

DIFFERENTIABILITY CONDITIONS AND BOUNDS ON SINGULAR POINTS

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It is well-known that a normal arc \mathcal{A}_4 of cyclic order four in the conformal plane contains at most finitely many singular points and in fact at most eleven. This bound can be reduced to four in the case of a strongly differentiable \mathcal{A}_4 . Using a characterization of singular points on such arcs this paper shows that strong differentiability is not a necessary condition for this bound. In fact a much weaker condition, viz., the existence of tangent circles, is sufficient to obtain four as the least upper bound.

In [4] a conformal proof is given for the following result. "A normal arc \mathcal{A}_4 of cyclic order four contains at most eleven singular points."

That a strongly differentiable ([3], 3.1) \mathcal{A}_4 contains at most four singular points can be found in 4.1.4.3 of [1] and in 3.6 of [4].

In §3 it is shown that assuming only Condition I ([3], 1.5) the maximum number of singular points on such arcs and curves is still four and that this is the best possible bound.

In order to obtain this result it is necessary to categorize the different possible types of singular points on such arcs. This characterization is similar to that of O. Haupt and H. Künneht for linearly singular points of arcs of linear order three ([1], 3.2.1).

The definitions and notations used in this paper can be found in [2] and [3]. We include the notations of ordinary and strong conformal differentiability for the reader's convenience. It is obvious that strong differentiability implies ordinary differentiability.

(a) A point p on an arc \mathcal{A} is said to be (conformally) *differentiable* if it satisfies two conditions:

Condition I. For every point $R \neq p$, and for every sequence of points $s \rightarrow p$, $s \in A$, $s \neq p$, there exists a circle C_0 such that $C(p, s, R) \rightarrow C_0$. C_0 is called the *tangent circle* of \mathcal{A} at p through R and is denoted $C(\tau, R)$.

Condition II. If $s \rightarrow p$, $s \neq p$, there exists a circle $C(p)$ such that $C(\tau, s) \rightarrow C(p)$. $C(p)$ is called the *osculating circle* of \mathcal{A} at p .

(b) A point p on an arc \mathcal{A} is said to be (conformally) *strongly differentiable* if it satisfies the following conditions:

Condition I'. Let $R \cong p$, $Q \rightarrow R$. If two distinct points u and v converge on \mathcal{A} to p , then $C(u, v, Q)$ always converges.

Condition II'. $C(t, u, v)$ converges if the three mutually distinct points t, u, v converge on \mathcal{A} to p .

1. **Types of singular points.** Hereafter, when considering a two-sided neighbourhood $L \cup \{z\} \cup M$ of an interior point z on an arc \mathcal{A} , we assume that \mathcal{A} has been oriented so that $L[M]$ is a preceding [proceeding] one-sided neighbourhood of z on \mathcal{A} .

DEFINITION. Let z be a singular point of an oriented arc \mathcal{A}_4 of cyclic order four. Then z is $(1, 3)$ [(3, 1); (2, 2)] singular provided that for any two-sided neighbourhood $L \cup \{z\} \cup M$ of z on \mathcal{A}_4 there exists a circle which meets L and M once and three times [three times and once; twice], respectively.

REMARK. (i) Counting multiplicities for this definition, each of the points in common with the circle and \mathcal{A}_4 must be intersection points.

(ii) For normal arcs \mathcal{A}_4 of cyclic order four, these are the only possible singular points. Otherwise, we would have what might be called a $(4, 0)$ or $(0, 4)$ singular point z . But z is elementary ([3], 5.1) since \mathcal{A}_4 contains only finitely many singular points. Thus a $(4, 0)$ or $(0, 4)$ singular point does not exist for normal arcs \mathcal{A}_4 of cyclic order four.

(iii) In each case there exists a limiting supporting general osculating circle of \mathcal{A}_4 at z which may be the point circle. If z is $(1, 3)$ or $(3, 1)$ singular, then one of the limiting supporting general osculating circles is the one-sided osculating circle of \mathcal{A}_4 at z from the right or left, respectively, since \mathcal{A}_4 is one-sidedly strongly differentiable at z ; cf. ([4], 3.4).

(iv) Let \mathcal{A}_4 be differentiable at a singular point z . Then the osculating circle supports \mathcal{A}_4 at z ; cf. ([2], 9) and ([3], Theorem 4). But \mathcal{A}_4 is one-sidedly strongly differentiable at z . Thus differentiable singular points are both $(3, 1)$ and $(1, 3)$ singular.

For the remainder of the paper \mathcal{A}_4 will denote a normal oriented arc of cyclic order four. Since end-points of such arcs are ordinary; cf. ([4], 3.3), we will restrict our attention to open arcs. The inclusion of end points will involve no additional singular points.

LEMMA 1.1. *Let z be a (2, 2) singular point on \mathcal{A}_4 which is neither (3, 1) nor (1, 3) singular. Then \mathcal{A}_4 does not satisfy Condition I at z .*

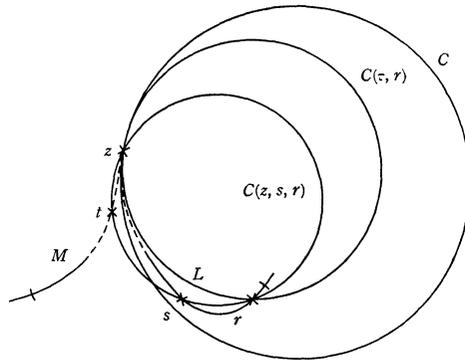
Proof. Since z is neither (3, 1) nor (1, 3) singular, the two one-sided osculating circles of \mathcal{A}_4 at z intersect \mathcal{A}_4 at z . Let us assume that \mathcal{A}_4 satisfies Condition I at z . Thus the family τ of tangent circles of \mathcal{A}_4 at z is a pencil of the second kind at z ; i.e., the circles of τ touch each other at z . There are two possible cases to consider.

Case (i). z is not a cusp point; i.e., the nontangent circles of \mathcal{A}_4 at z all intersect \mathcal{A}_4 at z . But then \mathcal{A}_4 satisfies Condition I' ([3], 3.1) at z ; cf. ([3], Theorem 5). Let $L \cup \{z\} \cup M$ be a sufficiently small two-sided neighbourhood of z on \mathcal{A}_4 . Since z is (2, 2) singular, but not (3, 1) or (1, 3) singular, there is a circle which intersects each of L and M at two points but no circle which meets L three times and M once, or L once and M three times. Call these points $l < l' < m' < m$.

Now let t move monotonically and continuously from l' on \mathcal{A}_4 toward z . Then there is a point $u \in C(l, t, m)$ on \mathcal{A}_4 which moves monotonically and continuously from m' toward z ; cf. ([4], 2.1). This movement will yield $t = u = z$ as t reaches z . Thus we obtain a general tangent circle K of \mathcal{A}_4 at z intersecting L at l and M at m . But K is then a tangent circle of \mathcal{A}_4 at z belonging to the pencil τ above, since Condition I' is satisfied at z . Also K supports \mathcal{A}_4 at z ; otherwise a suitable circle close to K would intersect \mathcal{A}_4 at three points close to z and also at l and m , thus violating the order of \mathcal{A}_4 . But then, since Condition I' is satisfied at z , a circle close to K through l, r, r' , where $r, r' \in L$ with r, r' close to z , will meet M at a point s close to m . Thus this circle meets L three times and M once; a contradiction.

Case (ii). z is a cusp point; i.e., the nontangent circles of \mathcal{A}_4 at z all support \mathcal{A}_4 at z . Let $L \cup \{z\} \cup M$ be a small two-sided neighbourhood of z on \mathcal{A}_4 . Let $r \in L$. Then the tangent circle $C(\tau, r)$ of \mathcal{A}_4 at z through r is close to the osculating circle C of $L \cup \{z\}$ at z . Since C intersects \mathcal{A}_4 at z , then $C(\tau, r)$ supports \mathcal{A}_4 at z . Next let s be close to z on $L, s \approx r$. Then the nontangent circle $C(z, s, r)$ is close to $C(\tau, r)$ and supports \mathcal{A}_4 at z , since z is a cusp point. It therefore intersects M at a point t . Hence a circle close to $C(z, s, r)$ will meet \mathcal{A}_4 at least five times; a contradiction.

Thus our assumption that \mathcal{A}_4 satisfies Condition I is incorrect and we have desired result.



2. Γ_a -singular points. Let a be the end-point of \mathcal{A}_4 with $a < s$ for all $s \in \mathcal{A}_4$. Let Γ_a be the system of circles passing through a . Then y is Γ_a -singular ([4], 2.3) if, for any neighbourhood N of y on \mathcal{A}_4 there exists a circle of Γ_a that meets N at least three times.

LEMMA 2.1. Let z_1, z_2 be two singular points of \mathcal{A}_4 . Also let a be the end-point of \mathcal{A}_4 with $a < z_1 < z_2$. Then

- (a) If z_1 and z_2 are (3, 1) and (1, 3) singular, respectively, then there is at least one Γ_a -singular point s on \mathcal{A}_4 with $z_1 \leq s \leq z_2$.
- (b) If z_1 and z_2 are (1, 3) and (3, 1) singular, respectively, then there is at least one Γ_a -singular point s on \mathcal{A}_4 with $z_1 < s < z_2$.
- (c) If z_1, z_2 are both (1, 3) singular, then there is at least one Γ_a -singular point s on \mathcal{A}_4 with $z_1 < s \leq z_2$.
- (d) If z_1, z_2 are both (3, 1) singular, then there is at least one Γ_a -singular point s on \mathcal{A}_4 with $z_1 \leq s < z_2$.

Proof. In all cases there exists at least one Γ_a -singular point s on \mathcal{A}_4 with $z_1 \leq s \leq z_2$; cf. ([4], 2.5). Hence (a) is obtained automatically.

To obtain (b) refer to the proof of 2.4 in [4], where $N_1 = L_1 \cup \{z_1\} \cup M_1$ and $N_2 = L_2 \cup \{z_2\} \cup M_2$ are two-sided neighbourhoods of z_1 and z_2 on \mathcal{A}_4 , respectively. Since z_1 is (1, 3) singular there exists a circle meeting L_1 once and M_1 three times. Now z_2 is (3, 1) singular. Hence there exists a circle meeting L_2 three times and M_2 once. The process used in the proof with respect to these two pairs of quadruplets enables one to construct a circle of Γ_a meeting the closed subarc \mathcal{B} of \mathcal{A}_4 bounded by z_1 and z_2 at three interior points. Then by 2.3 of [4], there exists at least one Γ_a -singular point s in the interior of \mathcal{B} ; i.e., $z_1 < s < z_2$. Arguments similar to those used above yield (c) and (d).

LEMMA 2.2. *Let $z_1 < z_2 < \dots < z_n$ be n singular points of \mathcal{A}_4 none of which are (2, 2) singular. Let a be the end-point of \mathcal{A}_4 with $a < z_1$. Then \mathcal{A}_4 contains at least $n - 1$ Γ_a -singular points.*

Proof. By our assumptions and remark (ii) of §1, z_i is (1, 3) singular but not (3, 1) singular, or (3, 1) singular but not (1, 3) singular, or both (3, 1) and (1, 3) singular; $i = 1, 2, \dots, n$. For $i = 1, 2, \dots, n - 1$, define the intervals

$$\langle z_i, z_{i+1} \rangle = \begin{cases} (z_i, z_{i+1}) \text{ if } z_i \text{ is (1, 3) singular and } z_{i+1} \text{ is (3, 1) singular,} \\ [z_i, z_{i+1}] \text{ if } z_i \text{ is not (1, 3) singular and } z_{i+1} \text{ is not (3, 1) singular,} \\ [z_i, z_{i+1}) \text{ if } z_i \text{ is not (1, 3) singular and } z_{i+1} \text{ is (3, 1) singular,} \\ (z_i, z_{i+1}] \text{ if } z_i \text{ is (1, 3) singular and } z_{i+1} \text{ is not (3, 1) singular.} \end{cases}$$

Then $\langle z_i, z_{i+1} \rangle$ and $\langle z_{i+1}, z_{i+2} \rangle$ are disjoint subsets of \mathcal{A}_4 for $i = 1, 2, \dots, n - 2$, and by Lemma 2.1 there is a Γ_a -singular point in $\langle z_i, z_{i+1} \rangle$ for $i = 1, 2, \dots, n - 1$. Hence \mathcal{A}_4 contains at least $n - 1$ Γ_a -singular points.

3. The main result. We are now ready to derive the desired result.

THEOREM 1. *A normal arc \mathcal{A}_4 of cyclic order four, satisfying Condition I at each point, contains at most four singular points.*

Proof. By Lemma 1.1, all the singular points of \mathcal{A}_4 are of type (3, 1) or (1, 3). Let us assume that \mathcal{A}_4 contains at least five singular points z_1, z_2, z_3, z_4, z_5 . If a is the end-point of \mathcal{A}_4 with $a < z_1$, then by Lemma 2.2 \mathcal{A}_4 contains at least four Γ_a -singular points.

Now let μ be a Möbius transformation mapping a to ∞ . Then removing $\mu(a)$ we have the open plane. But Γ_a -singular points now become linearly singular points (points of linear order three). Also let L_i be a small preceding neighbourhood of z_i and M a small preceding neighbourhood of z_5 . Then the arc $\mu(L_1 \cup [z_1, z_5] \cup M_5)$ is a bounded normal arc of linear order three containing at least four linearly singular points. This is known to be impossible; cf. ([1], 3.2.4).

Thus \mathcal{A}_4 contains at most four singular points.

A closed curve C_4 of cyclic order four is automatically normal.

Using the method of the proof of Theorem 1, one obtains the following result.

THEOREM 2. *A closed curve C_4 of cyclic order four satisfying Condition I contains at most four singular points.*

Finally, we should note that there are normal arcs and curves of cyclic order four satisfying Condition I which contain four singular points. For example, consider any arc of the ellipse given by the equation $3x^2 + 4y^2 = 12$ containing the points $P_1(0, \sqrt{3})$, $P_2(2, 0)$, $P_3(0, -\sqrt{3})$ and $P_4(-2, 0)$ or the ellipse itself. Each P_i , $i = 1, 2, 3, 4$ is a singular point of this arc.

Hence the maximum number of four singular points for normal arcs and curves of cyclic order four satisfying Condition I obtained in Theorems 1 and 2 is the best possible bound.

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Received January 30, 1975.

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