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ON THE ACTION OF THE DYER-LASHOF ALGEBRA IN $H_*(G)$

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Let G be the space of homotopy equivalences of S^n for $n \to \infty$. This is an infinite loop space, that is, it has definite deloopings. The first delooping of G is the classifying space for (stable) spherical fibrations.

The (mod. 2) homology ring of an infinite loop space is an algebra over the Dyer-Lashof algebra R of all primary homology operations. The principal result of this paper is the evaluation of the R-action in $H_*(G)$. The R-module $H_*(G)$ determines the R-module $H_*(G/O)$, where G/O is the homogeneous space associated with the infinite orthogonal subgroup of G. Let $\alpha: BSO \to G/O$ be a "solution" of the Adams conjecture in the 2-local category, and let $QH_*(G/O)$ be the R-module of indecomposable elements.

THEOREM. *The induced map $\alpha_*: H_*(BSO) \to Z_2 \otimes_R QH_*(G/O)$ is surjective, in fact $Z_2 \otimes_R QH_*(G/O) \cong QH_*(BSO)$.*

The basic method of the paper is to compare the Boardman-Vogt **[4]** infinite loop space structure on SG, called the *composition-structure* with the *loop-structure* on $Q(S^0) = \lim \Omega^n S^n$. The loop-structure is defined by the identification $Q(S^0) = \Omega^k \lim \Omega^n S^{n+k}$. Let

$$c: R \otimes H_*(SG) \to H_*(SG) \quad \text{and} \quad l: R \otimes H_*(Q(S^0)) \to H_*(Q(S^0))$$

denote the R-actions. The component $Q_0(S^0)$ of $Q(S^0)$ containing the constant map has the homotopy type of SG (the oriented homotopy equivalences) so that $H_*(SG) \cong H_*(Q_0(S^0))$. Roughly, our result on the R-module $H_*(SG)$ is that $c \equiv l$ modulo a certain "length" filtration and modulo totally decomposable elements, that is, decomposable elements of $H_*(SG)$ which are also decomposable in the loop product when considered as elements of $H_*(Q_0(S^0))$. The loop action l was essentially determined in **[10]**. The R-module $H_*(BSG)$ is an easy consequence of the main result.

THEOREM.

$$H_*(BSG) = H_*(BSO) \otimes E\{Q(a,a) \mid a = 1, 2, \cdots\} \otimes P,$$

where P is a (large) *polynomial algebra and $Q(a, a)$ are elements of degree* $2a + 1$.

The elements $Q(a, a)$ are connected via the homology operations, e.g. $\hat{Q}^{2a+2}(Q(a, a)) = Q(2a+1, 2a+1)$, where $\hat{Q}^{2a+2}$ is the indecomposable element in R of degree $2a+2$. When $a+1$ is a power of 2 the elements $Q(a, a)$ are particularly interesting; $Q(2^i-1, 2^i-1)$ is spherical if and only if the "Arf invariant one" conjecture has a positive answer in dimension $2^{i+1}-2$, that is, if and only if there is a smooth stably parallelizable $2^{i+1}-2$ dimensional closed manifold with *Arf* invariant one.

It is a result of Sullivan that in the 2-local category SG and G/O are products, $SG \cong J \times \operatorname{cok} J$ and $G/O \cong BSO \times \operatorname{cok} J$. From the corollary below it follows that this is not a splitting of H-spaces. Thus, neither BSG nor $B(G/O)$ splits as was expected.

COROLLARY. *In the 2-local category there is no H-map $f\colon BSO \to G/O$ with $f_*\colon H_2(BSO) \to H_2(G/O)$ nonzero.*

Any solution $\alpha\colon BSO \to G/O$ of the Adams conjecture maps $H_2(BSO)$ isomorphically to $H_2(G/O)$ and can therefore not be an H-map. From Sullivan's analysis of G/PL and Kirby-Siebenmann's result on TOP/PL it follows that G/TOP is 2-locally a product of Eilenberg-MacLane spaces,

$$G/TOP = \prod_{n \geqq 1} K(Z_2, 4n-2) \times \prod_{n \geqq 1} K(Z, 4n).$$

Boardman and Vogt has given G/TOP an infinite loop space structure.

COROLLARY. *In the 2-local category, $B^3(G/TOP)$ is not a product of Eilenberg-MacLane spaces.*

In a forthcoming paper we combine the "algebraic" point of view in this paper with "surgery" theory to obtain the structure of $H_*(G/TOP)$ as a module over the Dyer-Lashof algebra. In another paper we use the R-module structure of $H_*(BSG)$ to determine the higher torsion in $H_*(BSG; \mathbf{Z})$. These results are then used in **[15]** to evaluate (2-locally) the natural projection $SG \to G/TOP$ and in **[16]** to determine the 2-primary structure of the oriented PL bordism ring as well as the topological bordism ring in dimensions different from 4.

The paper is divided in 5 sections as follows:

§1 H^∞-structures
§2 Algebraic formulas
§3 The Dyer-Lashof algebra and its dual
§4 The homology operations in $H_*(SG)$
§5 The R-indecomposable elements of $H_*(G/O)$.

In §1 we define the various H^∞-structures under consideration and in §2 we collect the algebraic formulas on which this paper is based. Most important we establish a formula for evaluating the composition operations on loop products (the mixed cartan formula). J. P. May has more recently developed a very slick way of deriving the results of §1 and §2. However, for the applications to the R-module structure of $H_*(G/TOP)$ the unstable more geometric point of view taken in §1 still seems preferable. In §3 we compute the dual of the Dyer-Lashof algebra. Section 4 contains the main theorems of the paper: The R-module structure of $H_*(G)$ and $H_*(BG)$. Finally, in §5 we prove the corollaries listed above.

Most of the results of this paper were in one form or another contained in the author's doctoral thesis (University of Chicago, 1970) written under the guidance of J. P. May, who in every way possible supported this work. Most important, it was a conjecture of his relating to the R-module structure of $H_*(SG)$ which was our starting point.

Last, we point out once and for all that all homology and cohomology groups have Z_2 coefficients throughout the paper.

1. H^∞-structures. In this section we recollect the various results on H^∞-spaces needed in the rest of the paper. We are strongly inspired by the work of Boardman and Vogt **[4]**, May **[18]** and Tsuchiya **[28]** and claim little if any originality.

For any space X, the space $S^m \times X \times X$ has a free Z_2-action, $T(w; x, y) = (-w; y, x)$. The orbit space, called the quadratic construction, will be denoted $Ep^m(X)(m \leqq \infty)$ throughout the paper. A map $f: X \to Y$ induces a map $Ep^m(f): Ep^m(X) \to Ep^m(Y)$ defined as $Ep^m(f)(w; x_1, x_2) = (w; f(x_1), f(x_2))$. Following **[3]** and **[11]** we make the following definition

DEFINITION 1.1. An H_2^m-structure on an H-space $(X, \mu)(m \leqq \infty)$ is a map $\theta: Ep^m(X) \to X$ which (up to homotopy) satisfies

(i) $\theta(w; 1, 1) = 1$, $1 \in X$ the unit

(ii) $\theta \mid Ep^0(X) = \mu$.

A map $f: X \to Y$ between H_2^m-spaces is an H^m-*map* provided $f \circ \theta = \theta \circ Ep^m(f)$. Note, that if X has an H_2^m-structure for each m and if these fit together, that is, if the diagrams

$$\begin{tikzcd} Ep^m(X) \arrow[rr] \arrow[dr, "\theta^m"'] & & Ep^{m+1}(X) \arrow[dl, "\theta^{m+1}"] \\ & X & \end{tikzcd}$$

are homotopy commutative then X gets an H_2^∞-structure. Here $i\colon Ep^m(X) \to Ep^{m+1}(X)$ is induced from the inclusion which embeds S^m as equator in S^{m+1}.

An infinite loop space structure on a space X (for short, Ω^∞-structure) is an Ω-spectrum $\{B^nX\}$ with $B^0X = X$. In **[3]** and **[11]** it is shown that an Ω^∞-structure gives rise to an H_2^∞-structure; in fact it gives more. We are particulary interested in the "Adem diagram". To a group π, let $E\pi$ denote the infinite join $\pi * \pi * \cdots$ with diagonal π-action and orbit space $B\pi$. From an Ω^∞-structure on X Dyer and Lashof **[11]** construct a mapping $\theta_4\colon E\mathscr{S}_4 \times_{\mathscr{S}_4} X^4 \to X$ where $\mathscr{S}_4$ is the permutation group on 4 letters and they show that the diagram

$$(1.1)\qquad \begin{array}{ccc} Ep^\infty(Ep^\infty(X)) & \xrightarrow{j} & E\mathscr{S}_4 \times_{\mathscr{S}_4} X^4 \\ \downarrow Ep^\infty(\theta) & & \downarrow \theta_4 \\ Ep^\infty(X) & \xrightarrow{\theta} & X \end{array}$$

is homotopy commutative. Let $\tau = Z_2 \int Z_2 \subset \mathscr{S}_4$ denote the wreath product. Then $Ep^\infty(Ep^\infty(X)) = E\tau \times_\tau X^4$ and the map j above is induced from the inclusion $\tau \subset \mathscr{S}_4$.

DEFINITION 1.2. An H_2^∞-structure $\theta\colon Ep^\infty(X) \to X$ is called an H^∞-structure provided there exists a map $\theta_4\colon E\mathscr{S}_4 \times_{\mathscr{S}_4} X^4 \to X$ such that (1.1) is homotopy commutative.

The quadratic construction extends to bundles. If ξ is a bundle over X with total space $E(\xi)$ we let $Ep^m(\xi)$ denote the bundle over $Ep^m(X)$ with total space $Ep^m(E(\xi))$. When ξ is merely a spherical fibre space one uses this construction on the associated disc fibration. One has,

$$(1.2)\qquad \begin{aligned} Ep^m(\xi_1 \oplus \xi_2) &= Ep^m(\xi_1) \oplus Ep^m(\xi_2) \\ Ep^m(1) &= \zeta \oplus 1, \end{aligned}$$

where 1 is the trivial line bundle (real or complex) and ζ the canonical line bundle (real or complex) over $RP^m = Ep^m(*)$. For m even both $\tilde{KO}(RP^m)$ and $\tilde{K}(RP^m)$ are finite cyclic 2-groups generated by $\zeta - 1$. All the H^∞-structures we shall consider arise from framings of $Ep^m(2^n) = 2^n\zeta \oplus 2^n$, n large. We construct such framings explicitly using the Clifford algebra. Let $C(\mathbf{R}^n)$ denote the Clifford algebra of $\mathbf{R}^n$ equipped with the usual inner product. $C(\mathbf{R}^n)$ is generated by the orthonormal basis $e_1, \cdots, e_n$ in $\mathbf{R}^n$ with relations $e_i^2 = -1$ and $e_ie_j + e_je_i =$

0 (we allow $n = \infty$ and give in this case $\mathbf{R}^\infty$ and $C(\mathbf{R}^\infty)$ the weak topology, $\mathbf{R}^\infty = \lim \mathbf{R}^n$ etc.) It is well-known that $C(\mathbf{R}^n)$ inherits an inner product from $\mathbf{R}^n$. Suppose $m+1<n<\infty$ so that $S^m \subset \mathbf{R}^n \subset C(\mathbf{R}^n)$ and define

$$(1.3)\qquad \rho\colon S^m \to SO(C(\mathbf{R}^n)\oplus C(\mathbf{R}^n)) = SO(C(\mathbf{R}^{n+1}))$$

by $\rho(w)(x,y) = 1/\sqrt{2}(w\cdot(x-y), x+y)$. This is an equivariant map: $\rho(-w) = \rho(w)T$, where T is the twist map, $T(x,y)=(y,x)$. The adjoint of ρ defines a framing of $Ep^m(2^n)$ and a framing of $Ep^m(2^n)$ induces an equivariant map ρ as above.

Let $\tilde{G}(N)$ denote the space of selfmaps of S^{N-1}. For $m+1<n<\infty$ and $N=2^n$ we define

$$\hat{\theta}\colon Ep^m(\tilde{G}(N)) \to \tilde{G}(2N)$$

by $\hat{\theta}(w; f, g) = \rho(w)(f*g)\,\rho(w)^{-1}$, where $f*g$ denotes the join of f and g. When $m=N=\infty$, $\hat{\theta}$ is an H_2^∞-structure on $\tilde{G} = \lim \tilde{G}(N)$ with associated H-space structure induced from join of maps. It is well-known that this H-space structure on $\tilde{G}$ is equivalent to the H-space structure given by composition of maps.

DEFINITION 1.3. The H_2^∞-structure $\hat{\theta}$ above is called the composition-structure.

The same construction applies to the other mapping spaces such as the infinite orthogonal group O, the infinite unitary group U and the infinite homeomorphism group TOP, $TOP = \lim TOP(N)$, where $TOP(N)$ is the group of homeomorphisms of S^{N-1}.

The homotopy class of a map $S^{N-1}\to S^{N-1}$ is determined by its degree; thus $\pi_0(\tilde{G}(N)) = Z$. Let $\tilde{G}_i(N)$ denote the component of $\tilde{G}(N)$ consisting of maps of degree i and write $G(N) = \tilde{G}_1(N)\cup\tilde{G}_{-1}(N)$, $SG(N) = \tilde{G}_1(N)$. (If $N=\infty$ we usually write $\tilde{G}$ instead of $G(\infty)$ etc.) Spherical fibre spaces of (sphere) dimension $N-1$ is classified by $BG(N)$ and $\Omega BG(N) = G(N)$ (as H-spaces when $N=\infty$). Let ξ be a spherical fibre space over X. The virtual fibre space $Ep^m(\xi) - Ep^m(\dim\xi)$ is classified by a mapping $Ep^m(X)\to BG$. This induces an H_2^∞-structure on BG,

$$\hat{\theta}_{BG}\colon Ep^m(BG)\to BG.$$

The same construction applies to the other bundle theories, e.g. BO, BU and $BTOP$.

An H_2^m-structure on a space X induces by the following pointwise construction an H_2^m-structure on the loop space: $\theta^\Omega(w;\alpha,\beta)(t) =$

$\theta(w;\alpha(t),\beta(t))$. From Tsuchiya **[28]** we have $\hat{\theta}^{\Omega}_{BG}=\hat{\theta}$ ($\hat{\theta}$ as in Definition 1.3) and similarly for the other bundle theories.

Finally, we make explicit the H_2^∞-structures on the various homogeneous spaces G/TOP, G/O, TOP/O etc. Let $G/TOP(N)$ denote the fibre of the natural mapping from $BTOP(N)$ to $BG(N)$. Then $G/TOP(N)$ classifies $TOP(N)$-bundles equipped with a fibre homotopy framing. Suppose given such a bundle (or rather its associated disc bundle)

$$\begin{array}{l} E \xrightarrow{t} D^N \\ \downarrow \\ X\,, \end{array}$$

where $N=2^n$, $m+1\leqq n$ and $t:(E_x,\partial E_x)\to(D^N,S^{N-1})$ a homotopy equivalens of pairs for each $x\in X$. We construct a $G/TOP(2N)$ bundle over $Ep^m(X)$ as follows,

$$\begin{array}{l} Ep^m(E) \xrightarrow{Ep^m(t)} Ep^m(D^N) \xrightarrow{\rho} D^{2N} \\ \quad\downarrow \\ Ep^m(X). \end{array} \tag{1.5}$$

Associated with this construction we get a well-defined homotopy class $Ep^\infty(G/TOP)\to G/TOP$ which is an H_2^∞-structure with underlying H-space $(G/TOP,\oplus)$, $\oplus=$ Whitney sum of pairs (E,t). The other homogeneous spaces are treated in a similar fashion. We note that it is a direct consequence of the constructions that the various natural maps between the spaces under consideration are H^∞-maps (compare the introduction of **[8]**).

LEMMA 1.4. *The H_2^∞-structures above are all H^∞-structures.*

Proof. Let V be an inner product space isomorphic to $\mathbf{R}^\infty$ or one of the subspaces $\mathbf{R}^n$. It is a fundamental fact of **[4]** that the space $\mathcal{I}(V,\mathbf{R}^\infty)$ of isometric embeddings is contractible. The mapping constructed in (1.3) may be considered an equivariant map

$$\rho: S^\infty\to\mathcal{I}(\mathbf{R}^\infty\oplus\mathbf{R}^\infty,\mathbf{R}^\infty)$$

Now, $S^\infty\times S^\infty\times S^\infty\cong E(Z_2\int Z_2)$ (equivariantly) and we let

$$\varphi: S^\infty\times S^\infty\times S^\infty\to\mathcal{I}((\mathbf{R}^\infty)^4,\mathbf{R}^\infty)$$

be the map

$$\varphi(w;w_1,w_2)(u_1,u_2,u_3,u_4)=\rho(w;\rho(w_1;u_1,u_2),\ \rho(w_2;u_3,u_4))$$

since $\mathcal{T}((\mathbf{R}^\infty)^4,\mathbf{R}^\infty)$ is contractible and since $E\mathcal{S}_4$ and $E(Z_2\int Z_2)$ are free $\mathcal{S}_4$ and $Z_2\int Z_2$ spaces, respectively there exists a mapping

$$\bar{\varphi}\colon E\mathcal{S}_4\to\mathcal{T}((\mathbf{R}^\infty)^4,\mathbf{R}^\infty)$$

extending φ (up to equivariant homotopy). We define $\theta_4\colon E\mathcal{S}_4\times_{\mathcal{S}4}\tilde{G}^4\to\tilde{G}$ in analogy with Definition 1.3,

$$\theta_4(w;f_1,\cdots,f_4)=\bar{\varphi}(w)(f_1*\cdots*f_4)\ \bar{\varphi}(w)^{-1}.$$

It is now obvious that (1.1) is commutative so that $\tilde{G}$ is an H^∞-space. Essentially the same argument applies in all the other cases under consideration.

Boardman and Vogt **[4]** have introduced Ω^∞-structures on all the spaces (except $\tilde{G}$ which is not infinite loop space) we have given H_2^∞-structures. Boardman **[5]** pointed out to us

THEOREM 1.5. (Boardman) *The H^∞-structures above are induced from the Boardman-Vogt infinite loop space structures.*

Let $\tilde{F}(N)$ denote the space of basepoint preserving maps $S^N\to S^N$ and $\tilde{F}=\tilde{F}(\infty)=\lim\tilde{F}(N)$. The fibration $\tilde{F}(N)\to\tilde{G}(N+1)\xrightarrow{\text{eval}}S^N$ shows that $\tilde{F}\simeq\tilde{G}$. Since $\tilde{F}(N)=\Omega^NS^N$, $\tilde{F}=\lim\Omega^NS^N=Q(S^0)$. The space $\tilde{F}$ has an obvious Ω^∞-structure, called the *loop-structure*. The kth space in the defining Ω-spectrum is $\lim\Omega^{N-k}S^N$. The underlying H-space structure on $\tilde{F}$ is the loop sum of maps inducing sum in stable homotopy.

We close this section by listing diagrams (due to Milgram, May and Tsuchiya) which exploit the distributivity of the composition product over the loop sum. Suppose that X is an infinite loop space. Then X admits a right $\tilde{F}$-action, $c\colon X\times\tilde{F}\to X$ ($X=\Omega^nB^nX$ and $\Omega^nB^nX\times\tilde{F}(n)\to\Omega^nB^nX$ by composition). Let $\mu\colon X\times X\to X$ denote the underlying multiplication. Then (Milgram **[20]**)

$$\begin{array}{lccccc}
 & (X\times X)\times\tilde{F} & \xrightarrow{1\times\Delta} & X\times X\times\tilde{F}\times\tilde{F} & \longrightarrow & X\times\tilde{F}\times X\times\tilde{F}\\
(1.6) & \downarrow\mu\times1 & & & & \downarrow c\times c\\
 & X\times\tilde{F} & \xrightarrow{c} & X & \xleftarrow{\mu} & X\times X
\end{array}$$

is commutative.

Generalizing (1.6) we have the commutative diagram due to May

$$(1.7)\qquad \begin{array}{ccccc} Ep^{\infty}(X)\times\tilde F & \xrightarrow{1\times\Delta} & EP^{\infty}(X)\times\tilde F\times\tilde F & \longrightarrow & Ep^{\infty}(X\times\tilde F) \\ \downarrow \theta\times 1 & & & & \downarrow Ep^{\infty}(c) \\ X\times\tilde F & \xrightarrow{c} & X & \xleftarrow{\theta} & Ep^{\infty}(X) \end{array}$$

The next two diagrams, which express $\hat\theta\colon Ep^{\infty}(\tilde F)\to\tilde F$ (Definition 1.3, $\tilde F=\tilde G$) on a loop sum are due to Tsuchiya **[28]**. First, note that $S^{N-1}*S^{N-1}$ is equivariantly equivalent to $S(S^{N-1}\wedge S^{N-1})$. Therefore $\hat\theta$ in Definition 1.3 can be considered as a mapping

$$\hat\theta\colon Ep^{m}(\tilde F(N-1))\to\tilde F(2N-1);$$

$\hat\theta(w;f,g)$ is the composite

$$S(S^{N-1}\wedge S^{N-1})\xrightarrow{\rho(w)^{-1}}S(S^{N-1}\wedge S^{N-1})\xrightarrow{S(f\wedge g)}S(S^{N-1}\wedge S^{N-1})\xrightarrow{\rho(w)}S(S^{N-1}\wedge S^{N-1})$$

Consider two copies S_i^{∞} of S^{∞} and two set of elements $f_i,g_i\in\tilde F$, $i=1,2$. Let

$$T_1=S_1^{\infty}\wedge S_1^{\infty},\ T_2=S_2^{\infty}\wedge S_2^{\infty},\ T_3=(S_1^{\infty}\wedge S_2^{\infty})\vee(S_2^{\infty}\wedge S_1^{\infty})$$

and let $h\colon S^{\infty}\wedge S^{\infty}\to T_1\vee T_2\vee T_3$ be the composite

$$S^{\infty}\wedge S^{\infty}\xrightarrow{\nabla\wedge\nabla}(S_1^{\infty}\vee S_2^{\infty})\wedge(S_1^{\infty}\vee S_2^{\infty})\to T_1\vee T_2\vee T_3\xrightarrow{k}T_1\vee T_2\vee T_3,$$

where $k=(f_1\wedge g_1)\vee(f_2\wedge g_2)\vee(f_1\wedge g_2\vee f_2\wedge g_1)$. Let $p_i\colon T_1\vee T_2\vee T_3\to T_i$ be the "projection" and $d\colon T_3\to S^{\infty}\wedge S^{\infty}$ the folding map. We define

$$\vartheta_i\colon\tilde F^4\to\tilde F$$

by

$$\vartheta_i(f_1,f_2,g_1,g_2)=Sp_i\circ Sh\quad\text{for}\quad i=1,2$$
$$\vartheta_3(f_1,f_2,g_1,g_2)=Sd\circ Sp_3\circ Sh$$

In analogy with the definition of $\hat\theta$ we let

$$\hat\theta_i\colon Ep^{\infty}(\tilde F^2)\to\tilde F$$

be equal to

$$\hat\theta_i(w;(f_1,f_2),(g_1,g_2))=\rho(w)\circ\vartheta_i(f_1,f_2,g_1,g_2)\circ\rho(w)^{-1}.$$

The diagram below is homotopy commutative

$$
(1.8)\qquad
\begin{array}{ccc}
Ep^\infty(\tilde F^2) & \xrightarrow{\text{diag.}} & Ep^\infty(\tilde F^2)^3 \\
\downarrow Ep^\infty(l) & & \downarrow \hat\theta_1\times\hat\theta_2\times\hat\theta_3 \\
Ep^\infty(\tilde F) & \xrightarrow{\theta} \tilde F \xrightarrow{l(l\times 1)} & \tilde F^3.
\end{array}
$$

Here $l: \tilde F\times\tilde F\to\tilde F$ is the loop sum H-space structure. The map $\hat\theta_3(w;1,1,1,1): S^\infty\to\tilde F$ factors over RP^∞, say $\xi: RP^\infty\to\tilde F$, and we have the homotopy commutative diagram

$$
(1.9)\qquad
\begin{array}{ccc}
RP^\infty\times\tilde F^2 & \xrightarrow{1\times\Delta} & Ep^\infty(\tilde F^2) \\
\downarrow \xi\times Id & & \downarrow \hat\theta_3 \\
\tilde F\times\tilde F^2 & \xrightarrow{c(1\times c)} & \tilde F
\end{array}
$$

The category of H^∞-spaces admits products. If X and Y are H^∞-spaces then $X\times Y$ is an H^∞-space as follows

$$
Ep^\infty(X\times Y)\xrightarrow{D} Ep^\infty(X)\times Ep^\infty(Y)\xrightarrow{\theta\times\theta'} X\times Y,
$$

where $D(w;(x_1,y_1),(x_2,y_2))=((w;x_1,x_2),(w;y_1,y_2))$. We note that the diagonal $\Delta: X\to X\times X$ is always an H^∞-map and that the composition product and loop product on $\tilde F$ are H^∞-maps in the composition structure and loop structure, respectively.

2. Algebraic formulas. Every H^∞-structure on a space X gives rise to homology operations, that is, natural homomorphisms $Q^i: H_*(X)\to H_*(X)$ of degree i. These operations were first defined by Araki and Kudo **[3]** and later considered in more detail by Dyer and Lashof **[11]**. Before we give the definition we shall recall some facts about the homology of $Ep^m(X)$.

Let W be the standard $Z_2[Z_2]$-free resolution of Z_2; W_n a copy of the group ring $Z_2[Z_2]$ with generator, say e_n, $\partial e_n=(1+T)e_{n-1}$. The Eilenberg-Zilber map defines a chain equivalence

$$
W\underset{Z_2[Z_2]}{\otimes} C_*(X)\otimes C_*(X)\to C_*(Ep^\infty(X)),
$$

or more generally, a chain equivalence

$$
W^{[m]}\underset{Z_2[Z_2]}{\otimes} C_*(X)\otimes C_*(X)\to C_*(Ep^m(X)),
$$

where $W^{[m]}$ is the m-skeleton of W, $W^{[m]} = \bigotimes_{i \leqq m} W_i$, and $C_*(X)$ the chains of X. Every cycle $\bar{x} \in C_*(X)$ gives rise to cycles $e_p \otimes \bar{x} \otimes \bar{x}$ in $W^{[m]} \otimes_{Z_2[Z_2]} C_*(X) \otimes C_*(X)$ and therefore in $C_*(Ep^m(X))$. If $x \in H_*(X)$, then we write $e_p \otimes x \otimes x$ for the homology class of $e_p \otimes \bar{x} \otimes \bar{x}$ in $H_*(Ep^m(X))$, where $\bar{x}$ is a cycle representing x.

Similarly if $\bar{x}$ and $\bar{y}$ are cycles representing x and y in $H_*(X)$ then $e_m \otimes \bar{x} \otimes \bar{y} + e_m \otimes \bar{y} \otimes \bar{x}$ is a cycle in $W^{[m]} \otimes_{Z_2[Z_2]} C_*(X) \otimes C_*(X)$ whence a cycle in $C_*(Ep^m(X))$. The associated cohomology class is denoted $e_m \otimes x \otimes y + e_m \otimes y \otimes x$ or $e_m \otimes [x, y]$. Finally if $x, y \in H_*(X)$ then there is a class $e_0 \otimes x \otimes y$ in $H_*(Ep^m(X))$. We recall that, if $\{x_i\}$ is an additive basis of $H_*(X)$ then

$$e_0 \otimes x_i \otimes x_j,\ e_b \otimes x_i \otimes x_i,\ e_m \otimes [x_i, x_j];\ i < j \quad \text{and} \quad b \leqq m$$

is an additive basis for $H_*(Ep^m(X))$.

For every space X, $H_*(X)$ is a left A^{0p} module where A^{0p} denotes the opposite mod. 2 Steenrod algebra. Nishida **[22]** has computed the action of A^{0p} on $H_*(Ep^\infty(X))$. We shall need a slight extension of this namely to the finite case $H_*(Ep^m(X))$. The result is ($s = \deg x$)

(i) $$Sq^a(e_b \otimes x \otimes x) = \sum_{i \geqq 0} \binom{b - a + s}{a - 2i} e_{b-a+2i} \otimes Sq^i x \otimes Sq^i x + D_b(x),$$

where $D_b(x) = 0$ for $b < m$ and

$$D_m(x) = \sum_{i < [a/2]} e_m \otimes [Sq^i x, Sq^{a-i} x]. \tag{2.1}$$

Further,

(ii) $$Sq^a(e_0 \otimes x \otimes y) = \sum_{i \geqq 0} e_0 \otimes Sq^i x \otimes Sq^{a-i} y,$$

(iii) $$Sq^a(e_m \otimes [x, y]) = \sum_{i \geqq 0} e_m \otimes [Sq^i x, Sq^{a-i} y].$$

Here $\binom{i}{j}$ is the binomial coefficient (in Z_2) $i!/(j!(i-j)!)$. The proof of (i) in the case $b = m$ uses the computation of Nishida together with the observation that

$$(Ep^m(X), Ep^{m-1}(X)) \cong (D^m \times X \times X, S^{m-1} \times X \times X)$$

Also, (ii) and (iii) are obvious using the maps $X \times X \to Ep^m(X)$ and $S^m \times X \times X \to Ep^m(X)$. We leave the details to the reader.

The following formula, which is almost the definition of the Steenrod squares (see Steenrod-Epstein **[24]**), is of importance for our sequel computations. Let $d\colon S^\infty \times_{Z_2} X \to Ep^\infty(X)$ be the diagonal $d(w, x) = (w, x, x)$. Now $S^\infty \times_{Z_2} X = RP^\infty \times X$ and

$$d_*(e_b \otimes x) = \sum_{i \geqq 0} e_{b-s+2i} \otimes Sq^i x \otimes Sq^i x, \qquad s = \deg x. \tag{2.2}$$

Finally we list the coproduct in $H_*(Ep^m(X))$. Let $x, y \in H_*(X)$ with coproduct $\psi(x) = \Sigma x'_i \otimes x''_i$ and $\psi(y) = \Sigma y'_i \otimes y''_i$. Then

(2.3)
$$\psi(e_b \otimes x \otimes x) = \sum_{c=0}^{b} \sum_j (e_c \otimes x'_i \otimes x'_j) \otimes (e_{b-c} \otimes x''_i \otimes x''_j) + \bar{D}_b(x),$$

where $\bar{D}_b(x) = 0$ for $b < m$ and

$$\begin{aligned}\bar{D}_m(x) = \Sigma (e_m &\otimes [x'_i, x'_j]) \otimes (e_0 \otimes x''_i \otimes x''_j) \\ &+ (e_0 \otimes x'_i \otimes x'_j) \otimes (e_m \otimes [x''_i, x''_j]).\end{aligned}$$

Further,

$$\begin{aligned}\psi(e_m \otimes [x, y]) = \Sigma (e_m &\otimes [x'_i, x'_j]) \otimes (e_0 \otimes x''_i \otimes y''_j) \\ &+ \Sigma (e_0 \otimes x'_i \otimes y'_j) \otimes (e_m \otimes [x''_i, y''_j])\end{aligned}$$

and

$$\psi(e_0 \otimes x \otimes y) = \Sigma (e_0 \otimes x'_i \otimes y'_j) \otimes (e_0 \otimes x''_i \otimes \otimes y''_j).$$

The proof of (2.3) is standard and left to the reader.

Now, if $\theta: E_p^\infty(X) \to X$ is an H^∞-structure, define $Q^b: H_*(X) \to H_*(X)$, as

$$Q^b(x) = \theta_*(e_{b-s} \otimes x \otimes x), \qquad s = \deg x.$$

The Q^b is a homomorphism and natural with respect to induced H^∞-maps. We summarize some of the most important properties (compare Dyer-Lashof **[11]** and May **[17]**).

(Evaluation)

(2.4)
$$Q^b(x) = 0 \text{ if } b < \deg x$$
$$Q^b(x) = x^2 \text{ if } b = \deg x$$

(Stability)

(2.5)
$$\sigma_* Q^b(x) = Q^b \sigma_*(x).$$

where $\sigma_*: IH_*(\Omega X) \to H_*(X)$ is the homology suspension and the H^∞-structure on ΩX is the obvious one (cf. §1). From (2.1) we get

(Nishida relation)

(2.6)
$$Sq^a Q^b(x) = \sum_{t \geq 0} \binom{b-a}{a-2t} Q^{b-a+t}(Sq^t x).$$

(Coproduct formula)

(2.7)
$$\psi Q^b(x) = \sum Q^i(x'_j) \otimes Q^{b-i}(x''_j),$$

when $\psi(x) = \Sigma x'_j \otimes x''_j$.

From the "Adem-diagram" (see definition 1.2) one gets (Adem-relation)

(2.8)
$$Q^a Q^b(x) = \sum_{3t \geqq a+b} \binom{t-b-1}{2t-a} Q^{a+b-t} Q^t(x) \qquad (a > 2b).$$

This follows essentially from Adem's original computation of the Adem relations for the Steenrod operations. (For details see **[11]** and **[17]**).

If the H-map $X \times X \to X$ is an H^∞-map (and this is always the case if the H^∞-structure is associated to an Ω^∞-structure on X), then we have (**[11]**)

(Cartan-formula)

(2.9)
$$Q^b(xy) = \sum_{i=0}^{b} Q^i(x)\, Q^{b-i}(y).$$

Suppose now that θ: $Ep^\infty(X) \to X$ is induced from an Ω^∞-structure on X. If $x, y \in H_*(X)$ write $x * y$ for their Pontrjagin product. If $f \in H_*(\tilde{F})$, let xf be the the composition product, $xf = c_*(x \otimes f)$. The diagram (1.6) may be "evaluated" as follows (see Milgram **[21]**).

(2.10)
$$(x * y)f = \sum_i xf'_i * yf''_i,$$

where $\psi(f) = \Sigma f'_i \otimes f''_i$. Diagram (1.7) gives

(May's formula)

(2.11)
$$Q^i(x)f = \sum_i Q^{i+t}(xSq^t f).$$

Since this evaluation is somewhat more difficult than the previous ones and has not yet appeared in print, we go through it in some detail (the argument is due to May). The problem is to evaluate the upper horizontal line in (1.7). To this end there is a commutative diagram

(2.12)
$$\begin{array}{ccc} Ep^\infty(X) \times \tilde{F} & \xrightarrow{\Delta} & Ep^\infty(X \times \tilde{F}) \\ \downarrow D_1 & & \downarrow D_2 \\ Ep^\infty(X) \times (S^\infty \times_{Z_2} \tilde{F}) & \xrightarrow{1 \times d} & Ep^\infty(X) \times Ep^\infty(\tilde{F}), \end{array}$$

where

$$D_1((w;x,y),f) = ((w;x,y),(w,f)),$$
$$D_2(w;(x,f_1),(y,f_2)) = ((w;x,y),(w;f_1,f_2)),$$
$$\Delta((w;x,x),f) = (w;(x,f),(x,f)).$$

On homology level we have

$$D_{1*}((e_b \otimes x \otimes x) \otimes f) = \sum_i (e_{b-i} \otimes x \otimes x) \otimes (e_i \otimes f)$$

$$D_{2*}(e_b \otimes (x \otimes f) \otimes (x \otimes f)) = \sum_i (e_{b-i} \otimes x \otimes x) \otimes (e_i \otimes f \otimes f)$$

and by (2.2)

$$d_*(e_b \otimes x) = \sum_t e_{b+2t-s} \otimes Sq^t x \otimes Sq^t x, \qquad s = \deg x.$$

We notice that D_{2*} is a monomorphism so that it is enough to evaluate $((1 \times d) \circ D_1)_*$. We get

$$((1 \times d)D_1)_*((e_{b-q} \otimes x \otimes x) \otimes f)$$
$$= D_{2*}(e_{b-q+2t-s} \otimes (x \otimes Sq^t f) \otimes (x \otimes Sq^t f)),$$

$\deg x = q$, $\deg f = s$, and therefore using (2.12)

$$\Delta_*((e_{b-q} \otimes x \otimes x) \otimes f) = \sum_t e_{b-q+s+2t} \otimes (x \otimes Sq^t f) \otimes (x \otimes Sq^t f).$$

We can now complete the evaluation of (1.7). It is clear that

$$(c \circ (\theta \times 1))_*((e_{b-q} \otimes x \otimes x) \otimes f) = Q^b(x)f$$

and from above we get that

$$(\theta \circ Ep^\infty(c) \circ \Delta)_* ((e_{b-q} \otimes x \otimes x) \otimes f) = \sum_t Q^{b+t}(xSq^t f).$$

This completes the proof of (2.11).

The rest of this section is devoted to the proof of the "mixed Cartan formula," which evaluates the composition operations on loop products.

NOTATION. The product and operations associated with the loop-structure on $\tilde{F}(=\tilde{G})$ will be denoted $x * y$ and $Q^a(x)$, respectively. The product and operations associated to the composition structure is denoted $x \cdot y$ (or just xy) and $\hat{Q}^a(x)$.

For $k \in Z$, let $[k] \in H_0(\tilde{F}_k)$ be the nonzero element in the image of $H_0(\tilde{F}_k) \to H_0(\tilde{F})$, $\tilde{F}_k$ the kth component of $\tilde{F}$. Then [0] is the unit element of the loop product and [1] the unit element of the composition product. Further, if $a > 0$ then $Q^a([0]) = 0$, $\hat{Q}^a([1]) = 0$ and $\hat{Q}^a([0]) = 0$.

In dimension zero we get,

$$[k] * [l] = [k + l],\ [k] \cdot [l] = [kl]$$
$$\hat{Q}^0[k] = k^2,\ Q^0[k] = [2k]$$

In general if $x \in H_*(F_k) \subset H_*(\tilde{F})$ then $Q^b(x) \in H_*(\tilde{F}_{2k})$ and $\hat{Q}^b(x) \in H_*(\tilde{F}_{k^2})$. All components of $\tilde{F}$ belongs to the same homotopy type. On the algebraic side $[k] * (\ ): H_*(\tilde{F}_0) \to H_*(\tilde{F}_k)$ is an isomorphism.

In §1 we defined maps

$$\hat{\theta}_i: Ep^\infty(\tilde{F}^2) \to \tilde{F}, \qquad i = 1, 2 \text{ and } 3.$$

We get associated operations

$$\hat{Q}^b_i: H_*(\tilde{F}^2) \to \tilde{F},$$

$\hat{Q}^b_i(x \otimes y) = \hat{\theta}_{i*}(e_{b-\deg x-\deg y} \otimes (x \otimes y) \otimes (x \otimes y))$. It is an obvious consequence of the definitions that

$$\hat{Q}^b_1(x \otimes y) = \epsilon(y) \cdot \hat{Q}^b(x)$$
$$\hat{Q}^b_2(x \otimes y) = \epsilon(x) \cdot \hat{Q}^b(y),$$

$\epsilon: H_*(\tilde{F}) \to Z_2$ the augmentation. With this in mind one can now evaluate diagram (1.8). The proof is a simple application of (2.3).

PROPOSITION 2.1. *If $x, y \in H_*(\tilde{G})$ are classes with coproduct $\psi(x) = \Sigma x'_i \otimes x''_i$, $\psi(y) = \Sigma y'_j \otimes y''_j$ then*

$$\hat{Q}^a(x * y) = \sum \hat{Q}^{a_1}(x'_j) * \hat{Q}^{a_2}(y'_j) * \hat{Q}^{a_3}_3(x''_i \otimes y''_j),$$

the summation runs over all pairs (i, j) and triples (a_1, a_2, a_3) with $a_1 + a_2 + a_3 = a$.

The evaluation of (1.9) is essentially the same as that of (1.7) leading to May's formula. We leave the details to the reader, and just state the result.

PROPOSITION 2.2. *If $x, y \in H_*(\tilde{G})$ then*

$$\sum \hat{Q}_3^{a+t}(Sq^t(x \otimes y)) = \hat{Q}_3^a([1] \otimes [1])xy.$$

Finally, we shall make use of the inclusion map $J\colon O \to G$ (which is an H^∞-map) *to evaluate $\hat{Q}_3^b([1] \otimes [1])$. Recall the following facts:*

(i) $H_*(SO) = E\{u_1, u_2, \cdots\}$, $\psi(u_n) = \Sigma\, u_i \otimes u_{n-i}$

(ii) $H_*(BO) = Z_2[b_1, b_2, \cdots]$, $\psi(b_n) = \Sigma\, b_i \otimes b_{n-i}$

(iii) *There is exactly one nonzero primitive element s_n in $H_n(BO)$ and*

$$Sq^i s_{n+1} = \binom{n-i}{i} s_{n-i+1}.$$

(iv) *The homology suspension $\sigma_*\colon QH_*(SO) \to PH_*(BSO)$ is an isomorphism.*

Let $[-1] \in H_0(O)$ be the element with $J_*([-1]) = [-1]$.

LEMMA 2.3. *For all n, $\hat{Q}^n([-1]) = u_n$.*

Proof. A simple argument shows that $\sigma_*([-1]+[1]) = b_1$, $\sigma_*\colon IH_*(O) \to H_*(BO)$ the homology suspension. Hence by stability of the operations (and since $\hat{Q}^n([1]) = 0$, $n > 0$) $\sigma_*(\hat{Q}^n[-1]) = \hat{Q}^n b_1$. Now b_1 is primitive so that $\hat{Q}^n(b_1) = \lambda_{n+1} s_{n+1}$, $\lambda_{n+1} \in Z_2$. The Nishida relations together with (iii) above imply that λ_n is independent of n and therefore equal to 1 ($\lambda_1 = 1$, obviously). From (iv) above, $u_n + \hat{Q}^n[-1]$ is decomposable. Inductively we may assume $u_n + \hat{Q}^n[-1]$ is also primitive, but $H_*(SO)$ contains no nonzero decomposable primitives. This completes the proof.

Let $\chi\colon H_*(\tilde{G}) \to H_*(\tilde{G})$ be the canonical antiautomorphism in the Hopf algebra $(H_*(\tilde{G}), *)$. Since $Q^a([1] * [-1]) = \Sigma_{i=0}^a Q^i[1] * Q^{a-i}[-1]$ by the Cartan formula and since $Q^a([0]) = 0$ for $a > 0$ we see that

$$\chi(Q^a[1] * [-1]) = Q^a[-1] * [1].$$

Furthermore, if we set $x = [1]$, $y = [-1]$ in Proposition 2.1 then we get (since $\hat{Q}^a([0]) = 0$) that

$$\hat{Q}^a([-1]) * [-1] = \chi \hat{Q}_3^a([1] \otimes [-1]).$$

LEMMA 2.4. *For $x, y \in H_*(\tilde{G})$, $\hat{Q}_3^a(x \otimes y) = Q^a(xy)$.*

Proof. In light of May's formula (2.11) and Proposition 2.7 it is enough to prove that $Q^a([1]) = \hat{Q}_3^a([1] \otimes [1])$. From Milgram's and

May's computation of $H_*(SG)$ we know that $J_*(u_a) = Q^a[1] * [-1] \in H_*(SG)$, and therefore (Lemma 2.3) $\hat{Q}^a[-1] = Q^a[1] * [-1]$. The remarks we made above for the canonical antiautomorphism then complete the proof.

REMARK. There are now other approaches to showing that $\hat{Q}^a_3(x \otimes y) = Q^a(xy)$. First, it is enough to prove this for $y = [1]$. Secondly, $\hat{Q}^a_3(x \otimes 1)$ is the operation associated to the H^∞-structure $\hat{\theta}_3$: $E^\infty_p(\tilde{F}) \to \tilde{F}$ defined as follows: $\hat{\theta}_3(w; f_1, f_2)$ is the composite

$$S^\infty \xrightarrow{\rho(w)^{-1}} S^\infty \xrightarrow{\nabla} S^\infty \vee S^\infty \xrightarrow{f_1 \vee f_2} S^\infty \vee S^\infty \to S^\infty \xrightarrow{\rho(w)} S^\infty.$$

This is obviously an H^∞-structure extending the loop sum H-space structure. May **[18]** essentially takes this as a definition of the H^∞-structure associated to the infinite loop space structure on $Q(S^0)$. He then proves that this is the same as the H^∞-structure defined by Dyer and Lashof **[11]**.

We finally reformulate Proposition 2.1 to

THEOREM 2.5. (mixed Cartan formula). *Let $x, y \in H_*(\tilde{G})$ be elements with coproduct $\psi(x) = \Sigma x'_i \otimes x''_i$, $\psi(y) = \Sigma y'_j \otimes y''_j$. Then*

$$\hat{Q}^a(x * y) = \sum \hat{Q}^{a_1}(x'_i) * \hat{Q}^{a_2}(y'_j) * Q^{a_3}(x''_i y''_j),$$

where the sum is over all (i, j) and (a_1, a_2, a_3) with $a = a_1 + a_2 + a_3$.

3. The Dyer-Lashof algebra and its dual. Let $\mathcal{F}$ be the free graded associative algebra with unit generated by the symbols $Q^0, Q^1, \cdots, Q^i, \cdots$ where $\deg Q^i = i$. For any string of nonnegative integers $I = (i_1, \cdots, i_n)$, define $Q^I = Q^{i_1} \cdots Q^{i_n}$. We say that Q^I (or I) is *allowable* if $i_1 \leqq 2i_2$, $i_2 \leqq 2i_3, \cdots, i_{n-1} \leqq 2i_n$, and define the *excess* of Q^I (or I) to be

$$\operatorname{exc}(Q^I) = \sum_{j=1} i_j - 2i_{j+1} = i_1 - \sum_{j=2} i_j.$$

The *length* of Q^I, $l(Q^I)$, is the number of integers in I, i.e., $l(Q^I) = l(I) = n$ if $I = (i_1, \cdots, i_n)$. The *degree* of Q^I is Σi_j.

Let $\mathcal{I}$ be the ideal generated by the elements

(i) $r(a, b) = Q^a Q^b + \sum_t \binom{t-b-1}{2t-a} Q^{a+b-t} Q^t$, $a > 2b$.

(ii) Q^I, with $\operatorname{exc}(I) < 0$.

We notice that if $\theta: Ep^\infty(X) \to X$ is any H^∞-structure then every element of $\mathcal{I}$ acts on $H_*(X)$ as the zero homomorphism. In analogy with the case of cohomology operations we define the algebra of homology operations, $R = \mathcal{F}/\mathcal{I}$. We adopt May's terminology and call R the Dyer-Lashof algebra. Just as for the Steenrod algebra, the allowable elements of nonnegative excess form an additive basis for R.

Since the same H-space X can have serveral H^∞-structures the algebra R can act on $H_*(X)$ in more than one way.

Let us consider $\tilde{G}$ with the loop-structure and associated R-module structure. The computations of Dyer and Lashof (compare (5.1)) show that the evaluation map

$$e: R \to H_*(\tilde{G}), \quad e(r) = r([1]), \tag{3.1}$$

is a monomorphism (in fact, Im $e * H_0(\tilde{G}) = H_*(\tilde{G})$). It is a consequence (pointed out by J. P. May) that the ideal $\mathcal{I} \subset \mathcal{F}$ above is exactly the ideal of "universal" relations.

The free algebra $\mathcal{F}$ has, of course, the structure of a Hopf algebra, $\psi(Q^n) = \sum_{i=0}^n Q^i \otimes Q^{n-i}$. Either using the evaluation map e or by direct inspection one sees that $\mathcal{I}$ is a Hopf ideal, so that R becomes a Hopf algebra. We point out that R is not connected ($Q^0 \neq 1$), in fact R_0 is precisely the polynomial algebra generated by Q^0. There is a left action of A^{0p} on R, where A^{0p} is the opposite Steenrod algebra.

$$(3.2) \qquad \begin{aligned} Sq^a(Q^b) &= \binom{b-a}{a} Q^{b-a} \\ Sq^a(Q^b r) &= \sum \binom{b-a}{a-2t} Q^{b-a+t} Sq^t(r). \end{aligned}$$

(Compare the Nishida relations (2.6)). The evaluation map e shows that this is a legitimate definition.

Let $R[k]$ be the subvector space of R spanned by the elements Q^I of length k. It is clear that $R[k]$ is a sub-coalgebra of R and that it is closed under the left action of A^{0p}. In order to get a good grasp on R we shall now study the "dual Hopf algebra"

$$R^* = \lim_{\leftarrow n} \bigotimes_{k=1}^{n} R[k]^* = \prod_{k=1}^{\infty} R[k]^*.$$

The procedure is analogous to Milnor's procedure for studying the dual of the Steenrod algebra. A main point in Milnor's computation of A^* is the existence of a simple "test module", namely $H^*(RP^\infty)$. In our case there is no such simple test space; in fact the "simplest possible space would be $BO \times Z$", however $H_*(BO \times Z)$ is far too complicated to be of any use. Instead we construct, algebraically, a simple test module M which will then play the role of $H^*(RP^\infty)$.

Set

$$M = Z_2[b_0, b_1, \cdots], \qquad \deg b_i = 2^i - 1. \tag{3.3}$$

and give M the structure of an unstable algebra over the free Hopf algebra $\mathscr{F}$ as follows:

$$\begin{aligned} Q^n(b_i) &= b_i^2 \text{ if } n = 2^i - 1 \\ Q^n(b_i) &= b_{i+1} \text{ if } n = 2^i \\ Q^n(b_i) &= 0 \text{ otherwise} \\ Q^n(b' \cdot b'') &= \sum Q^i(b') \cdot Q^{n-i}(b''). \end{aligned} \tag{3.4}$$

LEMMA 3.1. *The above $\mathscr{F}$-module structure on M factors over R, so that M becomes an unstable algebra over R.*

Proof. From (3.4) we get

$$\begin{aligned} Q^{2n}Q^n(b_i) &= b_i^4 \text{ if } n = 2^i - 1 \\ Q^{2n}Q^n(b_i) &= b_{i+2} \text{ if } n = 2^i \\ Q^{2n+2}Q^n(b_i) &= b_{i+1}^2 \text{ if } n = 2^i - 1. \\ Q^{2n+1}Q^{n+1}(b_i) &= b_{i+1}^2 \text{ if } n = 2^i - 1. \end{aligned} \tag{3.5}$$

We notice that $Q^{2n+2}Q^n + Q^{2n+1}Q^{n+1} = 0$ by an Adem relation, and that except in the four cases in (3.5) $Q^{a_1}Q^{a_2}(b_i) = 0$. Since no Adem relation $r(p,q)$ contains the element $Q^{2n}Q^n$ it follows easily that $r(p,q)(b_i) \neq 0$ implies that $p = 2^{i+1}$, $q = 2^i - 1$, contradicting (3.5). Furthermore it is obvious that $Q^I(b_i) = 0$ if $\operatorname{exc}(Q^I) < 0$.

The proof is now easily completed employing the following two facts

(a) If $\psi(Q^I) = \Sigma\, Q^{I'} \otimes Q^{I''}$ and Q^I has negative excess, then for each term $Q^{I'} \otimes Q^{I''}$ either $Q^{I'}$ or $Q^{I''}$ has the same property.

(b) For each $a > 2b$,

$$\begin{aligned} \psi(r(a,b)) = \sum r(a_1, b_1) \otimes x_1 + \sum x_2 \otimes r(a_2, b_2) \\ + \sum Q^{I_1} \otimes y_1 \otimes \sum y_2 \otimes Q^{I_2}, \end{aligned}$$

where $r(a_1, b_1)$ and $r(a_2, b_2)$ are Adem relations and $\operatorname{exc}(Q^{I_1}) < 0$, $\operatorname{exc}(Q^{I_2}) < 0$. This proves the lemma.

Define sequences I_{ik} of length k as follows:

$$\begin{aligned} &I_{0k} = (0, \cdots, 0) \\ (3.6)\quad &I_{ik} = (2^{k-i-1}(2^i - 1), \cdots, 2(2^i - 1), 2^i - 1, 2^{i-1}, \cdots, 2, 1),\ i < k \\ &I_{kk} = (2^{k-1}, \cdots, 2, 1). \end{aligned}$$

It is easy to see (only using that $Q^I = 0$ in R if $\operatorname{exc}(I) < 0$ and the Adem relation $Q^1Q^0 = 0$) that the elements $Q^{I_{ik}} \in R[k]$ are primitive.

To every allowable sequence I of length k there are unique nonnegative integers λ_i such that $I = \Sigma_{i=1}^k \lambda_i I_{ik}$. We define an ordering in the set of admissible sequences of length k as follows: Write $I = \Sigma \lambda_i I_{ik}$ and $J = \Sigma \mu_i I_{ik}$ then $I \geqq J$ if $(\lambda_1, \cdots, \lambda_k) \geqq (\mu_1, \cdots, \mu_k)$ in the lexicographic ordering from the right.

Let $\xi_{ik} \in R[k]^*$ be dual to $Q^{I_{ik}}$, that is

$$\begin{aligned} &\langle \xi_{ik}, Q^{I_{ik}} \rangle = 1 \\ &\langle \xi_{ik}, Q^I \rangle = 0 \text{ if } I \text{ is allowable, } I \neq I_{ik}. \end{aligned}$$

The squaring map $\zeta^*\colon R[k] \to R[k+1]$, $\zeta^*(x) = Q^{\deg x}(x)$ defines a map of coalgebras, which maps $Q^{I_{ik}}$ to $Q^{I_{i,k+1}}$. Dually $\zeta\colon R[k+1]^* \to R[k]^*$ is a map of algebras which on generators takes the values

$$\begin{aligned} \zeta(\xi_{i,k+1}) &= \xi_{ik} \qquad i \leqq k \\ \zeta(\xi_{k+1,k+1}) &= 0. \end{aligned}$$

Suppose $\Lambda = (\lambda_1, \cdots, \lambda_k)$, $\lambda_i \geqq 0$, and write $\xi^\Lambda = \xi_1^{\lambda_1}, \cdots, \xi_k^{\lambda_k}$. Define $I(\Lambda) = \Sigma \lambda_i I_{ik}$; then $\langle \xi^\Lambda, Q^{I(\Lambda)} \rangle = 1$ but furthermore

LEMMA 3.2. *If $\langle \xi^\Lambda, Q^J \rangle = 0$ and J is allowable, then $J \geqq I(\Lambda)$.*

Proof. As used before, the elements $Q^{2n}Q^n$ never appear in any Adem relation. Therefore $\langle \xi_{kk}^n, Q^K \rangle = 0$, if K is not allowable. Let $J = \Sigma \mu_i I_{ik}$ be any allowable sequence with $\langle \xi^\Lambda, Q^J \rangle = 1$. Then

$$\langle \xi^{\Lambda'}, Q^{J'} \rangle = 1, \quad \text{where} \quad \Lambda' = (\lambda_1, \cdots, \lambda_{k-1}, 0)$$

and $J' = J - \lambda_k I_{kk}$. It follows that $\mu_k \geqq \lambda_k$. If $\mu_k = \lambda_k$ then $Q^J = \zeta^*(Q^{J''})$, $J'' = \Sigma_{i=1}^{k-1} \mu_i I_{i,k-1}$ and since $\zeta(\xi_{ik}) = \xi_{i,k-1}$ we get that $\langle \xi^{\Lambda''}, Q^{J''} \rangle = 1$, $\Lambda'' = (\lambda_1, \cdots, \lambda_{k-1})$. This in turn implies that $\mu_{k-1} \geqq \lambda_{k-1}$. Continuing in this fashion one proves that $(\mu_1, \cdots, \mu_k) \geqq (\lambda_1, \cdots, \lambda_k)$ or in other words, $J \geqq I(\Lambda)$.

COROLLARY 3.3. *$R[k]^* = Z_2[\xi_{ik}, \cdots, \xi_{kk}]$, where $\deg \xi_{ik} = 2^{k-i}(2^i - 1)$.*

Proof. From the previous lemma we have that $Z_2[\xi_{1k}, \cdots, \xi_{kk}]$ is contained in $R[k]^*$; but the dimensions of the two vector spaces are the same.

PROPOSITION 3.4. *The diagonal map $\Phi^*: R^* \to R^* \otimes R^*$ is given by*

$$\Phi^*(\xi_{a,a+b}) = \sum \xi_{l+i,l+i}^{2^{k+j}-2^k} \xi_{i,i+l}^{2^k} \otimes \xi_{j,j+k}$$

where the summation is over (i, j) and (l, k) with $i + j = a$ and $l + k = b$.

Proof. Let M be the R-module defined in (3.2) and (3.4) (compare Lemma 3.1), and let $\lambda: R \otimes M \to M$ be the R-action. Denote by $\lambda^*: M \to M \otimes R^*$ the dual action; explicitly

$$\lambda^*(m) = \sum m_i \otimes \eta_i \text{ if and only if } \lambda(r \otimes m) = \sum \langle r, \eta_i \rangle \cdot m_i$$

for all $r \in R$. Then there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\lambda^*} & M \otimes R^* \\ \downarrow \lambda^* & & \downarrow 1 \otimes \Phi^* \\ M \otimes R^* & \xrightarrow{\lambda^* \otimes 1} & M \otimes R^* \otimes R^* \end{array} \tag{3.7}$$

From (3.5) it is easily seen that if Q^I is allowable and if $Q^I(b_j) = b_{i+j}^{2^k}$ then $I = (2^i - 1)I_{i+k,i+k} + I_{i,i+k}$. Furthermore, Lemma 3.2 implies that if J is allowable and $\langle Q^J; \xi_{i+k,k+i}^{2^j-1} \xi_{i,i+k} \rangle = 1$ then $J = I$. Hence

$$\lambda^*(b_j) = \sum_{i,k} b_{i+j}^{2^k} \otimes \xi_{i+k,k+i}^{2^j-1} \xi_{i,k+i}.$$

The proof is then completed by exploiting the commutativity of (3.7):

$$\begin{aligned} (\lambda^* \otimes 1)\lambda^*(b_0) &= (\lambda^* \otimes 1) \sum_{j,k} b_j^{2^k} \otimes \xi_{j,k+j} \\ &= \sum_{j,k} \sum_{i,l} b_{i+j}^{2^{k+l}} \otimes \xi_{l+i,l+i}^{(2^i-1)2^k} \xi_{i,i+l}^{2^k} \otimes \xi_{j,j+k} \\ &= \sum_{a,c} b_a^{2^c} \otimes \sum_{\substack{i+j=a \\ l+k=c}} \xi_{l+i,l+i}^{2^{k+j}-2^k} \xi_{i,i+l}^{2^k} \otimes \xi_{j,j+k}. \end{aligned}$$

On the other hand

$$(1 \otimes \Phi^*)\lambda^*(b_0) = \sum_{a,c} b_a^{2^c} \otimes \Phi^*(\xi_{a,a+c}).$$

Now, compare the coefficients to $b_a^{2^c}$.

REMARK. It is obvious that $R[k]^* \cdot R[l]^* = 0$ if $k \neq l$. Proposition 3.4 and Corollary 3.3 therefore completely describe the structure of R^*.

We next turn to the structure of $R[k]^*$ as an A-module (A the mod. 2 Steenrod algebra). The commutative diagram

$$\begin{array}{ccc} R[k] & \xrightarrow{e} & H_*(\tilde{G}) \\ \downarrow \psi & & \downarrow \psi \\ R[k] \otimes R[k] & \xrightarrow{e\otimes e} & H_*(\tilde{G}) \otimes H_*(\tilde{G}), \end{array}$$

implies that $\psi: R[k] \to R[k] \otimes R[k]$ is a right A-module map, so that $R[k]^*$ is an algebra over A. Furthermore, it follows that $R[k]$ is an unstable *module* over A, that is, $Sq^a(\xi) = 0$ if $a > \deg \xi$.

PROPOSITION 3.5. *$R[k]^*$ is an unstable A-algebra. The action on the generators is*

(i) $Sq^a(\xi_{ik}) = \xi_{i+1,k}$ *if* $i < k$, $a = 2^{k-i-1}$

(ii) $Sq^a(\xi_{ik}) = \xi_{1k}\xi_{ik}$ *if* $i \leqq k$, $a = 2^{k-1}$

(iii) $Sq^a(\xi_{ik}) = 0$ *if* $a = 2^i$ *but* $a \neq 2^{k-1}, 2^{k-i-1}$.

Proof. Let us write $Q(I)$ instead of Q^I. Then from (3.2),

$$Sq^a(Q(I_{i+1,k})) = Q(I_{i,k}), \quad a = 2^{k-i-1}$$

$$Sq^a(Q(I_{1,k} + I_{i,k})) = Q(I_{i,k}), \quad a = 2^{k-1}.$$

Now, $\xi_{i+1,k}$ is the only element of $R[k]^*$ in degree $2^{k-i-1}(2^{i+1} - 1)$ and $\xi_{1k}\xi_{ik}$ is the only element of $R[k]^*$ in degree $2^{k-1} + 2^{k-i}(2^i - 1)$. This proves (i) and (ii). Further, notice that $R[k]^*$ has no elements of degree $2^j + 2^{k-i}(2^i - 1)$ if $j \neq 2^{k-1}$ and $j \neq 2^{k-i-1}$, so that (iii) follows. The properties (i), (ii), and (iii) imply that $Sq^a(\xi_{ik}) = \xi_{ik}^2$ if $a = \deg(\xi_{ik})$, and therefore that $Sq^a(\xi) = \xi^2$ in general since we have already observed that $R[k]^*$ is an algebra over A which is unstable as a module.

REMARKS. 1. May has generalized the above results to the modulo p case, p an odd prime. The answer is considerably more complicated.

2. T. Sugawara and H. Toda **[27]** have classified unstable polynomial algebras over the Steenrod algebra. The algebras $R[k]^*$ above are all simple of type E_{k+1} in their notation.

The various formulas developed in §2 and §3 can now be conviniently summarized. The group ring $Z_2[Z]$ has two multiplications,

$$[i] * [j] = [i + j] \quad \text{(loop product)},$$

$$[i] \cdot [j] = [ij] \quad \text{(composition product)},$$

and a comultiplication $\psi([i]) = [i] \otimes [i]$.

Consider bigraded vector spaces over Z_2, $A = \bigoplus {}_iA_j$, where i runs over all integers and j over all nonnegative integers. The vector space ${}_iA = \bigoplus_j {}_iA_j$ is called the ith component of A and $A_j = \bigoplus_i {}_iA_j$ are the elements of degree j. We shall assume that A is *connected* in the sence that $A_0 = Z_2[Z]$ and denote by $\epsilon : A \to Z_2[Z]$ the projection of A on A_0.

DEFINITION 3.6. A connected Hopf bialgebra (over Z_2) is a bigraded module as above together with linear (degree preserving) maps

$$* : A \otimes A \to A \quad \text{(loop product)}$$

$$\cdot : A \otimes A \to A \quad \text{(composition product)}$$

$$\psi : A \to A \otimes A \quad \text{(coproduct)}$$

subject to the following conditions

(i) $\epsilon : A \to Z_2[Z]$ preserve all 3 structures

(ii) $(A, *, \psi)$ and $(A, \cdot, \psi)$ are commutative and cocommuative Hopf algebras with unit [0] and [1], respectively

(iii) ${}_iA * {}_jA \subset {}_{i+j}A$, ${}_iA \cdot {}_jA \subset {}_{ij}A$ and $\psi({}_iA) \subset {}_iA \oplus {}_iA$

(iv) The composition product is distributive over the loop product, that is, the diagram

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "*\otimes 1"] \arrow[d, "1\otimes 1\otimes\psi"'] & A \otimes A \arrow[d, "\cdot"] \\
A \otimes A \otimes A \otimes A \arrow[d, "1\otimes T\otimes 1"'] & A \\
A \otimes A \otimes A \otimes A \arrow[r, "\cdot\otimes\cdot"] & A \otimes A \arrow[u, "*"']
\end{tikzcd}$$

is commutative.

It follows from (iv) that $x \cdot [0] = \epsilon(x) \cdot [0]$. All components of A are isomorphic coalgebras since ${}_0A \xrightarrow{*[i]} {}_iA \xrightarrow{*[-i]} {}_0A$ is the identity. Every

Hopf bialgebra gives rise to two (in general nonisomorphic) *connected* Hopf algebras, the zero component $({}_0A, *, \psi)$ and the 1-component $({}_1A, \cdot, \psi)$.

The "ground Hopf bialgebra" admits two actions of the Dyer-Lashof algebra,

$$l : R \otimes Z_2[Z] \to Z_2[Z] \qquad \text{(loop action)},$$
$$c : R \otimes Z_2[Z] \to Z_2[Z] \qquad \text{(composition action)},$$

defined by

$$l(Q^n \otimes x) = 0 \text{ for } n > 0, \; l(Q^0 \otimes [i]) = [2i] \text{ and}$$
$$c(Q^n \otimes x) = 0 \text{ for } n > 0, \; c(Q^0 \otimes [i]) = [i^2].$$

DEFINITION 3.7. A connected quasi R-Hopf bialgebra (over Z_2) is a connected Hopf bialgebra $(A, *, \cdot, \psi)$ together with two module structures

$$l : R \otimes A \to A \qquad \text{(loop action)}$$
$$c : R \otimes A \to A \qquad \text{(composition action)}$$

subject to the following conditions

(i) $\epsilon : A \to Z_2[Z]$ preserves the two actions
(ii) $(A, *, \psi)$ is an unstable Hopf algebra over R via l
(iii) $(A, \cdot, \psi)$ is an unstable Hopf algebra over R via c.
(iv) $l(R \otimes {}_iA) \subset {}_{2i}A$, $c(R \otimes {}_iA) \subset {}_{i^2}A$
(v) The diagram below is commutative

$$\begin{array}{lcl}
R \otimes A \otimes A & \xrightarrow{1 \otimes *} & R \otimes A \\
\quad \downarrow \psi & & \quad \downarrow c \\
R \otimes R \otimes R \otimes A \otimes A \otimes A \otimes A & & A \\
\quad \downarrow S & & \quad \uparrow * \\
(R \otimes A) \otimes (R \otimes A) \otimes (R \otimes A \otimes A) & & A \otimes A \\
\quad \downarrow 1 \otimes 1 \otimes (1 \otimes \cdot) & & \quad \uparrow * \otimes 1 \\
(r \otimes A) \otimes (R \otimes A) \otimes (R \otimes A) & \xrightarrow{c \otimes c \otimes l} & A \otimes A \otimes A
\end{array}$$

where $\psi = (\psi_R \otimes 1) \circ \psi_R \circ \psi_A \circ \psi_A$ and S is the obvious rearrangement of factors,

$$S(r_1 \otimes r_2 \otimes r_3 \otimes a_1 \otimes a_2 \otimes a_3 \otimes a_4)$$
$$= (r_1 \otimes a_1) \otimes (r_2 \otimes a_2) \otimes (r_3 \otimes a_3 \otimes a_4).$$

Diagram (v) is called the mixed Cartan formula. In the sequel we shall always write $Q^n(a)$ for $l(Q^n \otimes a)$ and $\hat{Q}^n(a)$ for $c(Q^n \otimes a)$.

REMARK. The reason for the terminology *quasi* R-Hopf bialgebra is that the term R-Hopf bialgebra should probably be reserved for the algebraic structure with includes the complicated formula, due to May and Tsuchiya, for evaluating $\hat{Q}^n(Q^m(x))$ (compare Proposition 4.11 and the remark following it).

We can now reformulate the major results of §2 in

THEOREM 3.8. $H_*(Q(S^0))$ *is a connected quasi*R*-Hopf bialgebra*

REMARK. M. Herrero **[13]** showed that $H_*(BO \times \mathbf{Z})$ is a connected quasi Hopf bialgebra; the loop product is induced from Whitney sum and the composition product from tensor product. J. P. May has informed us that a theory for homotopy — everything ring spaces has recently been developed. The homology of such a space is an R-Hopf bialgebra.

4. The homology operations in G. In this section we shall exploit the mixed Cartan formula proved in §2. Dyer and Lashof **[11]** computed $H_*(\tilde{G})$ as an algebra under the loop product and Milgram in the fundamental paper **[21]** showed how to transfer this information to $H_*(G)$ considered as an algebra with the composition product. The result of Dyer and Lashof is

$$H_*(\tilde{G}_0) = Z_2[Q^I[1] * [-2^{l(I)}] \mid I \text{ allowable, } \operatorname{exc}(I) > 0]. \tag{4.1}$$

Since the generators of $H_*(\tilde{G}_0)$ are expressed in terms of the loop-operations, the action of the loop-operations $Q^i : H_*(\tilde{G}_0) \to H_*(\tilde{G}_0)$ can be computed from the Adem relations and the Cartan formula (for the loop structure). To compute the "composition-operations" $\hat{Q}^i : H_*(\tilde{G}) \to H_*(\tilde{G})$ it is therefore sufficient to express $\hat{Q}^i$ in terms of Q^i. The mixed Cartan formula is a generalization of Milgram's formula (2.6) and our computations should be viewed as an extension of Milgram's way of computing $H_*(G)$.

Due to the considerable algebraic complications of the formulas, it is convenient to introduce a filtration of $H_*(\tilde{G})$. Following May, define a weight function,

$$(4.2)\quad \begin{array}{ll} \text{(i)} & w(Q^I[1]) = 2^{l(I)}, \quad \deg I > 0 \\ \text{(ii)} & w([n]) = 0, \quad n \in Z \\ \text{(iii)} & w(x * y) = w(x) + w(y) \\ \text{(iv)} & w(x + y) = \min(w(x), w(y)). \end{array}$$

The filtration associated to the weight function is

$$F^iH_*(\tilde{G}) = \{x \in H_*(\tilde{G}) \mid w(x) \geqq 2i\} \tag{4.3}$$

so that by (4.1), $F^0H_*(\tilde{G}) = H_*(\tilde{G})$ and $F^iH_j(\tilde{G}) = 0$ if $j < i$.

Let $I \subset H_*(\tilde{G}_j)$ be the vector space of positive dimensional elements, and write $I_j = I \cap H_*(\tilde{G}_j)$. The elements of $I * I$ will be called *loop decomposable* and the elements of $I \cdot I$ will be called *composition decomposable.* There are elements x of, say $I_0 * I_0$, such that $x * [1]$ is not composition decomposable — a fact which complicates computations.

DEFINITION 4.1. Let $\mathscr{D}_j$ be the vector space of elements d in $(I * I) \cap H_*(\tilde{G}_j)$ such that $d * [1-j]$ is decomposable in $H_*(SG)$ equipped with the composition product), and let $\mathscr{D} = \bigoplus \mathscr{D}_j$.

In the next few lemmas we examine the structure of $\mathscr{D}$.

LEMMA 4.2. *For any sequence $J, Q^J[1] \cdot Q^J[1]$ is loop decomposable.*

Proof. Let V_n be the subvector space of $H_*(\tilde{G})$ spanned by the elements $Q^J[1]$ with $l(J) = n$. The evaluation map $e: R[n] \to V_n \subset H_*(\tilde{G})$ defines an isomorphism between $R[n]$ and V_n. The composition product defines a product $V_n \otimes V_m \to V_{n+m}$ and therefore a product $\varphi: R[n] \otimes R[m] \to R[n+m]$, which is a coalgebra mapping. In particular, if $\alpha: R[n] \to R[2n]$ is the squaring map ($\alpha(r) = \varphi(r \otimes r)$), then $\alpha^*: R[2n]^* \to R[n]^*$ is a map of algebras. In §3 we examined the map $\zeta^*: R[k] \to R[k+1]$, $\zeta^*(r) = Q^{\deg r} \cdot r$. To prove the lemma we need to prove that $\operatorname{Im} \alpha \subseteq \operatorname{Im} \zeta^*$ or dually that the composite

$$\operatorname{Ker} \zeta \to R[2n]^* \xrightarrow{\alpha^*} R[n]^*$$

is the zero map. Now

$$\operatorname{Ker} \zeta = I(Z_2[\xi_{2n,2n}]) \cdot Z_2[\xi_{1,2n}, \cdots, \xi_{2n-1,2n}],$$

and $\alpha^*(\xi_{2n,2n}) = 0$, since $\deg(\xi_{2n,2n})$ is odd. This completes the proof.

Let $\xi: H_*(\tilde{G}) \to H_*(\tilde{G})$ be the squaring map in the loop product, $\xi(x) = x * x$ and $\hat{\xi}: H_*(\tilde{G}) \to H_*(\tilde{G})$ the squaring map in the composition product, $\hat{\xi}(x) = x \cdot x$.

LEMMA 4.3. *If $x \in H_*(\tilde{G}_{2k})$, then $\hat{\xi}(x) \in \operatorname{Im} \xi$.*

Proof. The previous lemma proves the statement in case $x \in \operatorname{Im} e$. Moreover, if $\hat{\xi}(x) \in \operatorname{Im} \xi$ and $\hat{\xi}(y) \in \operatorname{Im} \xi$ then $\hat{\xi}(x * y) \in \operatorname{Im} \xi$ by a trivial application of the mixed Cartan formula. Since $\operatorname{Im} e$ generates $H_*(\tilde{G}_{2*})$ under the loop product, this proves the lemma.

LEMMA 4.4. *If $x_i \in H_*(\tilde{G}_{2k})$ and $y_i \in H_*(\tilde{G}_{2k})$ and $\Sigma\, x_i y_i$ is loop decomposable, say $\Sigma\, x_i y_i = \Sigma\, z_j * z'_j$, then $\Sigma\, z_j z'_j \in \operatorname{Im} \xi$.*

Proof. If all x_i and y_i are in the image of the evaluation map $e: R \to H_*(\tilde{G})$ then the assertion follows from Lemma 4.2 and (2.11).

Suppose that $x = x' * x''$ and let $\psi(y) = \Sigma\, y'_k \otimes y''_k$. Distributivity yields $x \cdot y = \Sigma\, x' y'_k * x'' y''_k$. The coalgebra map ψ is cocommutative, hence $\Sigma\, x' y'_k \cdot x'' y''_k = \Sigma\, x' x'' y_0 y_0$, where $y_0 \otimes y_0$ is the "middle" term in $\psi(y)$, i.e., $\psi(y) = (1 + T)\eta(y) + y_0 \oplus y_0$, $\eta(y) \in H_*(\tilde{G}) \otimes H_*(\tilde{G})$ and T the twist map. But $y_0 y_0 \in \operatorname{Im} \xi$ by the lemma above and $\operatorname{Im} \xi$ is obviously an ideal under the composition product. This completes the proof, since $\operatorname{Im} e$ generates $H_*(\tilde{G}_{2k})$ under the loop product. We are now ready to characterize the vector space $\mathcal{D}$.

PROPOSITION 4.5. *Let x_i, y_i be elements of $I_{2k} \subset H_*(\tilde{G}_{2k})$. If $\Sigma\, x_i * y_i \in \mathcal{D}$ then $\Sigma\, x_i y_i$ is loop decomposable. Moreover, if we further assume that $\Sigma\, x_i * y_i \in F^3 H_*(\tilde{G})$ then the converse is also true.*

Proof. Both assertions are rather easy consequences of (2.10) and (2.11). We leave the first to the reader and prove the second. First, if $x, y \in I_0 \subset H_*(\tilde{G}_0)$ then

$$(x * [1])(y * [1]) + x * y * [1] + xy * [1] = \sum x'_k y'_l * x''_k * y''_l * [1]$$

where $\psi(x) = \Sigma\, x'_k \otimes x''_k$, $\psi(y) = \Sigma\, y'_l \otimes y''_l$ and $0 < \deg x''_k + \deg y''_l < \deg x + \deg y$. Note that if $x * y \in F^n H_*(\tilde{G})$ then each term in the sum $\Sigma\, x'_k y'_l * x''_k * y''_l * [1]$ is in $F^{n+1} H_*(\tilde{G})$. If $\deg y''_l > 0$ then $\Sigma\, (x'_k y'_l * x''_k) y''_l$ is loop decomposable (by Lemma 4.3 when $x''_k = [0]$) and if $\deg x''_k > 0$ then $\Sigma\, (x'_k y'_l * y''_l) x''_k$ is loop decomposable. Induction over the length filtration now gives

$$x * y * [1] = xy * [1] + (x * [1])(y * [1]) \qquad (\operatorname{mod} \mathcal{D}_1).$$

Second, if $x_i, y_i \in I_0 \subset H_*(\tilde{G}_0)$ and $\Sigma\, x_i y_i \in I_0 * I_0$ and $\Sigma\, x_i * y_i \in F^s H_*(\tilde{G})$ with $s \geqq 3$ then

$$\sum x_i * y_i * [1] \equiv \sum x_i y_i * [1] + \sum (x_i * [1])(y_i * [1]) \qquad (\operatorname{mod} \mathcal{D}_1).$$

Since $\Sigma x_i y_i * [1]$ is loop decomposable and because $\Sigma x_i y_i * [1] \in F^{s+1} H_*(\tilde{G})$ an induction over the length filtration completes the proof. The general case, $x_i, y_i \in I_{2k}$ is completely similar.

LEMMA 4.6. *If x and y are elements of $I_{2k} \subset H_*(\tilde{G}_{2k})$ then each of the two sums*

$$\sum_{i=0}^{n} \hat{Q}^i(x) Q^{n-i}(y) \quad \text{and} \quad \sum_{i=0}^{n} Q^i(x) Q^{n-i}(y)$$

is loop decomposable, in fact, a loop square.

Proof. The proof is essentially the same in the two cases. We give the argument for $\Sigma_{i=0}^{n} \hat{Q}^i(x) Q^{n-i}(y)$ and leave the other to the reader.

$$\begin{aligned}\sum_{i=0}^{n} Q^{n-i}(y)\hat{Q}^i(x) &= \sum \binom{i-t}{t-2s} Q^{n-i+t}(y\hat{Q}^{i-t}(Sq^t x)) \\ &= \sum 2^k Q^{n-k}(y\hat{Q}^{k+s}(Sq^s x)) \\ &= \sum Q^n(y\hat{Q}^s(Sq^s x)).\end{aligned}$$

The evaluation formula (2.4) together with the fact that $Sq^s(y) = 0$ if $2s > \deg y$ gives that $\hat{Q}^s(Sq^s x) = 0$ unless $2s = \deg y$ in which case $\hat{Q}^s(Sq^s x) = Sq^s x \cdot Sq^s x$. An application of Lemma 4.3 finishes the proof.

The mixed Cartan formula simplifies considerably modulo $\mathscr{D}$, explicitly if $x \in H_*(\tilde{G}_{2j})$ and $y \in H_*(\tilde{G}_{2i})$ then

$$\hat{Q}^a(x * y) \equiv \sum_{a_1+a_2=a} \hat{Q}^{a_1}(x) * \hat{Q}^{a_2}(y) * [8ij] + Q^a(xy) * [4(i^2+j^2)] \pmod{\mathscr{D}}$$

(4.4)

$$\hat{Q}^a(x * [1]) \equiv \hat{Q}^a(x) * [1+4j] + Q^a(x) * [1+4j^2] \pmod{\mathscr{D}}$$

The equations (4.4) are easy consequences of Proposition 4.5 and Lemma 4.6. We summarize some useful properties of $\mathscr{D}$ which in particular imply that $\mathscr{D}$ is an *R-ideal* with respect to both the loop structure and the composition structure.

PROPOSITION 4.7. *The vector space $\mathscr{D}$ satisfies*

(i) $I * I * I \subset \mathscr{D}$, $(I * I) \cdot I \subset \mathscr{D}$

(ii) $\hat{Q}^a(\mathscr{D}) \subset \mathscr{D}$, $Q^a(I * I) \subset \mathscr{D}$ *for every a.*

Proof. (i) is an easy consequence of Lemma 4.3 and Proposition 4.5. We leave the details to the reader.

To prove (ii) first observe that as a consequence of the previous lemma and of (i), $Q^a(x * y) \in \mathscr{D}$ when $x, y \in I \subset H_*(\tilde{G})$. In particular $Q^a(\mathscr{D}) \subset \mathscr{D}$.

Next, let $d \in \mathscr{D}_0 = \mathscr{D} \cap H_*(\tilde{G}_0)$. Then $d * [1]$ is decomposable in the composition product, $d * [1] \in I \cdot I$. From (4.4),

$$\hat{Q}^a(d * [1]) \equiv \hat{Q}^a(d) * [1] + Q^a(d) * [1] \pmod{\mathscr{D}}$$

Since $Q^a(d) \in \mathscr{D}_0$ this in particular means that $\hat{Q}^a(d) * [1] \in I \cdot I$. If $d = \Sigma\, x_i * y_i$ then again from (4.4),

$$\hat{Q}^a(d) \equiv Q^a(\Sigma\, x_i y_i) \pmod{I * I}$$

and $Q^a(\Sigma\, x_i y_i)$ is loop decomposable since $\Sigma\, x_i y_i \in I * I$ by Proposition 4.5; thus $\hat{Q}^a(d) \in I * I$. This proves that $\hat{Q}^a(\mathscr{D}_0) \subset \mathscr{D}_0$. In general if $d \in \mathscr{D}_j$ we write $d = d_0 * [j]$, $d_0 \in \mathscr{D}_0$ and apply (4.4) once more to see that $\hat{Q}^a(d) \in \mathscr{D}$. This completes the proof.

In **[14]** Kochman evaluated the homology operations in $H_*(SO)$. We use the H^∞-mapping $J\colon SO \to SG$ (where SG is equipped with the composition structure) to evaluate $\hat{Q}^a(Q^b[1]) \pmod{I * I}$. In principle the method evaluates $\hat{Q}^a(Q^b[1])$. However the computation would be awful. Since for every sequence $I, Q^I[1]$ can be decomposed,

$$Q^I[1] = \sum Q^{k_1}[1] \cdot \cdots \cdot Q_{k_l}[1]$$

knowledge of $\hat{Q}^a(Q^b[1])$ *in principle* evaluates $\hat{Q}^a(Q^I[1])$. The mixed Cartan formula then completely determines the action on $H_*(SG)$. Let $u_i \in H_*(SO)$ denote the i dimensional class in the image of the canonical map $RP^\infty \to SO$. (See **[24]**). Then

$$H_*(SO) = E\{u_1, u_2, \cdots\}, \qquad \psi(u_n) = \sum u_i \otimes u_{n-i}.$$

THEOREM 4.8. (Kochman). *If $a < 2b$ then*

$$Q^a(u_b) = \sum_{i=0}^{a-b-1} \sum_{j=0}^{b} \binom{a-i-j-1}{a-b-i-1} u_i\, u_j\, u_{a+b-i-j}, \quad (u_0 = 1).$$

This theorem completely determines the action $R \otimes H_*(SO) \to H_*(SO)$, because $u_b = \hat{Q}^b([-1])$ so that if $a > 2b$ then by (2.8)

$$\hat{Q}^a(u_b) = \sum \binom{t-b-1}{2t-a} \hat{Q}^{a+b-t}(u_t).$$

LEMMA 4.9.

$$Q^a[1]\cdot Q^b[1] = \sum_{2s\leqq a+b\leqq 3s}\left\{\binom{s}{a-s}+\binom{s}{b-s}+\binom{a+b-2s}{a-s}\right\}\cdot Q^{a+b-s}Q^s[1].$$

The proof consists of a routine computation with binomial coefficients. Notably the following formula of Adem **[2]** is used

$$\sum_{k=0}^{c}\begin{bmatrix}b-k\\k\end{bmatrix}\begin{bmatrix}a+k\\c-k\end{bmatrix}=\begin{bmatrix}a+b+1\\c\end{bmatrix},$$

where a, b and c are arbitrary integers and $\begin{bmatrix}a\\b\end{bmatrix}\in Z_2$ denotes the coefficient to t^b in $(1+t)^a \in Z_2[[t]]$.

THEOREM 4.10. (Milgram). *As Hopf algebras,*

$$H_*(SG) = E\{Q^a[1]*[-1]\mid a\geqq 1\}\otimes Z_2[Q^aQ^a[1]*[-3]\mid a\geqq 1]\otimes P,$$
$$P\otimes Z_2[Q^I[1]*[1-2^{l(I)}]\mid I \textit{ allowable}, \operatorname{exc}(I)>0,\ l(I)\geqq 2].$$

Moreover, the natural mapping J: $SO\to SG$ takes the generator $u_a\in H_a(SO)$ to $Q^a[1]*[-1]$.

Proof. The additive structure is as stated (see (4.1)). From Lemma 4.9 it follows that $Q^aQ^a[1]=Q^a[1]\cdot Q^a[1]$. An application of (2.10) and (2.11) then gives

$$(Q^a[1]*[-1])(Q^a[1]*[-1])=0.$$

One more application of (2.10) and (2.11) shows that the other generators are polynomial generators. The latter part of Theorem 4.10 follows by induction from the fact that in every degree $H_*(SG)$ has exactly one primitive element with zero square.

REMARK. We point out that the above evaluation of $H_*(SG)$ is a reformulation, due to May, of Milgram's original computation. (Compare **[10]**).

PROPOSITION 4.11. *For any integers a and b*

$$\hat{Q}^a(Q^b[1])\equiv\binom{a-1}{b}Q^{a+b}[1]*[2] \textit{ modulo } I*I.$$

Proof. Let us first assume $a \leqq 2b$. Let $\equiv$ denote equivalence modulo $I * I$. The mixed Cartan formula gives

$$\hat{Q}^a(Q^b[1] * [-1]) \equiv \hat{Q}^a(Q^b[1]) * [-3] + Q^aQ^b[1] * [-3];$$

it therefore suffices to evaluate $J_*\hat{Q}^a(u_b)$. From Kochmans theorem above

$$\begin{aligned} J_*(\hat{Q}^a u_b) \equiv & \sum_{i=1}^{a-b-1} Q^i[1] \sum_{j=1}^{b} \binom{a-i-j-1}{b-j} Q^j[1]Q^{a+b-i-j}[1] * [-5] \\ & + \sum_{j=1}^{b} \binom{a-j-1}{b-j} Q^j[1]Q^{a+b-j}[1] * [-3] \\ & + \sum_{i=1}^{a-b-1} \binom{a-i-1}{b} Q^i[1] \cdot Q^{a+b-i}[1] * [-3] \\ & + \binom{a-1}{b} Q^{a+b}[1] * [-1]. \end{aligned}$$

We then use Lemma 4.9 together with suitable summation formulas for binomial coefficients to get

$$\begin{aligned} J_*(\hat{Q}^a u_b) \equiv & \sum_{i=1}^{a-b-1} Q^i[1] \cdot Q^{a-i}Q^b[1] * [-7] \\ & + Q^aQ^b[1] * [-3] \\ & + \binom{a-1}{b} Q^{a+b}[1] * [-1]. \end{aligned}$$

But now,

$$\begin{aligned} \sum_{i=1}^{a-b-1} Q^i[1] \cdot Q^{a-i}Q^b[1] = & \sum_{i=1}^{a} \sum_{t \geqq 0} \binom{i-t}{t} Q^{a-i+t}(Q^{i-t}[1]Q^b[1]) \\ & + Q^{a-b}[1]Q^bQ^b[1] \equiv \sum_k \sum_i \binom{k}{i-k} Q^{a-k}(Q^k[1] \cdot Q^k[1]) = Q^aQ^b[2] \\ & \equiv 0. \end{aligned}$$

If $a > 2b$ we first use that $\hat{Q}^b[-1] = Q^b[1] * [-1]$ together with an Adem relation. This completes the proof.

REMARK. May and Tsuchiya using different methods have recently determined the rest of the terms in $\hat{Q}^a(Q^b[1])$. The result is

$$\hat{Q}^a(Q^b[1]) = \sum_{k=0}^{a+b/2} \binom{a-i-1}{b-i} Q^{a+b-i}[1] * Q^i[1].$$

Proposition 4.11 together with our good grasp on $\mathscr{D}$ makes the mixed Cartan formula an effective computational tool. From (4.4) we get

$$\hat{Q}^a(Q^I[1]*[1-2^{l(I)}]) \equiv \hat{Q}^a(Q^I[1])*[1-2^{2l(I)}] + Q^aQ^I[1]*[1-2^{l(I)+1}] \tag{4.5}$$

According to (2.11), the elements $Q^i[1] \in H_*(\tilde{G})$ generates all of $\mathrm{Im}(e: R \to H_*(\tilde{G}))$ under the composition product. Since $(I*I)\cdot I \subset \mathscr{D}$ the proposition above implies that for $p > 1$

$$\hat{Q}^a(Q^{i_1}[1]\cdots Q^{i_p}[1]) \equiv \sum \prod_{j=1}^{p} \binom{a_i-1}{i_j} Q^{i_1+a_1}[1]\cdots Q^{i_p+a_p}[1] \tag{4.6}$$

modulo $\mathscr{D}$, (summation over all sequences $(a_1,\cdots,a_p)$ with $a_1+\cdots+a_p = a$). In particular, we have proved

THEOREM 4.12. *For any sequence I of length l and any integer a,*

$$\hat{Q}^a(Q^I[1]*[1-2^l]) \equiv Q^aQ^I[1]*[1-2^{l+1}] + \sum Q^K[1]*[1-2^l]$$

modulo $\mathscr{D}$, where K runs over certain sequences of the same length as I.

The next theorem gives an affirmative answer to a conjecture of J. P. May.

THEOREM 4.13.

$$\begin{aligned} H_*(SG) = {} & E\{Q^a[1]*[-1] \mid a = 1,2,\cdots\} \\ & \otimes Z_2[Q^aQ^a[1]*[-3] \mid a = 1,2,\cdots] \otimes Z_2[\hat{Q}^I(Q^J[1]*[-3]) \mid l(J)=2, \\ & l(I)>0,\ (I,J)\ \textit{allowable},\ \mathrm{exc}\,(I,J)>0]. \end{aligned}$$

Proof. Theorem 4.12 together with Proposition 4.7 yields

$$\begin{aligned} \hat{Q}^I(Q^J[1]*[-3]) \equiv {} & Q^IQ^J[1]*[1-2^{l(I)+l(J)}] \\ & + \sum Q^K[1]*[1-2^{l(K)}] \text{ modulo } \mathscr{D} \end{aligned}$$

where the summation is over certain sequences K with $l(K) < l(I)+l(J)$. The result now follows from Theorem 4.10.

We next evaluate the Hopf algebra $H_*(BSG)$. Previously Milgram [21] determined the coalgebra structure or dually the cohomology algebra $H^*(BSG)$. However, the algebra structure of $H_*(BSG)$ is more important in connection with the determination of the various cobordism rings (see **[8]**). We let

$$\sigma\colon QH_*(SG) \to PH_*(BSG)$$

denote the homology suspension.

THEOREM 4.14. *As Hopf algebras*

$$\begin{aligned} H_*(BSG) = H_*(BSO) &\otimes E\{\sigma(Q^aQ^a[1]*[-3]) \mid a \geqq 1\} \\ &\otimes Z_2[\sigma(Q^aQ^b[1]*[-3]) \mid b < a \leqq 2b] \\ &\otimes Z_2[\sigma(Q^I[1]*[1-2^{l(I)}]) \mid I \text{ allowable}, \operatorname{exc}(I) > 1, l(I) > 2]. \end{aligned}$$

Proof. Consider the Eilenberg-Moore spectral sequence of the fibration $SG \to ESG \to BSG$ with

$$E^2 = \operatorname{Tor}_{H_*(SG)}(Z_2, Z_2)$$
$$E^\infty = E^0 H_*(BSG).$$

Since $H_*(SG) = H_*(SG) /\!\!/ H_*(SO) \otimes H_*(SO)$ and since $H_*(SG) /\!\!/ H_*(SO)$ is a polynomial algebra,

$$\begin{aligned} \operatorname{Tor}_{H_*(SG)}(Z_2, Z_2) = \operatorname{Tor}_{H_*(SO)}&(Z_2, Z_2) \\ &\otimes E\{s(x) \mid x \in Q(H_*(SG) /\!\!/ H_*(SO))\}. \end{aligned}$$

Here $s(x)$ has filtration degree 1 and total degree $\deg x + 1$.

The natural mapping $BSO \to BSG$ induces an injection in homology because the Stiefel-Whitney classes are fibre homotopy invariants. Therefore the elements of $\operatorname{Tor}_{H_*(SO)}(Z_2, Z_2)$ survive to E^∞. Since the rest of the generators of E^2 have filtration degree 1 the spectral sequence collapses, $E^2 = E^\infty$. This computes the additive structure, in fact the coalgebra structure of $H^*(BSG)$.

The multiplicative structure is determined from

(i) $\hat{Q}^{2a+1}(Q^aQ^a[1]*[-3]) \equiv 0 \pmod{\mathscr{D}}$

(ii) $\hat{Q}^a(Q^I[1]*[1-2^{l(I)}]) \equiv Q^aQ^I[1]*[1-2^{l(I)+1}] \pmod{\mathscr{D}}$

when $a = \deg Q^I + 1$ and $l(I) > 1$.

From (4.4) together with the Adem relation $Q^{2a+1}Q^a = 0$ we get modulo $\mathscr{D}$

$$\hat{Q}^{2a+1}(Q^aQ^a[1]*[-3]) \equiv \hat{Q}^{2a+1}(Q^aQ^a[1])*[-15].$$

But $Q^aQ^a[1] = Q^a[1]\cdot Q^a[1]$ (Lemma 4.9) and $\hat{Q}^{2a+1}(Q^a[1]\cdot Q^a[1]) = 0$ as a consequence of the Cartan formula. To prove (ii) we observe that when $l(I)=l$ then $Q^I[1]=\Sigma Q^{k_1}[1]\cdot\cdots\cdot Q^{k_l}[1]$. Because $a=k_1+\cdots+k_l+1$ and $l\geqq 2$ this implies that $Q^a(Q^I[1])\in(I*I)\cdot I\subset\mathcal{D}$. An application of (4.4) completes the proof of (ii) whence of the theorem.

5. The R-indecomposable elements of G/O. In §4 we saw that $QH_*(SG)$ is generated by a rather small set of elements under the action of the Dyer-Lashof algebra (Theorem 4.13). In this section we shall determine the minimal set of generating elements; in other words, we shall compute the Z_2 vector space $Z_2\otimes_R QH_*(SG)$ as well as the vector space $Z_2\otimes_R QH_*(G/O)$.

A simple argument using the Eilenberg-Moore spectral sequence of the fibration $SO\to SG\to G/O$ shows that $H_*(G/O)=H_*(SG)/\!/H_*(SO)$. From the previous section it follows that $H_*(SG)$ splits as an R-algebra,

$$H_*(SG)=H_*(SO)\otimes H_*(G/O).$$

Therefore

$$Z_2\otimes_R QH_*(SG)=Z_2\otimes_R QH_*(SO)\oplus Z_2\otimes_R QH_*(G/O).$$

From Theorem 4.8 it follows that $Z_2\otimes_R QH_*(SO)$ is a graded vector space with one generator in each dimension 2^i.

Suppose that $\Sigma Q^{a_i}Q^{b_i}=0$ in R and let c be an arbitrary integer. From (4.4) and the Propositions 4.7 and 4.11 we get in $H_*(SG)$:

$$\begin{aligned}&\hat{Q}^{a_i}(\hat{Q}^{b_i}(Q^c[1])*[-3])\\ &\quad\equiv \hat{Q}^{a_i}\hat{Q}^{b_i}(Q^c[1])*[-15]\ +Q^{a_i}(\hat{Q}^{b_i}(Q^c[1])\cdot[-3])*[25]\quad(\mathrm{mod}\ \mathcal{D})\\ &\quad\equiv \hat{Q}^{a_i}\hat{Q}^{b_i}(Q^c[1])*[-15]\ +\binom{b_i-1}{c}Q^{a_i}Q^{b_i+c}[1]*[-3]\quad(\mathrm{mod}\ \mathcal{D})\end{aligned}$$

so that when $\Sigma Q^{a_i}Q^{b_i}=0$ in R then

$$\begin{aligned}&\sum \hat{Q}^{a_i}(\hat{Q}^{b_i}(Q^c[1])*[-3])\\ &\quad\equiv\sum\binom{b_i-1}{c}Q^{a_i}Q^{b_i+c}[1]*[-3]\qquad(\mathrm{mod}\ \mathcal{D}).\end{aligned}\tag{5.1}$$

In particular, this shows that

$$\sum\binom{b_i-1}{c}Q^{a_i}Q^{b_i+c}[1]*[-3]$$

is zero in $Z_2\otimes_R QH_*(SG)$.

We use (5.1) in the following special cases (compare (2.4) and (2.8)):

(i) $n > 0$, $Q^{2n+2}Q^{n} + Q^{2n+1}Q^{n+1} = 0$ on classes of degree n,

(ii) $a > b$, $Q^{2a+1}Q^{b+1} + \lambda Q^{a+b+1}Q^{a+1} = 0$ on classes of degree b, $\lambda = a + b + 1$

(iii) $a > b + 1$, $Q^{2a}Q^{b+1} + Q^{a+b+1}Q^{a} = 0$ on classes of degree b.

Now, apply (5.1) with $c = n$ in case (i) and $c = b$ in case (ii) and (iii). We get the following equations in $Z_2 \otimes_R QH_*(SG)$

$$Q^{2n+1}Q^{2n+1}[1] * [-3] = 0 \text{ if } n > 0$$

$$(5.2) \quad Q^{2a+1}Q^{2b+1}[1] * [-3] = 0 \text{ if } a > b$$

$$Q^{2a}Q^{2b+1}[1] * [-3] = \binom{a-1}{b} Q^{a+b+1}Q^{a+b}[1] * [-3] \text{ if } a > b+1.$$

Next, let us take a look at the primitive elements in $H_*(SG)$. Let R_0 be the left ideal in R generated by the elements Q^{2a+1}, $a = 0, 1, \cdots$. It is easy to see from the Adem relations that R_0 is the set of $Q^I \in R$ with I allowable and not all entries even, or in other words, R_0 is the kernel of the halving map $\xi^*: R \to R$, dual to the squaring map in R^*.

PROPOSITION 5.1. *The primitive elements $PH_*(SG)$ are in one-to-one correspondence with R_0. In fact, there is a bijection $\beta: R_0 \to PH_*(SG)$ with the property that if $l(r) > 1$ then $\beta(r) \equiv e(r) * [1 - 2^{l(r)}]$ (modulo $\mathscr{D}$).*

Proof. The elements $Q^a[1] \in H_*(\tilde{G})$ generates under loop product a Hopf algebra isomorphic to $H_*(BO)$. If a is odd then there is an element $d \in I * I$ such that $Q^a[1] + d$ is primitive. But then for any $Q^I \in R$

$$Q^I Q^a[1] * [1 - 2^{l(I)+1}] + Q^I(d) * [1 - 2^{l(I)+1}],$$

is primitive. Also $Q^I(d) \in \mathscr{D}$ since d is loop decomposable. (Proposition 4.7). We define

$$\beta(Q^I Q^a) = Q^I Q^a[1] * [1 - 2^{l(I)+1}] + Q^I(d) * [1 - 2^{l(I)+1}].$$

To see that β is an isomorphism one uses the exact sequence of Milnor and Moore **[20]**,

$$PH_*(SG) \xrightarrow{\xi} PH_*(SG) \to QH_*(SG) \xrightarrow{\zeta} QH_*(SG) \to 0.$$

This completes the proof.

Let $q: PH_*(SG) \to Z_2 \otimes_R QH_*(SG)$ be the natural projection map.

COROLLARY 5.2.
(i) $\operatorname{Im}(q: PH_{2n+1}(SG) \to Z_2 \otimes_R QH_{2n+1}(SG)) = Z_2$, $n \geqq 1$
(ii) $\operatorname{Im}(q: PH_{2n}(SG) \to Z_2 \otimes_R QH_{2n}(SG)) = 0$, $n > 1$
(iii) $\operatorname{Im}(q: PH_2(SG) \to Z_2 \otimes_R QH_2(SG)) = Z_2$.
Moreover, $q(Q^{2n}Q^1[1] * [-3]) \neq 0$, $q(Q^1[1] * [-1]) \neq 0$ *and* $q(Q^1Q^1[1] * [-3] \neq 0$.

Proof. According to Theorem 4.12 it is enough to examine $q\beta(Q^J)$ for J of length 2. From (5.2) we then get that $\operatorname{Im}(q)$ is contained in the Z_2 vector space spanned by the elements $q\beta(Q^{n+1}Q^n)$, $q\beta(Q^1Q^1)$ and $q\beta(Q^1)$. But $Q^{n+1}Q^n = Q^{2n}Q^1$ by an Adem relation. Finally to prove that $q(Q^{2n}Q^1[1] * [-3]) \neq 0$ first observe that it suffices to prove this for $n+1$ a power of 2. This follows from the action of the Steenrod algebra: To every n there exists $\alpha \in A^{0p}$ and integer i so that $\alpha(Q^{2n}Q^1[1] * [-3]) = Q^{2(2^i-1)}Q^1[1] * [-3]$. Now, when $n+1$ is a power of 2 then $Q^a[1] \cdot Q^b[1] = Q^{n+1}Q^n[1] +$ other terms if and only if a or b are powers of 2. This together with Proposition 4.11 completes the proof.

We now turn to $Z_2 \otimes_R QH_*(G/O)$. With a slight abuse of notation we write $Q^I[1] * [1 - 2^{l(I)}]$, $l(I) \geqq 2$, for the generators of $H_*(G/O)$.

THEOREM 5.3. *The graded vector space $Z_2 \otimes_R QH_*(G/O)$ has one nonzero element in each dimension larger than one. In fact, $Q^{2^{i+1}m}Q^{2^i}[1] * [-3] \in H_*(G/O)$ represents the nonzero element in degree $2^i(2m+1)$, and $Q^{2^i}Q^{2^i}[1] * [-3]$ the nonzero element in degree 2^{i+1}.*

Theorem 5.3 is a consequence of Corollary 5.2 and Lemma 5.4 below.

Let A be a commutative and cocommutative (connected) Hopf algebra and R a cocommutative Hopf algebra (not necessarily connected). Suppose that A is a Hopf algebra over R. If X is a vector space, let $\Gamma(X)$ be the divided power algebra generated by X.

LEMMA 5.4. *Suppose that the squaring homomorphisms of A^* and R^* are injective. Let M be the image of the natural projection $P(A) \to Z_2 \otimes_R Q(A)$. Then $Z_2 \otimes_R Q(A) = Q\Gamma(M)$.*

Proof. There is an exact sequence

$$P(A) \to Q(A) \xrightarrow{\lambda} Q(A) \to 0$$

where λ is the dual of the squaring homomorphism. Since the halving homomorphism $\lambda: R \to R$ is also onto by assumption we get an exact sequence

$$Z_2 \otimes_R P(A) \xrightarrow{\pi} Z_2 \otimes_R Q(A) \xrightarrow{1\otimes\lambda} Z_2 \otimes_R Q(A) \to 0.$$

Hence $M \cong \ker(1 \otimes \lambda: Z_2 \otimes_R Q(A) \to Z_2 \otimes_R Q(A))$. Let us pick a map (extending $M \to Z_2 \otimes_R Q(A)$)

$$f: Q\Gamma(M) \to Z_2 \otimes_R Q(A)$$

with $f \circ \lambda = (1 \otimes \lambda) \circ f$, and consider the diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
M & \xrightarrow{Id} & M \\
\downarrow & & \downarrow \\
Q\Gamma(M) & \xrightarrow{f} & Z_2 \otimes_R Q(A) \\
\downarrow \lambda & & \downarrow 1 \otimes \lambda \\
Q\Gamma(M) & \xrightarrow{f} & Z_2 \otimes_R Q(A) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

There are exact sequences

$$0 \to \operatorname{Ker} \lambda \to \operatorname{Ker} \lambda^n \xrightarrow{\lambda} \operatorname{Ker} \lambda^{n-1} \to 0$$

and similarly for $1 \otimes \lambda$. The five lemma implies that

$$\operatorname{Ker} \lambda^n \xrightarrow{f} \operatorname{Ker} (1 \otimes \lambda)^n$$

is an isomorphism for each n and therefore that $Q\Gamma(M) \xrightarrow{f} Z_2 \otimes_R Q(A)$ is an isomorphism. This completes the proof.

Suppose that (ξ, t) is a pair consisting of a spin bundle with a homotopy framing $t: S(\xi) \to S^{8n-1}$. The Thom space $M(\xi)$ admits two KO-orientations, the Atiyah-Bott-Shapiro orientation $U_{\text{spin}} \in \widetilde{KO}(M(\xi))$ and the orientation U_t associated with the framing t. The quotient of these two classes defines an element $e(\xi, t) \in 1 + \tilde{KO}(X)$, $U_t = e(\xi, t) \cdot U_{\text{spin}}$. The space G/O is the fibre of the natural mapping $BSO \to BSG$. Hence it classifies oriented bundles with fibre homotopy framings. The inclusion $G/O \to BSO$ lifts to a unique map $G/O \to B\,\text{spin}$. Thus over a skeleton X of G/O we have a pair (ξ, t) of a spin bundle with a homotopy framing. The construction above defines an element $e \in 1 + \varprojlim \tilde{KO}$ (skeleton of G/O). From the splitting at each prime of G/O in BSO and a finite space it follows that the $\varprojlim^{(1)}$-term in Milnor sequence vanishes.

$$\varprojlim \tilde{KO}\,(\text{skeleton of } G/O) = [G/O,\ BSO].$$

In particular we get a unique homotopy class

$$e\colon G/O \to BSO.$$

This is an H-map when we equip BSO with the H-space structure coming from tensor products of bundles of virtual dimension 1. Following Sullivan we define the space $\operatorname{cok} J$ to be the fibre of e,

$$\operatorname{cok} J \to G/O \xrightarrow{e} BSO.$$

Sullivan further defines the space $\operatorname{Im} J$ to be the fiber of the mapping $\psi^3 - 1\colon BSO \to BSO$. In the rest of this section all spaces and maps are to be taken in the 2-local category. The (2-local) Adams conjecture asserts that the composite

$$BSO \xrightarrow{\psi^3 - 1} BSO \longrightarrow BSG$$

is null homotopic. This leads to the existence of a diagram of fibrations

$$\begin{array}{ccccc} \operatorname{Im} J & \longrightarrow & BSO & \xrightarrow{\psi^3-1} & BSO \\ \downarrow \tilde{\alpha} & & \downarrow \alpha & & \| \\ SG & \longrightarrow & G/O & \longrightarrow & BSO \end{array}$$

(α is called a solution to the Adams conjecture).

THEOREM 5.5 (Sullivan). *There are* (2-local) *homotopy equivalences*

$$G/O \simeq BSO \times \operatorname{cok} J$$

$$SG \simeq \operatorname{Im} J \times \operatorname{cok} J.$$

THEOREM 5.6. *For every solution α of the Adams conjecture,*

$$\beta: H_*(BSO) \xrightarrow{\alpha_*} H(G/O) \to Z_2 \otimes_R QH_*(G/O)$$

is an epimorphism.

Proof. The map $BSO \xrightarrow{\alpha} G/O \xrightarrow{e} BSO$ is a homotopy equivalence. This is the main point of Sullivans proof of Theorem 5.5. In particular α_*: $H_2(BSO) \to H_2(G/O)$ is an isomorphism.

Let $p_n \in PH_n(BSO)$ denote the unique nonzero primitive element. It is well known that

$$Sq^a(p_b) = \binom{b-a-1}{a} p_{b-a}.$$

Since $Sq^{2^i}(p_{2^{i+1}+1}) = p_{2^i+1}$ for $i > 0$ and $\beta(p_2) \neq 0$ it follows that $\beta(p_{2^{i+1}+1}) \neq 0$. The elements p_{2^i+1} generate $PH_{2*+1}(BSO)$ under the action of A^{0p}. From Theorem 4.15 we get that

$$Sq^{2a}(t_{2b+1}) = \binom{b-a}{a} t_{2b-2a+1},$$

where t_i is the nonzero element of $Z_2 \otimes_R QH_*(G/O)$ in degree i. Thus

$$\beta: PH_*(BSO) \to Z_2 \otimes_R QH_*(G/O)$$

is an isomorphism in odd degree. To complete the argument it suffices to observe that the halving mapping ζ: $H_{2n}(BSO) \to H_n(BSO)$ is onto $(\zeta(x) = Sq^n(x))$.

REMARK. One can show that there are decomposable elements $d \in H_*(BSO)$ with $\beta(d) \neq 0$. Thus β: $H_*(BSO) \to Z_2 \otimes_R QH_*(G/O)$ does not factor over $QH_*(BSO)$. Nevertheless there is of course a splitting $QH_*(BSO) \xrightarrow{s} H_*(BSO)$ so that

$$\beta s: QH_*(BSO) \to Z_2 \otimes_R H_*(BSO)$$

is an isomorphism.

PROPOSITION 5.7. *There is no H-map f: $BSO \to G/O$ with f_*: $H_2(BSO) \to H_2(G/O)$ nontrivial.*

Proof. The proof of the previous theorem implies that the composite $H_*(BSO) \xrightarrow{f} H_*(G/O) \to Z_2 \otimes_R QH_*(G/O)$ is an epimorphism. Therefore

$$f_*(p_9) = Q^8Q^1[1] * [-3] + \lambda \cdot Q^6Q^3[1] * [-3] + \text{decomposable terms.}$$

$(QH_9(G/O) = Z_2 \oplus Z_2)$. Applying Sq^1Sq^2 to this equation and using the Adem relation $Q^5Q^1 = Q^3Q^3$ we get

$$f_*(p_3^2) = Q^3Q^3[1] * [-3] + \text{decomposable term.}$$

Hence $f_*(p_3^2)$ is not decomposable and f cannot be an H-map.

REMARKS. Let $BO[n,\infty] \to BO$ denote the $(n-1)$-connected cover of BO. A slight extension of the argument above shows that the composite $BO[n,\infty] \to BO \to G/O$ is never an H-map. Further it is a consequence of Proposition 5.7 that G/O does not split as an H-space in the factors BSO and $\operatorname{cok} J$. Finally one may extend the arguments above to show that there is no H-splitting $SG \simeq \operatorname{Im} J \times \operatorname{cok} J$. This is all in strong contrast to the behavior at odd primes. When p is an odd prime one defines the space $\operatorname{Im} J$ as the fibre of $\psi^q - 1$: $BSO \to BSO$ where q is a topological generator of the p-adic integers. J. Tornehave has recently shown that when localized at p, $SG \simeq \operatorname{Im} J \times \operatorname{cok} J$ as infinite loop spaces for a suitable infinite loop space structure on $\operatorname{cok} J$.

The space G/TOP is (2-locally) a product of Eilenberg-MacLane spaces (Sullivan)

$$G/TOP = \prod_{n \geqq 1} K(Z_2, 4n-2) \times \prod_{n \geqq 1} K(Z, 4n).$$

The natural map τ: $SG \to G/TOP$ is an infinite loop map (Boardman-Vogt). Hence τ_*: $H_*(SG) \to H_*(G/TOP)$ commutes with homology operations. It is well-known that there exist elements in $\pi^S_{2^i-2}(S^0)$ with Arf invariant 1 if and only if there are spherical elements of $H_{2^i-2}(SG)$ which map nonzero to $H_{2^i-2}(G/TOP)$. In particular if $i = 2$ and 3 we do have such elements, $h(\eta^2) \in H_2(SG)$ and $h(\nu^2) \in H_6(SG)$ where h is the Hurewicz homomorphism. Spherical elements are primitive and annihilated by all Steenrod operations. It follows that

$$h(\eta^2) = Q^1Q^1[1] * [-3]$$

$$\text{(5.3)} \quad h(\nu^2) = Q^3Q^3[1] * [-3] + (Q^2Q^2[1] * [-3])(Q^1Q^1[1] * [-3]) + (Q^2Q^1[1] * [-3])(Q^2Q^1[1] * [-3]).$$

We can now prove

PROPOSITION 5.8. *The third Boardman-Vogt delooping*

$$B^3(G/TOP)$$

is not a product of Eilenberg-MacLane spaces in the 2-local category.

Proof. It suffices to prove that

$$\hat{Q}^4\colon H_2(G/TOP) \to H_6(G/TOP) \to QH_6(G/TOP)$$

is nonzero. If $B^3(G/TOP)$ was a product of Eilenberg-MacLane spaces then the iterated suspension σ^3: $QH_k(G/TOP) \to PH_{k+3}(G/TOP)$ would be injective. But $\hat{Q}^4$ is stable so that $\hat{Q}^4(\sigma^3(\iota_2)) = \sigma^3(\hat{Q}^4(\iota_2)) \neq 0$ which contradicts the excess relation. From the computations of §4 it is easy to see that

$$\hat{Q}^4(Q^1Q^1[1]*[-3]) = Q^3Q^3[1]*[-3] + (Q^2Q^2[1]*[-3])(Q^1Q^1[1]*[-3])$$

and from (5.3) it then follows that

$$\hat{Q}^4(\iota_2) = \iota_6 + \tau_*(Q^2Q^1[1]*[-3])^2$$

where ι_6 is the spherical class. Thus $\hat{Q}^4(\iota_2)$ is nonzero in $QH_6(G/TOP)$. This completes the proof.

REMARK. In a forthcoming paper we shall see that this pattern continues, $\hat{Q}^{2^i}(\iota_{2^i-2}) = \iota_{2^{i+1}-2}$. In fact we shall completely determine the action of the Dyer-Lashof algebra in $H_*(G/TOP)$.

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