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UNIFORM APPROXIMATION BY ELEMENTS OF A CONE OF REAL-VALUED FUNCTIONS

WALTER ROTH

The present paper gives conditions for approximation of all functions of a subspace M of $C(K)$, where K is a compact Hausdorff space by elements of an "admissible" subcone N of M. This implies generalizations of well-known theorems of Bauer and Stone-Weierstrass.

1. Preliminaries.

Let K be a compact Hausdorff space, M a linear space of real-valued continuous functions on K. A subcone N of M is said to be admissible if it separates the points of K and contains the constant functions. Denote by N^c the set of functions in $C(K)$ that are the pointwise sup of finitely many functions in N. Let $\underset{N}{<}$ be the relation on Ω_K (the set of probability measures on K) defined by: $\mu \underset{N}{<} \nu$ iff for every f in N $\mu(f) \leqq \nu(f)$. Then $\underset{N^c}{<}$ is an order relation on Ω_K. For every $\mu \in \Omega_K$ there exists an N^c-maximal (maximal with respect to $\underset{N^c}{<}$) measure $\nu \in \Omega_K$ such that $\nu \underset{N^c}{>} \mu$ (see [1], I.5) The N^c-maximal measures live on every Baire set containing the Choquet boundary $Ch_K(N)$ of N.

$Ch_K(N) = \{x \in K \mid \text{for every } \mu \in \Omega_K\ u \underset{N}{>} \epsilon_x \text{ implies } \mu = \epsilon_x\}$ (ϵ_x is the point measure at x). For $f \in C(K)$ the upper and lower envelopes are defined by

$$\bar{f}_N = \inf\{h \in -N \mid h > f\} \qquad \underline{f}_N = \sup\{h \in N \mid h < f\}$$

$\bar{f}_N$ resp. $\underline{f}_N$ are upper resp. lower-semi-continuous and coincide with f on $Ch_K(N)$. For every N^c-maximal measure $\mu \in \Omega_K$ we have $\mu(f) = \mu(\bar{f}_N) = \mu(\underline{f}_N)$.

2. An approximation-theorem: conditions for measures.

THEOREM 1. *Let M be a linear space of continuous real-valued functions on the compact Hausdorff space K and N an admissible cone. N is uniformly dense in M* (dense with respect to the sup-norm) *if and only if*

(1) *For all N^c-maximal measures μ, ν in Ω_K $\mu \underset{N}{>} \nu$ implies $\mu \underset{M}{>} \nu$, and one of the following conditions holds*

(2a) *Every M^c-maximal measure of Ω_K is N^c-maximal.*

(2b) *$f_N = f$ for every f in M.*

If K is metrizable or if $Ch_K(N)$ is compact (2a) *and* (2b) *can be replaced by* (2c) *$Ch_K(N) = Ch_K(M)$*

The proof will use

LEMMA 1. *N is uniformly dense in M if and only if for all μ, ν in Ω_K $\mu \underset{N}{>} \nu$ implies $\mu \underset{M}{>} \nu$.*

This Lemma is an immediate consequence of the Hahn-Banach Theorem and still holds if both N and M are admissible cones. Using the fact that the uniform closure of N in $C(K)$ is given by

$$\bar{N} = \{f \in C(K) \mid \text{such that for every measure } \lambda \text{ on } K\ \lambda(N) \geqq 0 \text{ implies } \lambda(f) \geqq 0.\}$$

we decompose each such measure $\lambda = \frac{1}{2}\|\lambda\|\ (\mu - \nu)$, where $\mu, \nu \in \Omega_K$ and deduce Lemma 1.

To prove the theorem

(α) suppose (1) and (2a) hold, $\mu, \nu \in \Omega_K$ such that $\mu \underset{N}{>} \nu$.

Let μ' be M^c-maximal and $\mu' \underset{M^c}{>} \mu$ ν' be M^c-maximal and $\nu' \underset{M^c}{>} \nu$. Then μ' and ν' are N^c-maximal (2a) and $\mu' \underset{N^c}{>} \mu$, $\nu' \underset{N^c}{>} \nu$ because $N^c \subset M^c$). M is a linear space, therefore $\underset{M}{<}$ implies $\underset{M}{>}$ (indeed $\underset{M}{<}$ is an equivalence relation $\widetilde{M}$) and we conclude

$$\left.\begin{array}{l}\mu'\widetilde{M}\mu\\ \nu'\widetilde{M}\nu\end{array}\right. \Rightarrow \mu'(f) = \mu(f) \geqq \nu(f) = \nu'(f) \text{ for every } f \text{ in } N.$$

i.e. $\mu' \underset{N}{>} \nu'$ which implies by (1) $\mu' \underset{M}{>} \nu'$, hence $\mu \underset{M}{>} \nu$. From Lemma 1 we deduce $M \subset \bar{N}$.

(β) (2b) implies (2a) because for every $f \in M$ the set $\{h \in N^c \mid h < f\}$ is directed upward converging pointwise (2b) to the continuous function f. By Dini's Lemma the convergence is uniform, hence $f \in \overline{N^c}$, $M \subset \overline{N^c}$, $M^c \subset \overline{N^c}$. N^c and M^c define the same order relation on Ω_K.

(γ) Suppose K is metrizable or $Ch_K(N)$ is compact. In this case $Ch_K(N) = Ch_K(M)$ implies (2a) because every measure in Ω_K is N^c-maximal (resp. M^c-maximal) then if and only if it lives on $Ch_K(N)$ (resp. $Ch_K(M)$).

As a corollary of Theorem 1 we find a well-known theorem (**[1]**, Th. II.4.5) on affine functions on a compact convex set:

COROLLARY 1. *Let A be the set of the affine continuous functions on the compact convex set X, $Ex(X)$ the extreme points of X, f: $Ex(X) \to \mathbf{R}$ a bounded continuous function* (in the relative topology on $Ex(X)$).

f can be extended to a function in A if and only if

(1) $\bar{f} = \underline{f}$ on $\overline{Ex(X)}$ $\quad (\bar{f} = \inf\{g \in A \mid g > f \text{ on } Ex(X)\}$
$\underline{f} = \sup\{g \in A \mid g < f \text{ on } Ex(X)\})$

and

(2) *For all A^c-maximal measures μ, ν in Ω_x $\mu \underset{A}{>} \nu$ implies $\mu(\bar{f}) = \nu(\bar{f})$.*

To prove the corollary suppose $K = \overline{Ex(X)}$, $N = A\,|_K$ and $M = N + \mathbf{R}\cdot f$, where $f = \underline{f}\,|_K = \bar{f}\,|_K \in C(K)$. Clearly N is closed in $C(K)$ and from Theorem 1 (2b) we see that N is uniformly dense in M, i.e. $N = M$.

REMARK. It is easy to show that if N is a linear space too, then condition (1) in Theorem 1 can be reduced to the comparison of maximal simplex measure (c.f. **[1]**, Chapter I.6), i.e.

(1′) For all N^c-maximal simplex measures μ, ν in Ω_K $\mu \underset{N}{>} \nu$ implies $\mu \underset{M}{>} \nu$.

3. Conditions for the order relation. In this section we are going to replace (1) in Theorem 1 by conditions in terms of the order relation $\leqq$ in the function sets N and M. We use a simple generalization of a theorem of Cartier, Fell and Meyer:

LEMMA 2. *Let M be a linear space of continuous real-valued functions on the compact Hausdorff space K, N an admissible cone in M. Suppose that for every function $f \in M$ the set*

$$\{g \in N \mid g < f\}$$

is directed upward. Then for all $\mu, \nu, \nu_1, \nu_2 \in \Omega_k$, $\lambda_1, \lambda_2 \geqq 0$ such that $\mu \underset{N}{>} \nu, \lambda_1 + \lambda_2 = 1$ and

$$\text{for every } f \text{ in } M \ \nu(f) = \lambda_1 \nu_1(f) + \lambda_2 \nu_2(f)$$

there exist $\mu_1, \mu_2 \in \Omega_K$ such that $\mu_1 \underset{N}{>} \nu_1$, $\mu_2 \underset{N}{>} \nu_2$ and

$$\text{for every } f \text{ in } M \ \mu(f) = \lambda_1 \mu_1(f) + \lambda_2 \mu_2(f).$$

To prove the lemma assume $\lambda_1, \lambda_2 > 0$. Define $\phi: M \times M \to \mathbf{R}$ by

$$\phi(f, g) = \lambda_1 \nu_1(\bar{f}_N) + \lambda_2 \nu_2(\bar{g}_N)$$

Then

(α) ϕ is positive subhomogeneous.

(β) For every $f \in M$, also $-f \in M$ and therefore

$\{h \in N \mid h < -f\}$ is directed upward, hence

$\{h \in -N \mid h > f\}$ is directed downward, hence

$$\begin{aligned} \mu(\bar{f}_N) &= \inf\{\mu(h) \mid h \in -M,\ h > f\} \\ &\leqq \inf\{\nu(h) \mid h \in -N,\ h > f\} = \nu(\bar{f}_N). \end{aligned}$$

(γ) Suppose $F = \{(f, f) \in M \times M\}$ and

$$\psi_0: F \to R, \qquad \psi_0(f, f) = \mu(f).$$

Then $\psi_0(f, f) = \mu(f) \leqq \mu(\bar{f}_N) \leqq \nu(\bar{f}_N) = \phi(f, f)$.

(δ) By the Hahn-Banach Theorem there exists a linear extension

$$\psi: M \times M \to \mathbf{R} \text{ of } \psi_0 \text{ such that } \psi \leqq \phi.$$

(ϵ) Let μ_1, μ_2 be probability measures on K such that for every f in M

$$\mu_1(f) = 1/\lambda_1 \psi(f, 0) \quad \text{and} \quad \mu_2(f) = 1/\lambda_2 \psi(0, f)$$

Then we conclude

For every f in M $\mu(f) = \psi_0(f,f) = \psi(f,f) = \lambda_1\mu_1(f) + \lambda_2\mu_2(f)$.

For every f in $-N$ $\mu_1(f) = 1/\lambda_1\psi(f,0) \leqq 1/\lambda_1\phi(f,0) = \nu_1(\bar{f}_N) = \nu_1(f)$.

Hence $\nu_1 \underset{N}{<} \mu_1$ and $\nu_2 \underset{N}{<} \mu_2$
which proves the lemma. Now we can state:

THEOREM 2. *Let M be a linear space of continuous real-valued functions on the compact Hausdorff space K, N an admissible cone in M. N is uniformly dense in M if and only if*

(1) *For every $f \in M$ the set*

$$\{g \in N \mid g < f\}$$

is directed upward and

(2) $Ch_K(N) = Ch_K(M)$.

Proof. Of course $M \subset \bar{N}$ implies (1) and (2). Suppose now (1) and (2) hold and

$$R = \{(\nu, \mu) \in M' \times M' \mid \quad \mu, \nu \geqq 0, \quad \|\mu\| = \|\nu\| = 1, \quad \nu \underset{N}{<} \mu\}$$

$$S = \{(\mu, \mu) \in M' \times M' \mid \quad \mu \geqq 0 \quad \|\mu\| = 1\}$$

Both R and S are compact convex in the weak topology on $M' \times M'$, $S \subset R$. We show $S = R$ and apply Lemma 1.

Lemma 2 shows that

$$Ex(R) \subset \{(\epsilon_x, \mu) \in R \mid x \in Ch_K(M)\}$$

because every $\mu \in M'$, $\mu \neq \epsilon_x$ for every $x \in Ch_K(M)$, is a convex combination of two different elements of M'. From $Ch_K(N) = Ch_K(M)$ we conclude now

$$(\epsilon_x, \mu) \in Ex(R) \Rightarrow \epsilon_x \underset{N}{<} \mu \Rightarrow \epsilon_x = \mu \Rightarrow (\epsilon_x, \mu) \in S,$$

hence $Ex(R) \subset S$, which implies $R = S$ and proves the theorem.

4. Applications: The case $M = C(X)$. If we choose $M = C(X)$ Theorem 2 leads to generalizations of theorems of Bauer and of Stone-Weierstrass.

THEOREM 3. *Let N be an admissible cone of continuous real-valued functions on the compact Hausdorff space K. The following conditions are equivalent*:

(1) *N is uniformly dense in CK).*

(2) *$\bar{N}$ is maximum-stable, i.e. $N^c \subset \bar{N}$, and $Ch_K(N) = Ch_K(N-N)$.*

(3) *For all $f, g \in N$ $\inf\{h \in N \mid h > f \vee g\} \in \bar{N}$ ($f \vee g$ is the pointwise sup of f and g)*

$$\text{and } Ch_K(N) = Ch_K(N-N) \quad \text{and} \quad \overline{Ch_K(-N)} = K.$$

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial. Suppose (2) holds: Theorem 2 shows that $N - N \subset \bar{N}$, hence $\bar{N}$ is a linear space. $N^c \subset \bar{N}$ therefore implies $(N^c - N^c) \subset \bar{N}$. From the classical Stone-Weierstrass theorem we know that $N^c - N^c$ is dense in $C(K)$ which proves $\bar{N} = C(K)$.

Suppose now (3) holds. We show that $N^c \subset \bar{N}$ and therefore (3) implies (2): Let $f, g \in N$. Then

$$h = \overline{f \vee g}_{(-N)} = \inf\{j \in N \mid j > f \vee g\} \in \bar{N}$$

Now $h|_{Ch_K(-N)} = f \vee g|_{Ch_K(-N)}$ and $\overline{Ch_K(-N)} = K$ imply $h = f \vee g \in \bar{N}$, hence $N^c \subset \bar{N}$.

Bauer's theorem (see **[1]**, Th. II.4.1) is a special case of the following.

THEOREM 4. *Let N be a uniformly complete admissible cone of continuous real-valued functions on the compact Hausdorff space X, $K = \overline{Ch_X(N-N)}$. The following conditions are equivalent*:

(1) $N|_K = C(K)$.

(2) *For all N^c-maximal measures μ, ν in Ω_x $\mu \underset{N}{>} \nu$ implies $\mu = \nu$ and $Ch_X(N) = K$.*

(3) *For all $f, g \in N$ $\inf\{h \in N \mid h > f \vee g\} \in N$ and $Ch_X(N) = \overline{Ch_X(N-N)}$ and $\overline{Ch_X(-N)} = K$.*

Proof. Clearly $N|_K$ is closed in $C(K)$. (1) $\Leftrightarrow$ (2) is an immediate consequence of Theorem 1. (1) $\Rightarrow$ (3) is trivial. To prove the implication (3) $\Rightarrow$ (1) we take f, $g \in N$. (3) guarantees the existence of an $h \in N$ such that $h = \inf\{j \in N \mid j > f \vee g\}$. From $\overline{Ch_K(-N)} = K$ we conclude now $h|_K = f \vee g|_K$. Hence $N|_K$ is maximum stable and Theorem (3) proves $N|_K = C(K)$. (Obviously $Ch_K(N|_K) = Ch_X(N)$.)

5. Examples.

1. Suppose K is the finite set $\{x_1, \cdots, x_n\}$, N an admissible cone of functions (here real-valued finite sequences) on K. Then $Ch_K(N)$ consists of all $x_j \in K$ such that there exists $f \in N$, such that $f(x_i) < f(x_j)$ for all $i \neq j$. (see [4], §7, Ex. 3). In this case Theorem 3 states:

$N = C(K) = \mathbf{R}^n$ iff for all $f, g \in N$ $\inf\{h \in N \mid h > f \vee g\} \in N$ and for every $x_j \in K$ there are $f, g \in N$ such that if $i \neq j$ $f(x_i) < f(x_j)$ and $g(x_i) > g(x_j)$.

2. Let K be the compact subset $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, 0\}$ of $\mathbf{R}$. Then $C(K)$ is the set of all convergent real-valued sequences.

$$N = \{(y)_n \in C(K) \mid (y)_n = (\alpha_1, \alpha_2, \cdots \alpha_k) + \lambda(1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \cdots, 0) + r,$$

$$\text{where} \quad r \in \mathbf{R},\ k \in \mathbf{N},\ \lambda \leqq \min\{0, \alpha_1, \alpha_2, \cdots \alpha_k\}\}$$

defines an admissible cone in $C(K)$ with the properties:

(1) $N \cap (-N) = \mathbf{R}$:
Let $f \in N \cap (-N)$. Then

$$f = (\alpha_1, \cdots \alpha_k) + \lambda(1, \tfrac{1}{2}, \cdots) + r = -(\beta_1, \cdots \beta_k) - \rho(1, \tfrac{1}{2}, \cdots) = r'$$

$$\Rightarrow \lambda = -\rho = 0 \quad \text{because} \quad \lambda, \rho \leqq 0$$

$$\Rightarrow \alpha_1, \cdots, \alpha_k \geqq 0, \qquad \beta_1, \cdots, \beta_k \geqq 0.$$

$$\Rightarrow \alpha_1 = -\beta_1 = 0, \cdots \qquad \alpha_k = -\beta_k = 0.$$

$$\Rightarrow f \in \mathbf{R}.$$

(2) Every $x \in K$ is a maximum point of an element of N. (For $x = 0$ choose $\alpha_i = 0$, $\lambda = -1$). Therefore $Ch_K(N) = K$ and $Ch_K(N) = Ch_K(N - N) = K$.

Every $1/n \in K$ is a minimum point of an element of N, hence $Ch_K(-N) = K$.

(3) N is maximum-stable.

From Theorem 3 we conclude that N is uniformly dense in $C(X)$.

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University of California, Berkeley