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FIXED POINT ITERATIONS OF NONEXPANSIVE MAPPINGS

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Let C be a boundedly weakly compact convex subset of a Banach space E. Suppose that each weakly compact convex subset of C possesses the fixed point property for nonexpansive mappings, and let $T: C \to C$ be nonexpansive. In this note it is shown (by a very simple argument) that if a sequence of iterates of T (generated with the aid of an infinite, lower triangular, regular row-stochastic matrix) is bounded, then T has a fixed point.

Dotson and Mann [3] proved this theorem under the additional assumption that E was uniformly convex. (Their complicated proof relied heavily on the uniform convexity of E.) We use our method also to establish a similar result (essentially due to Browder) for nonlinear nonexpansive semigroups.

Let C be a closed convex subset of a Banach space $(E, |\ \ |)$, and let $T: C \to C$ be nonexpansive (that is, $|Tx - Ty| \leqq |x - y|$ for all x and y in C). Let N denote the set of nonnegative integers, and suppose $A = \{a_{nk}: n, k \in N\}$ is an infinite matrix satisfying

$$a_{nk} \geqq 0 \quad \text{for all} \quad n, k \in N,$$
$$a_{nk} = 0 \quad \text{if} \quad k > n,$$
$$\sum_{k=0}^{n} a_{nk} = 1 \quad \text{for all} \quad n \in N,$$
$$\lim_{n \to \infty} a_{nk} = 0 \quad \text{for all} \quad k \in N.$$

If x_0 belongs to C, then a sequence $S = \{x_n: n \in N\} \subset C$ can be defined inductively by

$$x_n = a_{n0}\, x_0 + \sum_{k=1}^{n} a_{nk}\, T x_{k-1}, \qquad n \in N.$$

This iteration scheme is due to Mann [8].

It is not difficult to see that if T has a fixed point, then S is bounded. Dotson and Mann [3, Theorem 1] have proved that if E is uniformly convex, and if S is bounded for some x_0 in C, then T has a fixed point. Their proof is rather complicated and relies heavily on the uniform convexity of E. In this note we establish a far-reaching extension of the Dotson-Mann theorem in a very simple manner. We remark in passing that a special case of the Dotson-Mann result was independently established by Reinermann [**10**, p. 10]. He assumed that A is column-finite.

THEOREM 1. *Let C be a boundedly weakly compact convex subset of a Banach space E. Suppose that each weakly compact convex subset of C possesses the fixed point property for nonexpansive mappings, and that $T: C \to C$ is nonexpansive. If the sequence S defined above is bounded for some x_0 in C, then T has a fixed point.*

Proof. Pick a point y in C, and set $R = \limsup_{n\to\infty} |y - x_n|$. R is finite because S is bounded. Let $K = \{z \in C: \limsup_{n\to\infty} |z - x_n| \leqq R\}$. K is a non-empty bounded closed convex (hence weakly compact) subset of C. Now let z be in K. Then

$$\begin{aligned} |Tz - x_n| &\leqq a_{n0}|Tz - x_0| + \sum_{k=1}^{n} a_{nk}\,|Tz - Tx_{k-1}| \\ &\leqq a_{n0}|Tz - x_0| + \sum_{k=1}^{n} a_{nk}\,|z - x_{k-1}|. \end{aligned}$$

For each positive ϵ, there is $m(\epsilon) \in N$ such that $|z - x_k| < R + \epsilon$ for all $k > m$. Therefore we obtain for $n > m + 1$,

$$\begin{aligned} |Tz - x_n| &\leqq a_{n0}|Tz - x_0| + \sum_{k=1}^{m+1} a_{nk}\,|z - x_{k-1}| + \sum_{k=m+2}^{n} a_{nk}\,(R + \epsilon) \\ &\leqq a_{n0}|Tz - x_0| + \sum_{k=1}^{m+1} a_{nk}\,|z - x_{k-1}| + R + \epsilon \\ &= h(n) + R + \epsilon \end{aligned}$$

where $\lim_{n\to\infty} h(n) = 0$. Thus Tz belongs to K, and the result follows.

REMARK. S need not converge, even if E is a Hilbert space [**5**].

In the setting of Theorem 1, let $r_m = \inf\{r$: there exists $y \in C$ such that $|y - x_n| \leqq r$ for all $n \geqq m\}$, and $R = \lim_{n\to\infty} r_m$. Since C is convex and boundedly weakly compact, there is at least one point z in C such that $\limsup_{n\to\infty} |z - x_n| = R$. Such a point is called an asymptotic center of S with respect to C (cf. [**4**]). The proof of Theorem 1 shows that the set of asymptotic centers of S with respect to C is invariant under T (cf. [**9**, p. 253]). Consequently, it contains a fixed point of T. In particular, if the asymptotic center is unique (this indeed happens when E is uniformly convex, or more generally, uniformly convex in every direction **[2]**), then it is a fixed point of T. Note that a weakly compact convex subset of a Banach space which is uniformly convex in every direction has normal structure and therefore possesses the fixed point property for nonexpansive mappings **[6]**.

The idea of the proof of Theorem 1 can be also applied to a result on nonlinear nonexpansive semigroups which is essentially due to Browder **[1]**.

Recall that a nonexpansive semigroup on a subset D of a Banach space E is a function U: $[0, \infty) \times D \to D$ satisfying the following conditions:

$$U(t_1 + t_2, x) = U(t_1, U(t_2, x)) \quad \text{for} \quad t_1, t_2 \geqq 0$$

and $x \in D$,

$$|U(t, x) - U(t, y)| \leqq |x - y| \quad \text{for} \quad t \geqq 0$$

and x, $y \in D$,

$$U(0, x) = x \quad \text{for} \quad x \in D.$$

A semigroup U is called bounded if for each x in D there is $M(x)$ such that $|U(t, x)| \leqq M(x)$ for all $t \geqq 0$. It is said to have a fixed point x_0 if $U(t, x_0) = x_0$ for all $t \geqq 0$. If U has a fixed point, then it is clearly bounded. In order to prove the converse statement, we shall assume that D has the common fixed point property for nonexpansive mappings. This means that every commuting family of nonexpansive self-mappings of D has a common fixed point.

THEOREM 2. *Let C be a boundedly weakly compact convex subset of a Banach space E. Suppose that each weakly compact convex subset of C possesses the common fixed point property for nonexpansive mappings, and that U: $[0, \infty) \times C \to C$ is a nonexpansive semigroup. If U is bounded, then it has a fixed point.*

Proof. Fix a point x_0 in C, and let y be another point in C. $R = \limsup_{t\to\infty} |y - U(t, x_0)|$ is finite because the orbit $\{U(t, x_0)\colon t \geqq 0\}$ is bounded. Let $K = \{z \in C\colon \limsup_{t\to\infty} |z - U(t, x_0)| \leqq R\}$. K is a nonempty bounded closed convex (hence weakly compact) subset of C. If $z \in K$, $t_0 \geqq 0$, $\epsilon > 0$, and t is large enough, then

$$\begin{aligned} |U(t_0, z) - U(t, x_0)| &= |U(t_0, z) - U(t_0, U(t - t_0, x_0))| \\ &\leqq |z - U(t - t_0, x_0)| < R + \epsilon. \end{aligned}$$

Consequently, $U(t_0, z)$ also belongs to K. Thus K is invariant under the commuting family of nonexpansive mappings $\{U(t, \cdot): t \geqq 0\}$. Hence the result.

In the setting of Theorem 2 we can also define an asymptotic center, this time for $\{U(t, x_0): t \geqq 0\}$. If E is uniformly convex in every direction, this asymptotic center is unique. Moreover, a weakly compact convex subset of E has normal structure and therefore possesses the common fixed point property for nonexpansive mappings **[7]**. The proof of Theorem 2 shows that in this case the asymptotic center of $\{U(t, x_0): t \geqq 0\}$ is a fixed point of U.

REMARK. A version of Theorem 2 is true for arbitrary commutative semigroups of nonexpansive mappings.

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