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AN INVERSION OF THE S_2 TRANSFORM FOR GENERALIZED FUNCTIONS

L. S. DUBE

Define S_2 transform of a member f of a certain space of generalized functions as

$$F(x) = \langle f(t), K(t, x)\rangle$$

where

$$K(t, x) = \begin{cases} \dfrac{\log x/_t}{x - t}\,, & x \neq t \\ \dfrac{1}{x}\,, & x = t \end{cases}$$

$(0 < t < \infty, 0 < x < \infty)$.

It is shown that

$$\lim_{n\to\infty} H_{n,x}[F(x)] = f(x)$$

in the weak distributional sence. Here $H_{n,x}$ is a certain linear generalized differential operator.

1. Introduction. Schwartz [6, p. 248] first introduced the Fourier transform of distributions in 1947. Since then, extensions of the classical integral transformations to generalized functions have become of continuing interest. Some references to this effect are [2], [3], [4], [5], [8], [9] and [10]. The Stieltjes and iterated Stieltjes transforms of a function $f(t)$ have been defined respectively as

$$\tilde{f}(u) = \int_0^\infty \frac{f(t)}{u + t} dt\,, \quad u > 0$$

and

$$\tilde{\tilde{f}}(x) = \int_0^\infty \frac{du}{x + u} \int_0^\infty \frac{f(t)}{u + t} dt\,, \quad x > 0\,.$$

If it is permissible to change the order of integration in the above integral, one gets

$$\tilde{\tilde{f}}(x) = \int_{0+}^\infty \frac{\log x/t}{x - t} f(t) dt\,, \tag{1}$$

where $\log x/t/(x - t)$ is defined by its limiting value $1/x$ at $t = x$. (1) is referred to as the S_2 transform of the function $f(t)$ (see [1, p. 4]). The inversion formula for (1) due to Boas and Widder [1, p. 30] is given by

$$\lim_{n\to\infty} H_{n,x}[\tilde{\tilde{f}}(x)] = f(x) \,, \tag{2}$$

for almost all $x > 0$, where

$$H_{n,x}[\phi(x)] = \left(\frac{1}{n!(n-2)}\right)^2 [x^{2n-1}\{x^{2n-1}\phi^{(n-1)}(x)\}^{(2n-1)}]^{(n)} \,,$$

$n = 1, 2, \cdots$.

Pandey [4] has defined the Stieltjes transform of an arbitrary element f of a generalized function space $S'_\alpha(I)$ as

$$G(s) = \left\langle f(t), \frac{1}{s+t} \right\rangle \,, \tag{3}$$

for s lying in the complex plane with a cut along the negative real axis. He has also proved both complex and real (for $s > 0$) inversion formulae for the transform (3).

It is natural to ask whether one can extend the classical iterated Stieltjes transform to a space of generalized functions. If $G(u)$ is the Stieltjes transform of $f \in S'_\alpha(I)$ for $u > 0$, as defined by (3), it seems reasonable to define the iterated Stieltjes transform of f as

$$F(x) = \left\langle G(u), \frac{1}{x+u} \right\rangle, \quad x > 0 \,. \tag{4}$$

In order that the above definition be meaningful, $G(u)$ must belong to the space $S'_\alpha(I)$ as a regular generalized function. This we have not been able to show. In fact, $\int_0^\infty G(u)/(x+u)\, du$ ceases to exist in a neighborhood of zero, as $G(u) = 0(1/u)$, when $u \to 0+$ ([4, Corollary to Lemma 2a]). In this paper, we provide a partial solution to the present problem by defining the S_2 transform of generalized functions as in §3. In our definition of S_2 transform, the difficulty that occurs in justifying (4) does not arise. The inversion formula (2) is extended to a space of generalized functions in the sense of weak distributional convergence.

The notation and terminology will follow that of [3] and [11]. "I" denotes the open interval $(0, \infty)$ and t, x and u are real variables in I. If f is a generalized function, then $f(t)$ is used to indicate that the testing functions on which f is defined have t as their variable. The space of C^∞-funtions on I having compact support is denoted by $D(I)$ and its dual $D'(I)$ is the Schwartz distribution space.

2. The testing function space $S_\alpha(I)$. Let α be a fixed real number satisfying $0 < \alpha < 1$. $S_\alpha(I)$ is defined as the collection of all

C^∞-functions $\phi(t)$ on $I = (0, \infty)$ such that

$$\gamma_k(\phi) = \sup_{0<t<\infty} \left| t^\alpha \left(t \frac{d}{dt} \right)^k \phi(t) \right| < \infty ,$$

for each $k = 0, 1, 2, \cdots$.

The topology of $S_\alpha(I)$ is generated by the seminorms $\{\gamma_k\}$ [11, p. 8]. A sequence $\{\phi_n\}$ converges to a function ϕ in the topology of $S_\alpha(I)$ if and only if

$$t^\alpha \left(t \frac{d}{dt} \right)^k \phi_n(t) \longrightarrow t^\alpha \left(t \frac{d}{dt} \right)^k \phi(t)$$

as $n \to \infty$, uniformly in t, for each $k = 0, 1, 2, \cdots$. It turns out that $S_\alpha(I)$ is a locally convex, sequentially complete Hausdorff topological vector space. The dual space $S'_\alpha(I)$ consists of all linear continuous functionals on $S_\alpha(I)$. The space $D(I)$ is contained in $S_\alpha(I)$ and the topology of $D(I)$ is stronger than that induced on it by $S_\alpha(I)$. Hence the restriction of any $f \in S'_\alpha(I)$ to $D(I)$ is in $D'(I)$.

Regular generalized functions in $S'_\alpha(I)$. The regular generalized functions in $S'_\alpha(I)$ are characterized as follows:

If $f(t)$ is a locally integrable function such that $\int_0^\infty (|f(t)|/t^\alpha) dt < \infty$, then $f(t)$ generates a regular generalized function in $S'_\alpha(I)$ through the definition:

$$\langle f, \phi \rangle = \int_0^\infty f(t)\phi(t) dt , \quad \phi \in S_\alpha(I) .$$

The proof of the above statement follows easily in the lines of [11, V, p. 53].

Now define a function $K(t, x)$ on $(0 < t < \infty; 0 < x < \infty)$ as

$$K(t, x) = \begin{cases} \dfrac{\log x/_t}{x - t} , & t \neq x \\ 1/_x , & t = x . \end{cases} \tag{5}$$

For each fixed $x > 0$, $K(t, x)$ as a function of t belongs to $S_\alpha(I)$. In fact, taking the substitution $t - x = u$, the function $K(t, x)$ can be written as a power series in u with the centre $t = x$ and the radius of convergence x, which will imply that $K(t, x)$ is infinitely differentiable at $t = x$. That $K(t, x)$ is infinitely differentiable at $t \neq x$ is obvious. It follows now by a simple computation that $\gamma_k(K(t, x)) < \infty$ for a fixed $x > 0$ and for each $k = 0, 1, 2, \cdots$.

3. **The generalized S_2 transform.** For $f \in S'_\alpha(I)$, we define the S_2 transform of f as a function $F(x)$ obtained by applying f on the

kernel $K(t, x)$, i.e.

$$F(x) = \langle f(t), K(t, x) \rangle , \quad x > 0 , \tag{6}$$

where $K(t, x)$ is defined by (5).

The right hand side of (6) has a sense as $K(t, x)$ belongs to the testing function space $S_\alpha(I)$.

Following the technique used in [3, Th. 1] and applying the mathematical induction, one can show that $F(x)$ is an infinitely differentiable function and that

$$F^{(n)}(x) = \left\langle f(t), \frac{\partial^n}{\partial x^n} K(t, x) \right\rangle \quad \text{for each} \quad x > 0$$

and $n = 1, 2, \cdots$.

4. Inversion and uniqueness. Now we prove an inversion theorem for our generalized S_2 transform as follows:

THEOREM 1. *Let $f \in S'_\alpha(I)$ for $0 < \alpha < 1$ and let $F(x)$ be the S_2 transform of f as defined by* (5). *Then for an arbitrary $\phi(x) \in D(I)$ one has*

$$\langle H_{n,x} F(x), \phi(x) \rangle \longrightarrow \langle f, \phi \rangle \quad as \quad n \longrightarrow \infty$$

where the operator $H_{n,x}$ is defined by (3) *and the differentiation therein is understood in the distributional sense.*

Proof. By a simple computation, the operator $H_{n,x}$ can be expressed as a polynomial in $(x(d/dx))$ of degree $4n - 2$. Let us denote this polynomial by $P(x(d/dx))$. The theorem will be proved by justifying steps:

$$\langle H_{n,x} F(x), \phi(x) \rangle = \left\langle P\left(x \frac{d}{dx} \right) F(x), \phi(x) \right\rangle$$

$$= \int_0^\infty \left[P\left(x \frac{d}{dx} \right) F(x) \right] \phi(x) dx \tag{7}$$

$$= \int_0^\infty F(x) P\left(-x \frac{d}{dx} - 1 \right) \phi(x) dx \tag{8}$$

$$= \int_0^\infty \langle f(t), K(t, x) \rangle P\left(-x \frac{d}{dx} - 1 \right) \phi(x) dx \tag{8$'$}$$

$$= \left\langle f(t), \int_0^\infty K(t, x) P\left(-x \frac{d}{dx} - 1 \right) \phi(x) dx \right\rangle \tag{9}$$

$$\longrightarrow \langle f(t), \phi(t) \rangle , \quad \text{as} \quad n \longrightarrow \infty , \tag{10}$$

where $(4n-2)$ is the degree of the polynomial P.

The step (7) is obvious due to the fact that the function

$$P\left(x\frac{d}{dx}\right)F(x)$$

generates a regular distribution in $D'(I)$. The step (8) is obtained by applying integration by parts in (7) successively and using the fact that the support of $\phi(x)$ is contained in some open interval (a, b), $0 < a < b < \infty$, so that the limit terms in the integration vanish. The limits of integration in both (8)′ and (9) are essentially from a to b as the support of $P\left(-x\frac{d}{dx}-1\right)\phi(x)$ is contained in (a, b). Hence following the Riemann sum technique as used in [5, Th. 2] one can easily show that (8)′ equals (9). In order to show that (9)→(10), we need prove that

$$t^{\alpha}\left(t\frac{d}{dt}\right)^{k}\left[\int_0^\infty K(t, x)P\left(-x\frac{d}{dx} - 1\right)\phi(x)dx - \phi(t)\right] \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

uniformly for all $t \in (0, \infty)$, for each $k = 0, 1, 2, \cdots$.

Now

$$\left(t\frac{d}{dt}\right)\int_0^\infty K(t, x)P\left(-x\frac{d}{dx} - 1\right)\phi(x)dx \tag{11}$$

$$= \int_a^b \left(t\frac{d}{dt}\right)[K(t, x)]P\left(-x\frac{d}{dx} - 1\right)\phi(x)dx\ . \tag{12}$$

It can easily be checked that

$$\left(t\frac{d}{dt}\right)K(t, x) = \begin{cases} \left(-x\dfrac{d}{dx} - 1\right)K(t, x)\ , & \text{when} \quad t \neq x \\ \left(x\dfrac{d}{dx}\right)K(t, x)\ , & \text{when} \quad t = x\ . \end{cases}$$

Therefore (12) can be written without any change in the value of the integral as

$$\int_a^b \left[\left(-x\frac{d}{dx} - 1\right)K(t, x)\right]P\left(-x\frac{d}{dx} - 1\right)\phi(x)dx$$

$$= \int_a^b K(t, x)\left(x\frac{d}{dx}\right)\left[P\left(-x\frac{d}{dx} - 1\right]\phi(x)dx\ ,\right.$$

(by integration by parts)

$$= \int_a^b K(t, x)P\left(-x\frac{d}{dx} - 1\right)\left(x\frac{d}{dx}\right)\phi(x)dx$$

$$= \int_a^b \left[P\left(x\frac{d}{dx}\right)K(t, x)\right]\left(x\frac{d}{dx}\right)\phi(x)dx\ ,$$

(by integration by parts).

Hence applying $(t(d/dt))$ successively on the integral in (11) we get for any non-negative integer k

$$\left(t\frac{d}{dt}\right)^k \int_0^\infty K(t,x)P\left(-x\frac{d}{dx}-1\right)\phi(x)dx = \int\left[P\left(x\frac{d}{dx}\right)K(t,x)\right]\left(x\frac{d}{dx}\right)^k \phi(x)dx$$
$$= \int_0^\infty [H_{n,x}]K(t, x)\left(x\frac{d}{dx}\right)^k \phi(x)dx$$
$$= \int_0^\infty F_n(t, x)\left(x\frac{d}{dx}\right)^k \phi(x)dx ,$$

where

$$F_n(t, x) = d_n^2 x^{n-1} t^n \int_0^\infty \frac{u^{2n-1}}{(x+u)^{2n}(t+u)^{2n}} du ,$$

([1, Cor. 6.1.1, p. 20])

$$d_n = (2n-1)!\, c_n;\ c_1 = 1 \quad \text{and} \quad c_n = \frac{1}{n!(n-2)!} , \quad n \geqq 2 .$$

Also in view of [1, Lemma 7.2, p. 21], for $n \geqq 2$

$$\int_0^\infty F_n(t, x)dx = \left(\frac{n-1}{n}\right)^2 \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty .$$

Hence as $n \to \infty$,

$$\left(t\frac{d}{dt}\right)^k \left[\int_0^\infty K(t, x)P\left(-x\frac{d}{dx}-1\right)\phi(x)dx - \phi(t)\right]$$
$$= \int_0^\infty F_n(t, x)\left[\left(x\frac{d}{dx}\right)^k \phi(x) - \left(t\frac{d}{dt}\right)^k \phi(t)\right]dx;$$

here $(x(d/dx))^k\phi(x) \in D(I)$.

Now it suffices to show that

$$t^\alpha \int_0^\infty F_n(t, x)[\psi(x) - \psi(t)]dx \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \tag{13}$$

uniformly for all $t > 0$ for any $\psi(x) \in D(I)$.

Taking the substitution $x = ty$ and using the fact that $F_n(t, x)$ is homogeneous of degree-1 we get

$$\int_0^\infty F_n(t, x)[\psi(x) - \psi(t)]dx = \int_0^\infty F_n(1, x)[\psi(xt) - \psi(t)]dx$$
$$= \left(\int_0^{1-\eta} + \int_{1-\eta}^{1+\eta} + \int_{1+\eta}^\infty\right) F_n(1, x)[\psi(xt) - \psi(t)]dx$$

where η is taken be be a positive number less than 1/2.

In view of [1, Lemmas 7.2 and 8.2] it follows that

$$\int_0^{1-\eta} F_n(1, x)dx \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \tag{14}$$

$$\int_{1-\eta}^{1+\eta} F_n(1, x)dx \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty \tag{15}$$

and

$$\int_{1+\eta}^{\infty} F_n(1, x)dx \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty . \tag{16}$$

Let $\sup_{0<t<\infty} t^\alpha \psi(t) = M$, which clearly exists. Then

$$\left| t^\alpha \int_0^{1-\eta} F_n(1, x)[\psi(xt) - \psi(t)]dx \right| \leqq 2M \int_0^{1-\eta} F_n(1, x)dx \longrightarrow 0 \tag{17}$$

as $n \to \infty$ uniformly for all $t > 0$ in view of (14).

Similarly, using (16) we get

$$\left| t^\alpha \int_{1+\eta}^{\infty} F_n(1, x)[\psi(xt) - \psi(t)]dx \right| \longrightarrow 0 \tag{18}$$

as $n \to \infty$ uniformly for all $t > 0$.

Finally, in view of [7, Lemma 5, p. 287], and the fact that ψ has a compact support on I, for a given $\varepsilon > 0$ there exists a positive $\eta < 1/2$ such that

$$|t^\alpha[\psi(xt) - \psi(t)]| < \varepsilon ,$$

uniformly for all $t > 0$ and for all $x \in (1 - \eta, 1 + \eta)$. Hence the application of (15) leads to

$$\left| \int_{1-\eta}^{1+\eta} t^\alpha F_n(1, x)[\psi(xt) - \psi(t)]dx \right| < \varepsilon \int_{1-\eta}^{1+\eta} F_n(1, x)dx \longrightarrow \varepsilon \tag{19}$$

as $n \to \infty$ uniformly for all $t > 0$.

Combining (17), (18) and (19) in which ε is arbitrary, (13) is established. This completes the proof of the theorem.

THEOREM 2 (Uniqueness). *Let f and g be two members of $S'_\alpha(I)$ and let $F(x)$ and $G(x)$ be their S_2 transforms respectively as defined by* (5). *If $F(x) = G(x)$ for all $x > 0$ then $f = g$ in the sense of equality in $D'(I)$.*

Proof. For an arbitrary $\phi \in D(I)$,

$$\langle f - g, \phi \rangle = \lim_{n\to\infty} \langle H_{n,x}(F(x) - G(x)), \phi(x) \rangle ,$$
(by Theorem 1).
$$= 0 , \quad \text{since} \quad F(x) = G(x) \quad \text{for all} \quad x > 0 .$$

Hence $f = g$ in $D'(I)$.

An open problem. We state the following open problem related to the present work:

Can one justify the definition of the iterated Stieltjes transform of generalized functions as given by (4)? In order to do this, some modifications in the asymptotic order of $G(u)$, and in the characterization of regular generalized functions of $S'_\alpha(I)$ as given in §2, might be needed.

Granted that (4) is well defined, can one prove the equivalence of the S_2 and iterated Stieltjes transforms of generalized functions?

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CONCORDIA UNIVERSITY, MONTREAL, P. Q.