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ON M-PROJECTIVE AND M-INJECTIVE MODULES

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In this paper necessary and sufficient conditions are obtained for a direct sum $\bigoplus_{\alpha \in J} A_\alpha$ of R-modules to be M-injective in the sense of Azumaya. Using this result it is shown that if $\{A_\alpha\}_{\alpha \in J}$ is a family of R-modules with the property that $\bigoplus_{\alpha \in K} A_\alpha$ is M-injective for every countable subset K of J then $\bigoplus_{\alpha \in J} A_\alpha$ is itself M-injective. Also we prove that arbitrary direct sums of M-injective modules are M-injective if and only if M is locally noetherian, in the sense that every cyclic submodule of M is noetherian. We also obtain some structure theorems about Z-projective modules in the sense of Azumaya, where Z denotes the ring of integers. Writing any abelian group A as $D \oplus H$ with D divisible and H reduced, we show that if A is Z-projective then H is torsion free and every pure subgroup of finite rank of H is a free direct summand of H.

Most of these results were motivated by the results of B. Sarath and K. Varadarajan regarding injectivity of direct sums.

1. *M-projective and M-injective modules.* Throughout this paper R denotes a ring with $1 \neq 0$ and all the modules considered are left unitary modules over R. By an ideal in R we mean a left ideal in R. M denotes a fixed R-module. We first recall the notions of M-projective and M-injective modules originally introduced by one of the authors [1].

DEFINITION 1.1. An R-module H is called M-projective, if given a diagram

$$\begin{array}{ccccc} & & H & & \\ & & \downarrow{\scriptstyle f} & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

of maps of R-modules with the horizontal sequence extact, $\exists$ a map $h: H \to M$ such that $\varphi \circ h = f$.

The notion of an M-injective module is defined dually.

REMARK 1.2. Regarding R as a left module over itself in the usual way it turns out that R-injective modules are the same as the injective modules over R. However R-projective modules are not the same as projective modules over R.

LEMMA 1.3. *Every divisible abelian group D is Z-projective.*

Proof. Trivial consequence of the fact $\mathrm{Hom}(D, Z) = 0 = \mathrm{Hom}(D, Z_k)$ whenever D is divisible.

REMARK 1.4. We know that projective modules over Z are free. Hence no divisible abelian group $D \neq 0$ is projective over Z.

LEMMA 1.5. *Suppose H is a torsion free abelian group with the property that every pure subgroup of rank 1 of H is a free direct summand of H. Then every pure subgroup of finite rank of H is also a free direct summand of H.*

Proof. By induction on the rank of the subgroup. Let S be a pure subgroup of H of rank k with $k>1$. We can pick a pure subgroup B of S of rank 1. Then B is also pure in H and hence by assumption B is free abelian and $H = C \oplus B$ for some C. Since $S \supset B$ we get $S = (S \cap C) \oplus B$. Now $S \cap C$ is of rank $(k-1)$ and pure in S and hence pure in H. By the inductive hypothesis $S \cap C$ is free abelian and $H = (S \cap C) \oplus L$ for some L. From $C \supset S \cap C$ we now $C = (S \cap C) \oplus (L \cap C)$. Thus $S = (S \cap C) \oplus B$ is free abelian and

$$\begin{aligned} H &= C \oplus B = (S \cap C) \oplus (L \cap C) \oplus B \\ &= (S \cap C) \oplus B \oplus (L \cap C) = S \oplus (L \cap C)\,. \end{aligned}$$

DEFINITION 1.6. We say that a torsion free abelian group H has property (P) if every pure subgroup of finite rank of H is free and a direct summand of H.

Given any abelian group A we can write A as $D \oplus H$ where D is the maximal divisible subgroup of A and H is reduced. Also $H \cong A/D$ is well-determined up to an isomorphism. We will refer to any group isomorphic to H as the reduced part of A.

THEOREM 1.7. *Suppose H is reduced abelian group which is Z-projective. Then H is torsion-free with property (P).*

Proof. It is well-known that a reduced abelian group which is not torsion-free admits of a nonzero finite cyclic direct summand [3, Th 9, p. 21]. Clearly the identity map $Z_m \to Z_m$ (for $m \geqq 1$) can not be lifted to a map $Z_m \to Z$. This proves that Z_m is not Z-projective. Hence if a reduced abelian group H is Z is Z-projective it has to torsion free.

For any $a \neq 0$ in H let $S_a = \{x \in H \mid x$ and a linearly dependent over $Z\}$. Then it is trivial to see that S_a is a pure subgroup of rank 1 in H. Moreover S_α is reduced since H is. Hence $\exists$ a prime p such that $S_\alpha \neq pS_\alpha$. Let $c \in S_\alpha$ be such that $c \notin pS_\alpha$. Since S_α is a pure subgroup of H we see that $c \notin pH$. Hence $\eta(c) \neq 0$ where $\eta: H \to H/pH$ denotes the canonical quotient map. Regarding H/pH as a vector space over Z_p we can comylete $\eta(c)$ to a basis $\{\eta(c)\} \cup \{u_j\}_{j \in J}$ of H/pH over Z_p. Let $\theta: H/pH \to Z_p$ be the Z_p-linear map determined by $\theta(\eta(c)) = 1 \in Z_p$ and $\theta(u_j) = 0$ for all $j \in J$. The Z-projectivity of H now yields a map $h: H \to Z$ with

$$\begin{tikzcd} & H \arrow[dl, dashed, "h"'] \arrow[d, "\theta\circ\eta"] & \\ Z \arrow[r, "\varphi"'] & Z_p \arrow[r] & 0 \end{tikzcd}$$

commutative, where $\varphi: Z \to Z_p$ is the canonical quotient map. From $\varphi h(c) = \theta \circ \eta(c) = 1 \in Z_p$ it follows that $\varphi h(c) \neq 0$. Hence $g = h\backslash S_\alpha$: $S_\alpha \to Z$ is a non-zero homorphism. It follows that $\operatorname{Im} g = kZ$ for some integer $k \geqq 1$. Composing g with the obvious isomorphism $kZ \cong Z$ we get an epimorphism $g': S_\alpha \to Z$. Since Z is free the sequence $S_\alpha \xrightarrow{g'} Z \to 0$ splits. S_α being a torsion-free group of rank 1 it now follows that $S_\alpha \xrightarrow{g'} Z$ is an isomorphism. Thus for $\alpha \neq 0$ in H the subgroup S_α is isomorphic to Z.

Our next step is to show that S_α is a direct summand of H. Let c be a generator for $S_\alpha \cong Z$ and $V = \{\alpha \in \operatorname{Hom}(H, Z) \mid \alpha(c) \neq 0\}$. From what we have seen already V is a nonempty set. Let $l = \min_{\alpha \in V} |\alpha(c)|$. We will show that $l = 1$. Suppose on the contrary $l > 1$. There definitely exists an element $\alpha \in V$ such that $\alpha(c) = l$. Let p be a prime divisor of l and $l = kp$. Now $c \notin pS_\alpha$. The argument used already yields a map $h: H \to Z$ such that $\varphi h(c) = 1 \in Z_p$. This means $h(c) = np + 1$ for some $n \in Z$. Writing $n = kd + r$ with $d \in Z$ and r an integer satisfying $0 \leqq r < k$ consider the element $h - d\alpha \in \operatorname{Hom}(H, Z)$. Now, $\{h - d\}(c) = np + 1 - dl = np + 1 - dkp = rp + 1$. Clearly, $0 < rp + 1 < rp + p = (r + 1)p \leqq kp = l$. Thus $\beta = h - d\alpha$ is in V and $|\beta(c)| = rp + 1 < l$, contradicting the definition of l. This contradiction proves that $h = 1$. It now follows that $\exists$ an $\alpha: H \to Z$ with $\alpha(c) = 1$, in which case $\exists$ a splitting $\mu: Z \to H$ for α with $\mu(1) = c$. Hence $S_\alpha = \mu(Z)$ is a direct summand of H.

It is clear that every pure subgroup of rank 1 of H is of the form S_α for some $\alpha \neq 0$ in H. Now appealing to Lemma 1.5 we immediately see that H has property (P).

COROLLARY 1.8. *Let $A = D \oplus H$ with D the maximal divisible subgroup of A. If A is Z-projective then H is torsion-free and* *has property (P).*

COROLLARY 1.9. *A finitely generated abelian group A is Z-projective $\Leftrightarrow A$ is free of finite rank.*

COROLLARY 1.10 *Suppose H is a reduced decomposable torsion-free abelian group.* (i.e., H is the direct sum of rank 1 torsion-free abelian groups). *Then H is Z-projective $\Leftrightarrow H$ is free.*

PROPOSITION 1.11. *Let p be a prime. An abelian group A is Z_{p^∞}-injective if and only if $A \cong (\bigoplus_{\alpha \in J} Z_{p^\infty}) \oplus B$, a direct sum of copies of Z_{p^∞} with an abelian group B having no p-torsion.*

Proof. Suppose $A \cong (\bigoplus_{\alpha \in J} Z_{p^\infty}) \oplus B$ with B having *no* p-torsion. Since $\bigoplus_{\alpha \in J} Z_{p^\infty}$ is divisible, it is injective over Z and hence Z_{p^∞}-injective as well. The only subgroups of Z_{p^∞} are Z_{p^∞} and Z_{p^k} for some integer $k \geqq 1$. When B has no p-torsion $\operatorname{Hom}(Z_{p^k}, B) = 0 = \operatorname{Hom}(Z_{p^\infty}, B)$. This proves that B is Z_{p^∞}-injective.

Conversely, assume A to be Z_{p^∞}-injective. Let $\alpha \in A$ be an element in the p-primary torsion of A. Suppose the order of α is p^k. Then $\exists$ a homomorphism $Z_{p^k} \xrightarrow{f} A$ carrying the element 1 of Z_{p^k} to α. Since A is Z_{p^∞}-injective $\exists$ an extension $g\colon Z_{p^\infty} \to A$ of f. Then $\operatorname{Im} g$ is divisible, $\alpha \in \operatorname{Im} g$ and $\operatorname{Im} g$ is in the p-primary torsion of A. This proves that the p-primary torsion of A is divisible. Since any divisible subgroup of A is a direct summand of A and since any divisible p-primary abelian group is a direct sum of copies of Z_{p^∞} it follows that $A \cong (\bigoplus_{\alpha \in J} Z_{p^\infty}) \oplus B$ with B having no p-torsion.

We now recall the definitions of an M-epimorphism and an M-monomorphism due to one of the authors [1], and state two results due to him.

DEFINITION 1.12. (i) Let A, B be R-modules and $\theta\colon A \to B$ an epimorphism. θ is said to be an M-epimorphism if $\exists$ a map $\psi\colon A \to M$ such that $\operatorname{Ker}\theta \cap \operatorname{Ker}\psi = 0$.

(ii) Let $\alpha\colon A \to B$ be a monomorphism. α is called an M-monomorphism if $\exists$ a map $\beta\colon M \to B$ such that $\operatorname{Im}\alpha$ and $\operatorname{Im}\beta$ together generate B.

PROPOSITION 1.13 [1], [5]. *The following conditions on an R-module H are equivalent.*

(1) *H is M-projective*

(2) *Given any M-epimorphism $\theta\colon A \to B$ and any $f\colon H \to B$* $\exists$ *a map $h\colon H \to A$ such that $\theta \circ h = f$*

(3) *Every M-epimorphism $\theta\colon C \to H$ splits.*

PROPOSITION 1.14. *Dual of Proposition* 1.13.

DEFINITION 1.15. For any module H let $C^p(H)$(respy $C^i(H)$) = the class of all modules M such that H is M-projective (respy M-injective). For any module M let $C_p(M)$ (respy $C_i(M)$) denote the class of M-projective (respy M-injective) modules.

PROPOSITION 1.16 [1], [5].

(1) *$C^p(H)$ is closed under submodules, homomorphic images and the formation of finite direct sums.*

(2) *$C^i(H)$ is closed under submodules, homomorphic images and arbitrary direct sums.*

(3) *$C_p(H)$(respy $C_i(H)$) is closed under direct sums* (respy direct products) *and direct summands* (respy direct factors)

REMARKS.

1.17. In general $C^p(H)$ is not closed under formation of arbitrary direct sums. For instance let $R = Z$ and $H = Q$ the additive group of the rationals. From Lemma 1.3 we see that Q is Z-projective. Thus $Z \in C^p(Q)$. Let J be an *infinite* set and for each $\alpha \in J$ let $M_\alpha = Z$. Then each $M_\alpha \in C^p(Q)$. Clearly Q is a quotient of $\bigoplus_{\alpha \in J} M_\alpha$ and the identity map of Q can not be lifted to a map of Q into $\bigoplus_{\alpha \in J} M_\alpha$. This means $\bigoplus_{\alpha \in J} M_\alpha \notin C^p(Q)$.

1.17′. Since $C^p(H)$ is closed under submodules from 1.17 it follows that $C^p(H)$ in general is not closed under formation of arbitrary direct products.

1.18. In general $C^i(H)$ is not closed under formation of arbitrary direct products. Let $R = Z$ and $H = Z$. From Proposition 1.11 we have $Z_{p^\infty} \in C^i(Z)$. Let $M = \prod_p Z_{p^\infty}$, the direct product taken over all primes. It is known and quite easy to see that $\exists$ a subgroup of M which is isomorphic to Q. If $M \in C^i(Z)$ from (2) of Proposition 1.16 it would that $Q \in C^i(Z)$. Since the identy map of Z can not be extended to a map of Q into Z it follows that Z is not Q-injective. In other words $Q \notin C^i(Z)$. This in turn implies $M \notin C^i(Z)$.

2. **M-injectivity of direct sums.** For any module A and any $x \in A$ we denote the left annihilator $\{\lambda \in R \mid \lambda x = 0\}$ of x by L_x.

DEFINITION 2.1. An element $x \in A$ is said to be dominated by M if $L_x \supset L_m$ for some $m \in M$.

Given a family $\{A\}_{\alpha \in J}$ of modules let x be the element of $\prod_{\alpha \in J} A_\alpha$ whose α-component is x_α. Let $I_x = \{\lambda \in R \mid \lambda \boldsymbol{x} \in \bigoplus_{\alpha \in J} A_\alpha\}$.

DEFINITION 2.2. We call $\boldsymbol{x} \in \prod_{\alpha \in J} A_\alpha$ a special element if $I_x x_\alpha =$

0 for almost all α. In otherwords $\exists$ a finite subset F of J such that $\lambda x_\alpha = 0$ for all $\lambda \in I_x$ and for all $\alpha \notin F$.

PROPOSITION 2.3. *A is M-injective $\Leftrightarrow$ A is Rm-injective for all $m \in M$.*

Proof. This is an easy consequence of 1.16 (2). The implication $\Rightarrow$ follows from the closedness of $C^i(A)$ under submodules. As for $\Leftarrow$, by the closedness of $C^i(A)$ under direct sums it follows that A is $\bigoplus_{m \in M} Rm$-injective. Since M is a homomorphic image of $\bigoplus_{m \in M} Rm$ and since $C^i(A)$ is closed under homomorphic images, it follows that A is M-injective.

THEOREM 2.4. *$\bigoplus_{\alpha \in J} A_\alpha$ is M-injective $\Leftrightarrow$ each A_α is M-injective and every element of $\prod_{\alpha \in J} A_\alpha$ dominated by M is special.*

Proof. $\Rightarrow$: Let $\boldsymbol{x} \in \pi A_\alpha$ be dominated by M, that is, there is an $m \in M$ such that $L_m \subset L_x$. This implies that the mapping $\lambda m \to \lambda \boldsymbol{x} (\lambda \in R)$ is well defined and gives a homomorphism $f: Rm \to \pi A_\alpha$. The image of the submodule $I_x m$ by f is clearly $I_x \boldsymbol{x} (\subset \bigoplus A_\alpha)$. Thus the restriction of f to $I_x m$ is regarded as a homomorphism $I_x m \to \bigoplus A_\alpha$. Since $\bigoplus A_\alpha$ is Rm-injective, this homomorphism can be extended to a homomorphism $Rm \to \bigoplus A_\alpha$ which means that there exists an $u \in \bigoplus A_\alpha$ such that $\lambda \boldsymbol{x} = \lambda \boldsymbol{u}$ for all $\lambda \in I_x$. It follows then that $I_x \boldsymbol{x}\alpha = I_x u_\alpha$ for all $\alpha \in J$. But since $u\alpha = 0$ for almost all α, it follows that $I_x x\alpha = 0$ for almost all α too, i.e., $\boldsymbol{x}$ is special.

$\Leftarrow$: Let $m \in M$ and consider the cyclic submodule Rm of M. Let I be a left ideal of R. Then IM is a submodule of Rm. (Conversely every submodule of Rm is of the form Im with a suitable left ideal I). Let there be given a homomorphism $h: Im \to \bigoplus A_\alpha$. Then since $\bigoplus A_\alpha \subset \pi A_\alpha$ and πA_α is M-whence Rm-injective, h can be extended to a homomorphism $Rm \to \pi A_\alpha$. Let $\boldsymbol{x} \in \pi A_\alpha$ be the image of m. Then the homomorphism is given by $\lambda m \to \lambda \boldsymbol{x} (\lambda \in R)$. There fore it follows that $I\boldsymbol{x} = h(Im) \subset \bigoplus A_\alpha$ whence $I \subset I_x$. On the other hand, since clearly $L_m \subset L_x$, $\boldsymbol{x}$ is dominated by M and thus $\boldsymbol{x}$ is special by assumption, i.e., $I_x \boldsymbol{x}_\alpha = 0$ whence $Ix_\alpha = 0$ for almost all α. Let $\boldsymbol{u}$ be the element of $\bigoplus A_\alpha$ whose α-component is x_α or 0 according as $Ix_\alpha \neq 0$ or $Ix_\alpha = 0$. Then it is clear that $\lambda \boldsymbol{u} = \lambda \boldsymbol{x}$ for all $\lambda \in I$. Further, it is also clear that $L_m \subset L_x \subset L_u$ and therefore the mapping $\lambda m \to \lambda \boldsymbol{u} (\lambda \in R)$ is well defined. This mapping gives a homomorphism $f: Rm \to \bigoplus A_\alpha$ which is an extension of h, because $f(\lambda m) = \lambda \boldsymbol{u} = \lambda \boldsymbol{x}$ for all $\lambda \in I$. This implies that $\bigoplus A_\alpha$ is Rm-injective and so is M-injective (by Proposition 2.3).

THEOREM 2.5. *The direct sum of any family of M-injective modules is M-injective $\Leftrightarrow$ every cyclic submodule of M is noetherian.*

Proof. $\Leftarrow$. Let $\{A_\alpha\}$ be a family of M-injective modules. Let x be an element of πA_α dominated by M; thus there is an $m \in M$ such that $L_m \subset L_x$. Consider $I_x m$. Since clearly $L_x \subset I_x$ whence $L_m \subset I_x$, it follows that $I_x/L_m \cong I_x m$. On the other hand, $I_x m$ is a submodule of the Noetherian module Rm. Hence I_x/L_m is finitely generated, i.e., there exist a finite number of elements $\lambda_1, \lambda_2, \cdots, \lambda_n$ of I_x such that

$$I_x = R\lambda_1 + R\lambda_2 + \cdots + R\lambda_n + L_m \, .$$

It follows therefore $I_x x_\alpha = R\lambda_1 x_\alpha + R\lambda_2 x\alpha + \cdots + R\lambda_n x_\alpha$ for all components x_α. Since, however, for each i, $\lambda_i x_\alpha = 0$ for almost all α, it follows that $I_x x_\alpha = 0$ for almost all α, that is, x is special. Thus $\bigoplus A_\alpha$ is M-injective by Theorem 2.4.

$\Rightarrow$. Let Rm, $m \in M$ be any cyclic submodule of M. Then $R/L_m \cong Rm$, and there is a $(1-1)$ correspondence between the left ideals of R containing L_m and submodules of Rm. Thus in order to show that Rm is noetherian it is sufficient to prove that there is no properly ascending infinite sequence of ideals of R containing L_m. Suppose there exists an infinite sequence $L_m \subset I_1 \subset I_2 \subset I_3 \subset \cdots$ of ideals I_j with $I_j \neq I_{j+1}$ for every $j \geqq 1$. Let $B_j = R/I_j$, η_j: $R \rightarrow B_j$ the canonical projection. Let A_j be the injective hull of B_j. Then each A_j is M-injective also. By assumption $\exists$ an $m \in M$ s.t. $I_1 \supset L_m$. The element $x = (x_j)_{j \geqq 1}$ of $\prod_{j \geqq 1} A_j$ where $x_j = \eta_j$ (1) is clearly dominated by M. For any $\lambda \in I_j$ we have $\lambda x_k = 0$ for $k \geqq j$. Hence $I_j \subset I_x$ for all $j \geqq 1$. Let λ_j be any element of I_{j+1} which is not in I_j. Then $\lambda_j x_j \neq 0$ and $\lambda_j \in I_x$. This proves that $I_x x_j \neq 0$ for every $j \geqq 1$. This means x is not a special element and hence by theorem 2.4, $\bigoplus_{j \geqq 1} A_j$ is not M-injective. This proves the implication $\Rightarrow$.

REMARK 2.6. A result of H. Bass [2] asserts that arbitrary direct sums of injective modules over R are injective $\Leftrightarrow$ R is noetherian. Theorem 2.5 is a generalization of this result of H. Bass. When $M = R$ we get the result of Bass.

THEOREM 2.7. *Suppose $\{A_\alpha\}_{\alpha \in J}$ is a family of R-modules such that for every countable subset K of J, $\bigoplus_{\alpha \in K} A_\alpha$ is M-injective. Then $\bigoplus_{\alpha \in J} A_\alpha$ is itself M-injective.*

Proof. Assume that $\bigoplus_{\alpha \in J} A_\alpha$ is not M-injective. Then, by Theorem 2.4, there exists an $x \in \prod_{\alpha \in J} A_\alpha$ which is dominated by M but is not special, i.e., $I_x x_\alpha \neq 0$ for infinitely many $\alpha \in J$. Let K be an infinite countable subset of the infinite set $\{\alpha \in J \mid I_x x_\alpha \neq 0\}$. Let $\boldsymbol{y}$ be element of $\prod_{\alpha \in K} A_\alpha$ whose α-component y_α is equal to x_α for all $\alpha \in K$. Then clearly $I_x \subset I_y$, so that it follows that $\boldsymbol{y}$ is dominated by M and $I_y y_\alpha = I_y x_\alpha \neq 0$ for all $\alpha \in K$. This implies again by Theorem 2.4 that $\bigoplus_{\alpha \in K} A_\alpha$ is not M-injective (because each A_α is M-injectiue by the assumption of our theorem). This is a contradiction, and so the proof is completed.

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