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CONCERNING SIU'S METHOD FOR SOLVING $y'(t) = F(t, y(g(t)))$

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A procedure is given, which is parameterized by a certain set of real-valued functions, that yields sufficient conditions on each of g and F to guarantee a solution to $y'(t)=F(t,y(g(t)))$.

The following is the main result.

THEOREM 1. *Suppose that f is a real-valued continuous function with connected domain J of real numbers so that*

(1) $0 \in J$ *and* $f(0) > 0$

(2) f *is increasing on* $J \cap [0, +\infty)$

(3) f *is decreasing on* $J \cap (-\infty, 0]$

(4) $0 < k < 1$, *if the range of* f *is unbounded; and* $k = 1$ *if the range of* f *is bounded*

(5) B *is a Banach space,* $F: J \times B \rightarrow B$, *and* $N: J \rightarrow R$

(6) F *is continuous and there is a constant* C *so that*

$$\left|\int_0^x ||F(s, 0)||ds\right| \leqq Cf(x), \text{ for all } x \text{ in } J .$$

(7) $||F(t, x) - F(t, y)|| \leqq N(t)||x - y||$, *for* $t \in J$ *and* $x, y \in B$

(8) N *is positive and Lebesgue integrable or subintervals of* J

(9) g *is any continuous function from* J *into* J *so that* $g(x) \in f^{-1}[f(0), k|f'^{\operatorname{sign} x}(x)|/N(x)]$ *for all* $x \in J$. ($f'^{\operatorname{sign} x}$ denotes the right-hand derivative if $x > 0$ and it denotes the left-hand derivative if $x < 0$.)

(10) $q \in B$.

Then, there is a unique function $y: J \rightarrow B$ *so that* $y'(t) = F(t, y(g(t)))$, $y(0) = q$ *and* $||y(t)|| \leqq \text{Constant} \cdot f(t)$, *for all* t *in* J.

LEMMA. *If f satisfies conditions* (1), (2), *and* (3) *in the statement of Theorem* 1, *then* $\int_0^x f'^{\operatorname{sign} s}(s)\, ds$ *exists in the Lebesgue sense and is less than or equal to* $f(x) - f(0)$, *for each* $x \in J$.

Proof of Lemma. Suppose $x > 0$. Let $f_n(s) = [f(s + 1/n) - f(s)]n$. Then $f'^{+}(s) = \lim f_n(s)$ for almost all $s > 0$. Clearly each f_n is summable, because each f_n is continuous. Also, for each n

$$\begin{aligned}\int_0^x f_n(s) &= \int_0^x n\left[f\left(s + \frac{1}{n}\right) - f(s)\right]ds \\ &= n\int_{1/n}^{x+1/n} f(s)ds - n\int_0^x f(s)ds\end{aligned}$$

$$= n\int_x^{x+1/n} f(s)ds - n\int_0^{1/n} f(s)ds \text{ (which approaches } f(x) - f(0))$$
$$\leqq 2\cdot\{\text{sup of } f \text{ on } [0, x+1]\}$$

Thus, by Fatou's lemma [see 3, page 39], f'^{+} is summable on $[0, x]$ for all $x > 0$ and $x \in J$ and

$$\int_0^x f'^{+}(s)ds \leqq \liminf \int_0^x f_n(s)ds \leqq f(x) - f(0) .$$

The proof is similar for $x < 0$.

Proof of Theorem 1. Let $\|\cdot\|$ be the norm of B and define $|\cdot|$ to be the norm defined by $|z| = \sup\{\|z(x)\|/f(x): x \in J\}$ for each z continuous from J into B such that this supremum exists. Let Y denote the Banach space of all such z, with norm $|\cdot|$. For each $z \in Y$ and $x \in J$, let $(Tz)(x) = q + \int_0^x F(s, z(g(s)))ds$. Suppose $z, w \in Y$. Then

$$\begin{aligned}\|(Tz)(x) - (Tw)(x)\| &= \left\|\int_0^x [F(s, z(g(s))) - F(s, w(g(s)))]ds\right\| \\ &\leqq \left|\int_0^x N(s)\|z(g(s)) - w(g(s))\|ds\right| \\ &\leqq \left|\int_0^x N(s)f(g(s))ds\right| |z - w| .\end{aligned}$$

Thus, $|Tz - Tw| \leqq \sup\left\{\left|\int_0^x N(s)f(g(s))ds\right|/f(x)\right\}|z - w|$. The following shows that T is a contraction.

Case 1. Suppose $x > 0$. Then if $0 \leqq s \leqq x$, $f(g(s)) \in [f(0), kf'^{+}(s)/N(s)]$. Thus, $\left|\int_0^x N(s)f(g(s))ds\right| = \int_0^x N(s)f(g(s))ds \leqq \int_0^x kf'^{+}(s)ds \leqq k(f(x) - f(0))$, by the lemma.

Case 2. Suppose $x < 0$. Then if

$$x \leqq s \leqq 0,\ f(g(s)) \in [f(0), -kf'^{-}(s)/N(s)] .$$

Thus,

$$\left|\int_0^x N(s)f(g(s))ds\right| = \int_x^0 N(s)f(g(s))d \leqq s\, k\int_0^x f'^{-}(s)ds \leqq k(f(x) - f(0)) ,$$

by the lemma.

Thus, in either case $c = \sup\left\{\left|\int_0^x N(x)f(g(s))ds\right|\Big/ f(x)\right\} \leqq \sup\{k(1 - f(0)/f(x))\}$. So, if the range of f is unbounded, $c \leqq k$ and if the range of f is bounded by L and $k = 1$, then $c \leqq 1 - f(0)/L$. So T

has contraction constant c. The zero function Z is in Y, because $(TZ)(x) = q + \int_0^x F(s, 0)ds$ and $||(TZ)(x)||/f(x) \leqq [||q|| + Cf(x)]/f(x) \leqq ||q||/f(0) + C$. Now if $w \in Y$, $||(Tw)(x)||/f(x) \leqq |Tw - TZ| + |TZ|$. Thus $Tw \in Y$. So, by Banach's contraction mapping principle, T has a unique fixed point in Y. This proves Theorem 1.

REMARKS. (1) Given any g one may find an appropriate N and apply Theorem 1, by requiring $N(x) \leqq k|f'^{\mathrm{sign} x}(x)|/f(g(x))$.

(2) At any particular x, there is an f so that $f'^{\mathrm{sign} x}(x) = \infty$ and so that t does not have infinite derivative in a deleted neighborhood of x. For this type f, $g(x)$ could be any number.

In [5], Siu essentially uses the method of Theorem 1 with $f(x) = \exp(|x|/e)$ to obtain:

THEOREM 2 (Siu). *If $|g(x)| \leqq |x| + c$, where $0 < c < 1/e$, for all real numbers x, then $y' = y(g)$, $y(0) = q$ has unique solution subject to $||y(x)|| \leqq \mathrm{constant} \cdot \exp(|x|/e)$*

In [4], the author proves:

THEOREM 3. *If $\{I(i)\}$ is a sequence of intervals so that $I(0) = \{0\} \subseteqq I(i) \subseteqq I(i + 1)$, $I(i) = [a(i), b(i)]$, and $\max\{a(i - 1) - a(i), b(i) - b(i - 1)\} < 1$ for each positive integer i; then $y' = y(g)$, $y(0) = q$ has unique solution on $\cup\{I(i)\}$, whenever g is countinuous and $g(I(i)) \subseteqq I(i)$ for each positive integer i.*

The following theorem is comparable to each of Theorem 2 and Theorem 3.

THEOREM 4. *Suppose the hypothesis of Theorem 3 holds and k is in $(0, 1)$ such that $\max\{a(i - 1) - a(i), b(i) - b(i - 1)\} < k$ for each positive integer i. Then, for each positive integer i, there exists $\delta(i) > 0$ such that if g is a continuous function from $\cup\{I(i)\}$ into $\cup\{I(i)\}$ such that $g(I(i)) \subseteqq [a(i) - \delta(i), b(i) + \delta(i)]$, then $y'(t) = F(t, y(g(t)))$, $y(0) = q$ has a solution on $\cup\{I(i)\}$ for any F such that $N = 1$, where F and N satisfy the conditions listed for them in the hypothesis of Theorem 1.*

Proof of Theorem 4. Let f be a positive continuous piecewise linear function with domain $\cup\{I(i)\}$ such that f has slope $M(i)$ or $(b(i - 1), b(i)))$ and slope $-M(i)$ on $(a(i), a(i - 1))$ where the sequence $\{M(i)\}$ is chosen such that for each nonnegative integer n,

$$M(n+1)$$
$$> \max \left\{ \left[f(0) + \sum_{i=1}^{n} M(i)(a(i-1) - a(i)) \right] \Big/ [k - (a(n) - a(n+1))] \,, \right.$$
$$\left. \left[f(0) + \sum_{i=1}^{n} M(i)(b(i) - b(i-1)) \right] \Big/ [k - (b(n+1) - b(n))] \right\} .$$

Let

$$\delta(n+1) = \min \left\{ a(n+1) - a(n+2),\, b(n+2) - b(n+1) \,, \right.$$
$$\left[kM(n+1) - \left[f(0) + \sum_{i=1}^{n+1} M(i)(a(i-1) - a(i)) \right] \right] \Big/ M(n+2) \,,$$
$$\left. \left[kM(n+1) - \left[f(0) + \sum_{i=1}^{n+1} M(i)(b(i) - b(i-1)) \right] \right] \Big/ M/(n+2) \right\} .$$

It follows that the hypotheses of Theorem 1 hold.

REMARK. The solution in Theorem 4 is unique in the Banach space Y of Theorem 1, which depends on f.

The following is a straightforward application of Theorem 1.

THEOREM 5. *Suppose*

(1) $$-1 < -k < a < 0 < b < k < 1$$

(2) $$m \geqq \max \{1/(k+a),\, 1/(k-b)\}$$

(3) $$nk \geqq \max \{(1 - ma)/(1 + mb),\, (1 + mb)(1 - ma)\}$$

(4) $$l(x) = \begin{cases} x - (\log nk)/n & , \quad x \leqq a \\ a - (\log km/(1 - ma))/n & , \quad a < x < b \\ -x + a + b - (\log kn(mb + 1)/(1 - ma))/n \,, & b \leqq x \end{cases}$$

and

$$u(x) = \begin{cases} -x + a + b + (\log nk/(1 - ma)/(1 + mb))/n \,, & x \leqq a \\ b + (\log \mathrm{km}/(1 + mb))/n & , \quad a < x < b \\ x + (\log nk)/n & , \quad b \leqq x \end{cases}$$

(5) $l \leqq g \leqq u$ *and* g *is continuous* .

Then, there is a unique solution to $y'(t) = F(t, y(g(t)))$, $y(0) = q$ where F and N satisfy the hypotheses of Theorem 1, $N = 1$ and, $\|y(x)\| \leqq \text{constant} \cdot f(x)$, for all real numbers x, where

$$f(x) = \begin{cases} (1 - ma) \exp(-n(x - a)) \,, & x \leqq a \\ 1 - mx & , \quad a \leqq x \leqq 0 \\ 1 + mx & , \quad 0 \leqq x \leqq b \\ (mb + 1) \exp(n(x - b)) & , \quad b \leqq x \end{cases}$$

REMARK. If $a = -b$, then $u = -l$.

The following is a generalization of a theorem by D. R. Anderson [1].

COROLLARY TO THEOREM 5. *Suppose* $0 < \beta < 1$, $\varepsilon > 0$, *and*

$$|g(x)| \leqq \begin{cases} -x + \dfrac{1}{e} - \varepsilon & x \leqq -\beta \\ \beta + \left(\log \dfrac{1}{\beta}\right) \Big/ e - \varepsilon & -\beta < x < \beta \\ x + \dfrac{1}{e} - \varepsilon & \beta \leqq x . \end{cases}$$

Then, there is a solution to $y'(t) = F(t, y(g(t)))$, $y(0) = q$ *for* $N = 1$ *any* y *subject to* $||y(x)|| \leqq$ *constant*$\cdot f(x)$ *for an appropriate* f.

Proof. Straightforward.

REMARK. As β approaches 0, g is allowed to become indefinitely large at 0.

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