-system contained in $\mathscr{F}$, we see by induction that ϕ can be extended to a partial density $\delta\colon \mathscr{F} \to \mathfrak{A}$. From the definition of ϕ it follows immediately that $U \subset \phi(U) = \delta(U)$ for all $U \in \mathscr{B}$.

The following two propositions are stated without proofs. The proofs can be found in [4], although a slightly different notation is used there.

PROPOSITION 2. ([4], p. 64, Lemma 4.6). *Let* $(\mathfrak{A}_n)_{n \in \mathbf{N}}$ be a sequence of σ-fields of subsets of X with

$$\mathfrak{A}_n \subset \mathfrak{A}_{n+1} \subset \mathfrak{A}$$

and $\delta_n : \mathfrak{A}_n \to \mathfrak{A}$ a partial density with

$$\delta_{n+1} \mid \mathfrak{A}_n = \delta_n$$

for every $n \in \mathbf{N}$.

Denote by $\mathfrak{A}_\infty$ *the* σ*-field generated by* $\cup \{\mathfrak{A}_n : n \in \mathbf{N}\}$. *Then there exists a partial density* $\delta_\infty\colon \mathfrak{A}_\infty \to \mathfrak{A}$ *satisfying* $\delta_\infty | \mathfrak{A}_n = \delta_n$ *for all* $n \in \mathbf{N}$.

Though the above proposition is not explicity stated in his paper Traynor proved it independently in [8].

PROPOSITION 3. ([4], p. 61, Satz. 4.3 and Lemma 4.4). *Let* $\mathscr{L}$ *be a* σ*-field,* $A_0 \in \mathfrak{A}$ *arbitrary, and* $\delta : \mathscr{L} \to \mathfrak{A}$ *a partial density.*

Then $\mathscr{L}_0 := \{(B \cap A_0) \cup (B' \setminus A_0)\colon B, B' \in \mathscr{L}\}$ *is a* σ*-field containing* $\mathscr{L} \cup \{A_0\}$ *and there exists a partial density* $\delta_0\colon \mathscr{L}_0 \to \mathfrak{A}$ *extending* δ. δ_0 *can be defined by*

$$\delta_0((B \cap A_0) \cup (B' \setminus A_0))$$
$$= (\delta((B \cap B_1) \cup (B' \setminus B_1)) \cap A_0) \cup (\delta((B \setminus B_2) \cup (B' \cap B_2)) \setminus A_0),$$

where B, B' *are elements of* $\mathscr{L}$ *and where* $B_1 \in \mathscr{L}$ *satisfies* $A_0 \subsetneq B_1$ *and* $B_1 \subsetneq A$ *for every* $A \in \mathscr{L}$ *with* $A_0 \subsetneq A$ *while* $B_2 \in \mathscr{L}$ *satisfies* $CA_0 \subsetneq B_2$ *and* $B_2 \subsetneq C$ *for every* $C \in \mathscr{L}$ *with* $CA_0 \subsetneq C$.

THEOREM 1. *There exists a strong lower density* $\rho : \mathfrak{A} \to \mathfrak{A}$.

Proof. Let $\mathscr{B}$ be a countable base for the topology of X and $\mathscr{F}$ the field generated by $\mathscr{B}$. According to Proposition 1 there is a partial density $\delta : \mathscr{F} \to \mathfrak{A}$ such that $U \subset \delta(U)$ for every $U \in \mathscr{B}$.

Let $f:\mathbf{N}\to\mathscr{F}$ be a bijection and $\mathfrak{A}_n$ the field generated by $\{f(m):0\leqq m\leqq n\}$. It is clear that $\mathfrak{A}_n\subset\mathscr{F}$ is a σ-field. Define $\delta_n:\mathfrak{A}_n\to\mathfrak{A}$ by $\delta_n=\delta\,|\,\mathfrak{A}_n$. Then $(\mathfrak{A}_n,\delta_n)_{n\in\mathbf{N}}$ fulfills the assumptions of Proposition 2. Thus there is a partial density δ_∞: $\mathfrak{A}_\infty\to\mathfrak{A}_\infty$ extending δ, where $\mathfrak{A}_\infty$ is the σ-field generated by $\mathscr{F}=\bigcup\{\mathfrak{A}_n:n\in\mathbf{N}\}$. Using Zorn's lemma and Propositions 2 and 3 we will show that δ_∞ can be extended to a density $\rho:\mathfrak{A}\to\mathfrak{A}$.

To this purpose define

$$\mathscr{T}:=\{(\mathscr{L},\psi):\mathscr{L}\ \sigma\text{-field containing } \mathfrak{A}_\infty \text{ and } \psi:\mathscr{L}\to\mathfrak{A}$$
$$\text{a partial density with } \psi\,|\,\mathfrak{A}_\infty=\delta_\infty\}.$$

For $(\mathscr{L},\psi),(\mathscr{L}',\psi')\in\mathscr{T}$ let $(\mathscr{L},\psi)\leqq(\mathscr{L}',\psi')$ denote the fact that $\mathscr{L}\subset\mathscr{L}'$ and $\psi'\,|\,\mathscr{L}=\psi$. It is easy to check that $\leqq$ is an order-relation on $\mathscr{T}$.

Next we will prove that $\mathscr{T}$ is inductively orderby by $\leqq$.
Let $\mathscr{K}\subset\mathscr{L}$ be any totally ordered subset. The following two cases have to be considered.

(1) If for every sequence $(\mathscr{L}_n,\psi_n)_{n\in N}$ in $\mathscr{K}$ there is a $(\mathscr{L},\psi)\in\mathscr{K}$ with $(\mathscr{L}_n,\psi_n)\leqq(\mathscr{L},\psi)$ for all $n\in\mathbf{N}$, then $\bar{\mathscr{L}}:=\bigcup\{\mathscr{L}:(\mathscr{L},\psi)\in\mathscr{K}\}$ is a σ-field containing $\mathfrak{A}_\infty$ and $\bar{\psi}:\bar{\mathscr{L}}\to\mathfrak{A}$ defined by $\bar{\psi}\,|\,\mathscr{L}=\psi$ for every $(\mathscr{L},\psi)\in\mathscr{K}$ is a partial density extending δ_∞. Hence we have $(\bar{\mathscr{L}},\bar{\psi})\in\mathscr{T}$ and $(\mathscr{L},\psi)\leqq(\bar{\mathscr{L}},\bar{\psi})$ for all $(\mathscr{L},\psi)\in\mathscr{K}$.

(2) If there exists a sequence $(\mathscr{L}_n,\psi_n)_{n\in\mathbf{N}}$ in $\mathscr{K}$ such that for each $(\mathscr{L},\psi)\in\mathscr{K}$ there is an $n\in\mathbf{N}$ with $(\mathscr{L},\psi)\leqq(\mathscr{L}_n,\psi_n)$ we have

$$\bigcup\{\mathscr{L}:(\mathscr{L},\psi)\in\mathscr{K}\}=\bigcup\{\mathscr{L}_n:n\in\mathbf{N}\}=\bar{\mathscr{L}}.$$

Let $\mathscr{L}_\infty$ denote the σ-field generated by $\bar{\mathscr{L}}$. From Proposition 2 we conclude the existence of a partial density $\psi_\infty:\mathscr{L}_\infty\to\mathfrak{A}$ which extends all ψ_n and hence we get $(\mathscr{L},\psi)\leqq(\mathscr{L}_\infty,\psi_\infty)\in\mathscr{T}$ for all $(\mathscr{L},\psi)\in\mathscr{K}$.

Since $(\mathfrak{A}_\infty,\delta_\infty)\in\mathscr{T}$ and thus $\mathscr{T}\neq\emptyset$, Zorn's lemma gives us the existence of a maximal element $(\bar{\mathscr{F}},\rho)$ in $\mathscr{T}$.
From Proposition 3 we conclude $\bar{\mathscr{F}}=\mathfrak{A}$ and hence ρ is a density. To prove the theorem it remains to check that ρ is strong. To this end let U be any open subset of X. Then

$$U=\bigcup\{B\in\mathscr{B}:B\subset U\}$$

for $B\in\mathscr{B}$ and $B\subset U$ we have

$$\rho(U)\supset\rho(B)=\delta_\infty(B)=\delta(B)=\delta(B)\supset B$$

and thus

$$U=\bigcup\{B\in\mathscr{B}:B\subset U\}\subset\rho(U).$$

Hence ρ is strong.

The main theorem. To verify the main theorem we need one more lemma which is an immediate consequence of a more general theorem of von Neumann and Stone ([**6**], p. 372, Th. 18). Easier direct proofs were given by Sion in [**7**], Gapaillard in [**3**], Traynor in [**8**] and the author in [**4**]. Let us restate this lemma but omit the proof.

LEMMA 3. *If* $(X, \mathfrak{A}, \mu)$ *is complete and* $\delta : \mathfrak{A} \to \mathfrak{A}$ *a density then there is a lifting* $\phi : \mathfrak{A} \to \mathfrak{A}$ *with* $\delta(A) \subset \phi(A)$ *for every* $A \in \mathfrak{A}$.

THEOREM 2. *If* $(X, \mathfrak{A}, \mu)$ *is complete then there exists a strong lifting* $\phi ; \mathfrak{A} \to \mathfrak{A}$.

Proof. According to Theorem 1 there is a strong density $\delta : \mathfrak{A} \to \mathfrak{A}$. From Lemma 3 we get a lifting $\phi ; \mathfrak{A} \to \mathfrak{A}$ satisfying $\delta(A) \subset \phi(A)$ for all $A \in \mathfrak{A}$. Since δ is strong we therefore conclude $U \subset \delta(U) \subset \phi(U)$ for every open set $U \subset X$.

Hence ϕ is a strong lifting.

Notes.

(1) All previous statements of this paper remain true for σ-finite measure spaces because for any σ-finite measure on a σ-field there exists a finite measure with the same nullsets.

(2) The methods of proofs show that for an arbitrary finite or σ-finite measure space $(Y, \mathscr{L}, \nu)$ and any countable $\mathscr{F} \subset \mathscr{L}$, such that for each finite $\mathscr{G} \subset \mathscr{F}$ with $\nu(\cap \mathscr{G}) = 0$ we have $\cap \mathscr{G} = \emptyset$, there is a density $\delta : \mathscr{L} \to \mathscr{L}$ satisfying $F \subset \delta(F)$ for all $F \in \mathscr{F}$.

If $(Y, \mathscr{L}, \nu)$ is furthermore complete then there is even a lifting $\phi : \mathscr{L} \to \mathscr{L}$ with these properties.

(3) Let X be a second countable topological space, $\mathfrak{A}$ the Borel-field in X and $\mathfrak{W}$ a σ-ideal in $\mathfrak{A}$ containing no nonempty open sets. Then $\mathfrak{A}$ has at most the power of the continuum. Assuming the continuum hypothesis and using the same methods as in the first part of this paper we get a strong density $\delta ; \mathfrak{A} \to \mathfrak{A}$ with respect to $\mathfrak{W}$. From a lifting theorem of von Neumann-Stone (certificate [**6**] or [**4**], p. 47, Satz 3.3) we may conclude (using the continuum hypothesis another time) that there is even a strong lifting $\phi : \mathfrak{A} \to \mathfrak{A}$.

(4) The existence of a strong lifting was well-known for separable metric spaces with finite outer regular measures strictly positive on non-empty open sets (see Ionescu-Tulcea [**5**] and Sion [**7**]) and for locally compact metrizable spaces with (not necessarily σ-finite) Radon measures whose support is the whole space (see Ionescu-Tulcea [**5**]). The first fact is a special case of the main theorem of this paper while the second fact can be derived from this theorem using Proposition 2, p. 108 in Ionescu-Tulcea [**5**], and observing that any metrizable compact space is separable.

(5) Bichteler proves in [**1**] a strong-lifting-theorem implying the existence of a strong lifting for locally compact metrizable spaces with

Radon measures whose support is the whole space. But the result of this paper are no (immediate) consequences of Bichteler's theorem.

(6) Eifrig [2] gives a proof of the existence of a strong lifting for the interval [0, 1] using methods of nonstandard-analysis.

(7) For applications of strong-lifting-theorems to integral representations and disintegration of measures see Ionescu-Tulcea [5] and Sion [7].

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