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CONVOLUTION MULTIPLIERS AND DISTRIBUTIONS

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In this paper, in a purely algebraic way, Schwartz distributions in several variables are generalized in accordance with their homomorphism interpretation proposed by R. A. Struble.

0. Introduction. R. A. Struble in [10] has shown that Schwartz distributions can be characterized simply as mappings, from the space $\mathscr{D}$ of test functions into the space $\mathscr{E}$ of smooth functions, which commute with ordinary convolution. This new view of distributions has turned out to be very useful [11, 12] and motivated us to give a simple generalization for distributions which is closely related to Mikusiński operators and convolution quotients of other types [11, 12, 4, 13]. The method employed here is an appropriate modification of a general algebraic method [5, 2, 8].

Mappings which commute with convolution are called convolution multipliers here. (Distributions can be characterized as convolution multipliers, Mikusiński operators themselves are convolution multipliers.)

In §1, convolution multipliers from various subsets of $\mathscr{D}$ into $\mathscr{E}$ are discussed. We are primarily concerned with their maximal extensions.

In §2, a module $\mathfrak{M}$ of certain maximal convolution multipliers is constructed and investigated from an algebraic point of view.

In §3, Schwartz distributions are embedded and characterized in $\mathfrak{M}$. For example, we prove that distributions are the only continuous elements of $\mathfrak{M}$. Finally, we show that there are elements in $\mathfrak{M}$ which are not distributions.

To illustrate the appropriateness of our generalizations, we refer to the following facts:

One of the difficulties in working with Schwartz distributions is that only distributions Λ satisfying $\Lambda * \mathscr{D} = \mathscr{D}$ are invertible in $\mathscr{D}'$. Whereas, distibutions Λ satisfying $\Lambda * \mathscr{D} \subset \mathscr{D}$ such that $\Lambda * \mathscr{D}$ has no proper annihilators in $\mathscr{E}$ are invertible in $\mathfrak{M}$. (The heat operator in two dimensions [1] seems to be a distribution which is not invertible in $\mathscr{D}'$, but is invertible in $\mathfrak{M}$.)

There are regular Mikusiński operators [1] which are not distributions. Whereas, normal Mikusiński operators [11] can be embedded in $\mathfrak{M}$.

1. Convolution multipliers and their maximal extensions. Let k be a fixed positive integer, $\mathbf{R}^k$ be the k-dimensional Euclidean space and $\mathbf{C}$ be the field of complex numbers.

Let $\mathscr{E}$ be the set of all infinitely differentiable functions from $\mathbf{R}^k$ into $\mathbf{C}$, and let $\mathscr{D}$ be the subset of $\mathscr{E}$ consisting of all those functions with compact support.

It is known that with the pointwise linear operations and convolution $\mathscr{D}$ is a commutative complex algebra without proper zero-divisors, and $\mathscr{E}$ is a complex vector space and a $\mathscr{D}$-module.

DEFINITION 1.1. Let

$$M = \{F: D_F \subset \mathscr{D} \to \mathscr{E} : \forall\, \varphi, \psi \in D_F : F(\varphi) * \psi = \varphi * F(\psi)\}.$$

PROPOSITION 1.2. *Let $F \in M$. Then F has a maximal extension in M.*

Proof. Let $\mathscr{F} = \{G \in M : F \subset G\}$. Under set inclusion $\mathscr{F}$ is a novoid partially ordered set. Moreover, if $\mathscr{B}$ is a chain in $\mathscr{F}$, then $\cup\, \mathscr{B}$ is an upper bound for it in $\mathscr{F}$. (To prove this use (2.19) Theorem in **[3]**.) Thus, by Zorn's lemma $\mathscr{F}$ has a maximal member.

DEFINITION 1.3. Let $D \subset \mathscr{D}$. Then D is said to be normal **[11]** if and only if $D \neq \phi$ and D has no proper annihilators in $\mathscr{E}$, i.e., $f \in \mathscr{E}$ and $f * D = \{0\}$ imply that $f = 0$.

LEMMA 1.4. *Let $D \subset \mathscr{D}$. Suppose that for every $0 < \epsilon \in \mathbf{R}^1$ there exists $\varphi \in \mathscr{D}$ such that $0 \leqq \varphi$ or $\varphi \leqq 0$ and the diameter of the support of φ is less than ϵ. Then D is normal. In particular, $\mathscr{D}$ is normal.*

Proof. See the proof of 1.5. Lemma in **[4]**.

PROPOSITION 1.5. *Let $D \subset \mathscr{D}$. Then every $F \in M$ with domain D has a unique maximal extension in M if and only if D is normal.*

Proof. Suppose first that every $F \in M$ with domain D has a unique maximal extension in M. To prove that D is normal suppose that $f \in \mathscr{E}$ such that $f * D = \{0\}$. Let F and 0 be the functions defined on $\mathscr{D}$ by $F(\varphi) = f * \varphi$ and $O(\varphi) = 0$. Then F and 0 are maximal extensions of $F \mid D$ in M. Thus, $f * \varphi = 0$ for all $\varphi \in \mathscr{D}$. Hence, it follows that $f = 0$.

To prove the converse suppose that D is normal and $F \in M$ with domain D. We prove that

$$\bar{F} = \{(\varphi, f) \in \mathscr{D} \times \mathscr{E} : \forall\, \sigma \in D : f * \sigma = \varphi * F(\sigma)\}$$

is the unique maximal extension of F in M.

To prove that $\bar{F}$ is a function, suppose that (φ, f_1), $(\varphi, f_2) \in \bar{F}$. Then

$$f_1 * \sigma = \varphi * F(\sigma) \quad \text{and} \quad f_2 * \sigma = \varphi * F(\sigma)$$

for all $\sigma \in D$. Hence, it follows that $(f_1 - f_2) * \sigma = 0$ for all $\sigma \in D$. This implies that $f_1 = f_2$.

Now, we prove that $\bar{F}$ is a multiplier. For this, suppose that $\varphi, \psi \in D_{\bar{F}}$. Then

$$\bar{F}(\varphi) * \sigma = \varphi * F(\sigma) \quad \text{and} \quad \bar{F}(\psi) * \sigma = \psi * F(\sigma)$$

for all $\sigma \in D$. Hence, it follows that

$$(\bar{F}(\varphi) * \psi - \varphi * \bar{F}(\psi)) * \sigma = 0$$

for all $\sigma \in D$. This implies that $\bar{F}(\varphi) * \psi = \varphi * \bar{F}(\psi)$.

The remaining part of the proof is quite obvious.

REMARK 1.6. In the following, we shall not make use of Proposition 1.2. and 1.5. They are only to make clear and complete our treatment.

DEFINITION 1.7. Let

$$\mathcal{M} = \{F \in M \colon D_F \text{ is normal}\}.$$

For any $F \in \mathcal{M}$, let

$$\bar{F} = \{(\varphi, f) \in \mathcal{D} \times \mathcal{E} \colon \forall\, \sigma \in D_F \colon f * \sigma = \varphi * F(\sigma)\}.$$

THEOREM 1.8. *Let $F \in \mathcal{M}$. Then $\bar{F}$ is the unique maximal extension of F in $\mathcal{M}$.*

Proof. See the second part of the proof of Proposition 1.5.

COROLLARY 1.9. *Let $F \in \mathcal{M}$. Then $D_{\bar{F}}$ is an algebra ideal in $\mathcal{D}$, and $\bar{F}$ is a vector space and a $\mathcal{D}$-module homomorphism.*

Proof. Easy computation.

COROLLARY 1.10. *Let $F, G \in \mathcal{M}$. Suppose that $D \subset D_F \cap D_G$ such that D is normal and $F(\varphi) = G(\varphi)$ for all $\varphi \in D$. Then $\bar{F} = \bar{G}$.*

Proof. Trivial.

2. A module of convolution multipliers.

DEFINITION 2.1. Let

$$\mathfrak{M} = \{\bar{F}\colon F \in \mathcal{M}\}$$

and

$$\mathfrak{N} = \{\bar{\Phi}\colon \Phi \in \mathcal{M}, \Phi(D_\Phi) \subset \mathscr{D}\}.$$

For any $F, G \in \mathfrak{M}$ and $\Phi \in \mathfrak{N}$, let

$$F + G = \overline{F + G}$$

and

$$\Phi * F = F * \Phi = \overline{F \circ \Phi}.$$

THEOREM 2.2. *With + and $*$, $\mathfrak{N}$ is a commutative ring with unity and without proper zero-divisors, and $\mathfrak{M}$ is a unitial $\mathfrak{N}$-module.*

Proof. In the course of this proof we shall often use the following obvious fact: If $D_1, D_2 \subset \mathscr{D}$ are normal then $D_1 * D_2$ is normal. Moreover, if in addition $D_1 * \mathscr{D} \subset D_1$ and $D_2 * \mathscr{D} \subset D_2$, then $D_1 * D_2 \subset D_1 \cap D_2$, and so $D_1 \cap D_2$ is also normal.

Suppose that $F, G \in \mathfrak{M}$ and $\Phi, \Psi \in \mathfrak{N}$. From the definition of $\mathfrak{M}$ it follows immediately that $\Phi^{-1}(\mathscr{D})$ and $\Psi^{-1}(\mathscr{D})$ are normal algebra ideals in $\mathscr{D}$.

The first step is to prove that $f + G$, $F * \Phi \in \mathfrak{M}$ and $\Phi + \Psi$, $\Phi * \Psi \in \mathfrak{N}$. *For example, we prove that* $F * \Phi \in \mathfrak{M}$. For all $\varphi \in D_F$ and $\psi \in \Phi^{-1}(\mathscr{D})$ we have

$$(F \circ \Phi)(\varphi * \psi) = F(\Phi(\varphi * \psi)) = F(\varphi * \Phi(\psi)) = F(\varphi) * \Phi(\psi).$$

Hence, it follows that $D_F * \Phi^{-1}(\mathscr{D}) \subset D_{F\circ\Phi}$. Thus $D_{F\circ\Phi}$ is normal. Moreover, for all $\varphi, \psi \in D_{F\circ\Phi}$, we have

$$(F \circ \Phi)(\varphi) * \psi = F(\Phi(\varphi)) * \psi = F(\Phi(\varphi) * \psi) = F(\varphi * \Phi(\psi))$$
$$= \varphi * F(\Phi(\psi)) = \varphi * (F \circ \Phi)(\psi).$$

Thus $F \circ \Phi \in \mathcal{M}$, and so $F * \Phi = \overline{F \circ \Phi} \in \mathfrak{M}$.

The next step is to prove the required commutative, associative and distributive laws for + and $*$. For example, we prove that $F * (\Phi + \Psi) = F * \Phi + F * \Psi$. Clearly, $D_F * (\Phi^{-1}(\mathscr{D}) \cap \Psi^{-1}(\mathscr{D}))$ is normal and for all $\varphi \in D_F * (\Phi^{-1}(\mathscr{D}) \cap \Psi^{-1}(\mathscr{D}))$, we have

$$\begin{aligned}(F*(\Phi+\Psi))(\varphi) &= (F\circ(\Phi+\Psi))(\varphi) = F(\Phi(\varphi)+\Psi(\varphi))\\ &= F(\Phi(\varphi))+F(\Psi(\varphi))\\ &= (F\circ\Phi+F\circ\Psi)(\varphi) = (F*\Phi+F*\Psi)(\varphi).\end{aligned}$$

Hence, using Corollary 1.10., we get $F*(\Phi+\Psi)=F*\Phi+F*\Psi$.

Now, let 0 and 1 be the functions defined on $\mathscr{D}$ by

$$0(\varphi)=0 \quad\text{and}\quad 1(\varphi)=\varphi.$$

Then $0, 1\in\mathfrak{N}$ and

$$F+0=\overline{F+0}=\bar{F}=F,$$

$$F+(-F)=\overline{F+(-F)}=\overline{0\,|\,D_F}=0$$

$$F*1=\overline{F\circ 1}=\bar{F}=F.$$

Finally, we prove that $\mathfrak{N}$ has no proper zero-divisors. For this suppose that $\Phi*\Psi=0$. Then for all $\varphi\in\Phi^{-1}(\mathscr{D})$ and $\psi\in\Psi^{-1}(\mathscr{D})$, we have

$$(\Phi*\Psi)(\varphi*\psi)=\Phi(\varphi)*\Psi(\psi)=0.$$

Since $\mathfrak{N}$ is commutative, it is no restriction to assume that $\Psi\neq 0$. Then there exists $\psi_\circ\in\Psi^{-1}(\mathscr{D})$ such that $\Psi(\psi_\circ)\neq 0$. Moreover, for all $\varphi\in\Phi^{-1}(\mathscr{D})$, we have $\Phi(\varphi)*\Psi(\psi_\circ)=0$. Thus $\Phi(\varphi)=0$ for all $\varphi\in\Phi^{-1}(\mathscr{D})$, and so $\Phi=0$.

LEMMA 2.3. (i) *Let $F\in\mathfrak{M}$. Then F is injective if and only if the range of F has no proper annihilators in $\mathscr{D}$.*

(ii) *Let $\Phi\in\mathfrak{N}$. Then Φ is injective if and only if $\Phi\neq 0$.*

Proof. (i) Suppose first that F is injective. Let $0\neq\varphi\in D_F$, $\chi\in\mathscr{D}$ and suppose that $\chi*F(D_F)=\{0\}$. Then

$$F(\chi*\varphi)=\chi*F(\varphi)=0.$$

Hence, since F is injective and $F(0)=0$, it follows that $\chi*\varphi=0$. This implies that $\chi=0$.

Suppose now that $F(D_F)$ has no proper annihilators in $\mathscr{D}$. Let $\varphi_1,\varphi_2\in D_F$ and suppose that $F(\varphi_1)=F(\varphi_2)$. Then

$$\varphi_1*F(\psi)=F(\varphi_1)*\psi=F(\varphi_2)*\psi=\varphi_2*F(\psi)$$

i.e., $(\varphi_1-\varphi_2)*F(\psi)=0$ for all $\psi\in D_F$. Hence, by the assumption, it follows that $\varphi_1=\varphi_2$.

(ii) The necessity is trivial. Suppose now that $\Phi \neq 0$. Then there exists $\varphi \in \Phi^{-1}(\mathscr{D})$ such that $\Phi(\varphi) \neq 0$. This implies that $\Phi(D_\Phi)$ has no proper annihilators in $\mathscr{D}$.

THEOREM 2.4. *Let $\Phi \in \mathfrak{N}$. Then Φ is invertible in $\mathfrak{N}$ if and only if the range of Φ has normal intersection with $\mathscr{D}$.*

Proof. Suppose first that Φ is invertible in $\mathfrak{N}$. Then there exists $\Psi \in \mathfrak{N}$ such that $\Phi * \Psi = 1$. Thus, for all $\varphi \in D_\Phi * \Psi^{-1}(D)$, we have

$$(\Phi * \Psi)(\varphi) = \Phi(\Psi(\varphi)) = 1(\varphi) = \varphi.$$

Hence $D_\Phi * \Psi^{-1}(\mathscr{D}) \subset \Phi(D_\Phi) \cap \mathscr{D}$. Thus $\Phi(D_\Phi) \cap \mathscr{D}$ is normal.

Suppose now that $\Phi(D_\Phi) \cap \mathscr{D}$ is normal. Then $\Phi \neq 0$, and so Φ is injective. Thus Φ^{-1} is a function. Moreover, for any $\varphi, \psi \in \Phi(D_\Phi) \cap \mathscr{D}$ we have

$$\Phi(\Phi^{-1}(\varphi) * \psi) = \Phi(\Phi^{-1}(\varphi)) * \psi = \varphi * \psi$$

and

$$\Phi(\varphi * \Phi^{-1}(\psi)) = \varphi * \Phi(\Phi^{-1}(\psi)) = \varphi * \psi,$$

i.e., $\Phi^{-1}(\varphi) * \psi = \varphi * \Phi^{-1}(\psi)$. Thus $\Phi^{-1} | \Phi(D_\Phi) \cap \mathscr{D} \in \mathscr{M}$. We prove that $\overline{\Phi^{-1} | \Phi(D_\Phi) \cap \mathscr{D}}$ is the inverse of Φ in $\mathfrak{N}$. Clearly, $D_\Phi * (\Phi(D_\Phi) \cap \mathscr{D})$ is normal, and for all $\varphi \in D_\Phi * (\Phi(D_\Phi) \cap \mathscr{D})$ we have

$$(\Phi * \overline{\Phi^{-1} | \Phi(D_\Phi) \cap \mathscr{D}})(\varphi) = \Phi(\Phi^{-1}(\varphi)) = \varphi = 1(\varphi).$$

This implies that $\Phi * \overline{\Phi^{-1} | \Phi(D_\Phi) \cap \mathscr{D}} = 1$.

EXAMPLES 2.5. Let $0 \neq \varphi \in \mathscr{D}$ such that $\int_{\mathbf{R}^k} \varphi = 0$, and let Φ be the function defined on $\mathscr{D}$ by $\Phi(\psi) = \varphi * \psi$. Then $0 \neq \Phi \in \mathfrak{N}$, but Φ is not invertible in $\mathfrak{N}$.

Namely, $\Phi(D_\Phi) \cap \mathscr{D} = \varphi * \mathscr{D}$ is not normal, since $f * (\varphi * \mathscr{D}) = \{0\}$ for all constant $f \in \mathscr{E}$.

3. Embedding of distributions. Let $\mathscr{D}'$ be the set of all Schwartz distributions on $\mathbf{R}^k$, and let $\mathscr{E}'$ be the subset of $\mathscr{D}'$ consisting of all those distributions with compact support.

It is known that, under addition and convolution, $\mathscr{E}'$ is a commutative ring with unity and without proper zero-divisors, and $\mathscr{D}'$ is a unitial $\mathscr{E}'$-module. Moreover, $\mathbf{C}$ and $\mathscr{E}$ are embedded in $\mathscr{D}'$ such that $\mathbf{C}, \mathscr{D} \subset \mathscr{E}'$.

DEFINITION 3.1. For any $\Lambda \in \mathscr{D}'$, let F_Λ be the function defined on $\mathscr{D}$ by

$$F_\Lambda(\varphi) = \Lambda * \varphi.$$

THEOREM 3.2. *The mapping defined on $\mathscr{D}'$ by*

$$\Lambda \to F_\Lambda$$

is an injection from $\mathscr{D}'$ into $\mathfrak{M}$ taking $\mathscr{E}'$ into $\mathfrak{M}$ such that

$$F_{\Lambda_1+\Lambda_2} = F_{\Lambda_1} + F_{\Lambda_2} \quad \textit{for all} \quad \Lambda_1, \Lambda_2 \in \mathscr{D}'$$

and

$$F_{\Lambda_1 * \Lambda_2} = F_{\Lambda_1} * F_{\Lambda_2} \quad \textit{for all} \quad \Lambda_1 \in \mathscr{E}' \quad \textit{and} \quad \Lambda_2 \in \mathscr{D}'.$$

Proof. Easy computation.

REMARK 3.3. Theorem 3.2. allows us to more and less identify Λ with F_Λ. But, for the easier understanding, we shall not make this identification complete.

THEOREM 3.4. *Let $F \in \mathfrak{M}$. Then $F = F_\Lambda$ for some $\Lambda \in \mathscr{D}'$ if and only if $D_F = \mathscr{D}$.*

Proof. See Lemma 3 in **[10]**.

In the following theorem $\mathscr{E}$ and $\mathscr{D}$ are supposed to be equipped with their usual topologies **[9]**.

THEOREM 3.5. *Let $F \in \mathfrak{M}$. Then $F = F_\Lambda$ for some $\Lambda \in \mathscr{D}'$ if and only if F is continuous.*

Proof. Suppose first that $F = F_\Lambda$ for some $\Lambda \in \mathscr{D}'$. Then $F(\varphi) = \Lambda * \varphi$ for all $\varphi \in \mathscr{D}$. Thus, by 6.33 Theorem in **[9]**, F is continuous.

Suppose now that F is continuous. Let λ be the function defined on $\check{D}_F = \{\check{\varphi} : \varphi \in D_F\}$ by

$$\lambda(\varphi) = F(\check{\varphi})(0).$$

Then λ is a continuous linear functional on the linear subspace $\check{D}_F$ of $\mathscr{D}$. Thus, by 3.6 Theorem in **[9]**, there exists $\Lambda \in \mathscr{D}'$ such that $\lambda \subset \Lambda$. Moreover, we have

$$F_\Lambda(\varphi)(t) = (\Lambda * \varphi)(t) = \Lambda(\mathcal{T}_t \check{\varphi}) = \lambda(\mathcal{T}_t \check{\varphi}) = F((\mathcal{T}_t \check{\varphi})^\vee)(0)$$
$$= F(\mathcal{T}_{-t}\varphi)(0) = (\mathcal{T}_{-t} F(\varphi))(0) = F(\varphi)(t)$$

for all $\varphi \in D_F$ and $t \in \mathbf{R}^k$. Hence, it follows that $F = F_\Lambda$.

REMARK 3.6. The proof of Theorem 3.5 shows that we might have used an apparently weaker continuity property of F in Theorem 3.5., if we had taken the topology of the pointwise convergence on $\mathbf{R}^k$ instead of the usual topology of $\mathscr{E}$ **[12]**.

LEMMA 3.7. *Let $\Phi \in \mathfrak{R}$. Suppose that $(\sigma)_{n=1}^\infty$ is an approximate identity (delta sequence* **[12]**) *such that $(\sigma_n)_{n=1}^\infty \subset D_\Phi$ and $\{\Phi(\sigma_n)\}_{n=1}^\infty \subset \mathscr{D}$. Then $\Phi(D_\Phi) \subset \mathscr{D}$.*

Proof. Let $\varphi, \psi \in D_\Phi$ such that $\Phi(\psi) \in \mathscr{D}$. Then

$$\Phi(\varphi) * \sigma_n = \varphi * \Phi(\sigma_n) \quad \text{and} \quad \Phi(\sigma_n) * \psi = \sigma_n * \Phi(\psi)$$

for all n. Hence,

$$\operatorname{supp} \Phi(\varphi) * \sigma_n \subset \operatorname{supp} \varphi + \operatorname{supp} \Phi(\sigma_n)$$

and, using Lions' theorem **[12]**,

$$[\operatorname{supp} \Phi(\sigma_n)] + [\operatorname{supp} \psi] = [\operatorname{supp} \sigma_n] + [\operatorname{supp} \Phi(\psi)]$$

for all n. Thus, we have

$$\operatorname{supp} \Phi(\varphi) * \sigma_n \subset \operatorname{supp} \varphi + [\operatorname{supp} \Phi(\psi)] - [\operatorname{supp} \psi] + [\operatorname{supp} \sigma_n]$$

for all n. Therefore, there exists a bounded set $B \subset \mathbf{R}^k$ such that

$$\operatorname{supp} \Phi(\varphi) * \sigma_n \subset B$$

for all n. Hence, since $\lim_{n\to\infty} \Phi(\varphi) * \sigma_n = \Phi(\varphi)$, it follows that

$$\operatorname{supp} \Phi(\varphi) \subset B.$$

EXAMPLE 3.8. Let Q be the quotient field of $\mathscr{D}$. Suppose that $q \in Q$ such that D_q contains an approximate identity, but $D_q \neq \mathscr{D} \setminus \{0\}$. (To make it clear, recall that Q consists of all

$$\frac{\varphi}{\psi} = \{(\sigma, \chi) \in (\mathscr{D}\setminus\{0\}) \times \mathscr{D} : \chi * \psi = \sigma * \varphi\}$$

such that $\varphi, \psi \in \mathcal{D}$ and $\psi \neq 0$. Concerning the existence of such a q see [1].) Then, it is clear that $q \in \mathcal{M}$, and so $\bar{q} \in \mathfrak{M}$. Moreover, by Lemma 3.7., we have $\bar{q}(D_{\bar{q}}) \subset \mathcal{D}$. Therefore $\bar{q} = q \cup \{(0, 0)\}$. Hence, since $D_q \neq \mathcal{D}\setminus\{0\}$, it follows that $D_{\bar{q}} \neq \mathcal{D}$. Thus, by Theorem 3.4., there is no $\Lambda \in \mathcal{D}'$ such that $\bar{q} = F_\Lambda$.

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