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# SOLVABILITY OF CONVOLUTION EQUATIONS IN $\mathscr{K}_p'$, $p > 1$

S. SZNAJDER AND Z. ZIELEZNY

**Let $S$ be a convolution operator in the space $\mathscr{K}_p'$, $p > 1$, of distributions in $R^n$ growing no faster than $\exp(k|x|^p)$ for some $k$. A condition on $S$ introduced by I. Cioranescu is proved to be equivalent to $S*\mathscr{K}_p' = \mathscr{K}_p'$.**

We denote by $\mathscr{K}_p'$, $p > 1$, the space introduced in [4] and consisting of distributions in $R^n$ which "grow" no faster than $\exp(k|x|^p)$, for some $k$.

I. Cioranescu [1] characterized distributions with compact support, i.e. in the space $\mathscr{E}'$, having fundamental solutions in $\mathscr{K}_p'$. We recall that a distribution $E$ is a fundamental solution for $S \in \mathscr{E}'$ if

$$S*E = \delta ,$$

where $\delta$ is the Dirac measure and * denotes the convolution. Cioranescu proved that, if $S$ is a distribution in $\mathscr{E}'$ and $\hat{S}$ its Fourier transform, the following conditions are equivalent:

(a) There exist positive constants $A$, $N$, $C$ such that

$$\sup_{x \in R^n, |x| \leqq A[\log(2+|\xi|)]^{1/q}} |\hat{S}(\xi + x)| \geqq \frac{C}{(1 + |\xi|)^N}, \ \xi \in R^n ,$$

where $1/p + 1/q = 1$.

(b) $S$ has a fundamental solution in $\mathscr{K}_p'$.

In this paper we study the solvability of convolution equations in $\mathscr{K}_p'$. If $\mathscr{O}_C'(\mathscr{K}_p' : \mathscr{K}_p')$ is the space of convolution operators in $\mathscr{K}_p'$, we ask the question: Under what condition on $S \in \mathscr{O}_C'(\mathscr{K}_p' : \mathscr{K}_p')$ is $S*\mathscr{K}_p' = \mathscr{K}_p'$? The last equation means that the mapping $u \rightarrow S*u$ of $\mathscr{K}_p'$ into $\mathscr{K}_p'$ is surjective.

We prove the following theorem which extends the results of Cioranescu mentioned above.

THEOREM. *If $S$ is a distribution in $\mathscr{O}_C'(\mathscr{K}_p' : \mathscr{K}_p')$ then each of the conditions* (*a*) *and* (*b*) *is equivalent to each of the following ones*:

(a′) *There exist positive constants $A'$, $N'$, $C'$ such that*

$$\sup_{z \in C^n, |z| \leqq A'[\log(2+|\xi|)]^{1/q}} |\hat{S}(\xi + z)| \geqq \frac{C'}{(1 + |\xi|)^{N'}} ; \quad \xi \in R^n ,$$

*where* $1/p + 1/q = 1$.

(c) $S*\mathscr{K}_p' = \mathscr{K}_p'$.

REMARK. For $p = 1$ a similar theorem was proved in [5].

Before presenting the proof we state the basic facts about the spaces $\mathscr{K}_p'$ and $\mathscr{O}_C'(\mathscr{K}_p' : \mathscr{K}_p')$; for the proofs we refer to [4].

We denote by $\mathscr{K}_p$ the space of all functions $\varphi \in C^\infty(R^n)$ such that

$$v_k(\varphi) = \sup_{x \in R^n, |\alpha| \leqq k} e^{k|x|^p} |D^\alpha \varphi(x)| < \infty , \qquad k = 0, 1, \cdots ,$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and

$$D^\alpha = \left(\frac{1}{i} \frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{1}{i} \frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{1}{i} \frac{\partial}{\partial x_n}\right)^{\alpha_n} .$$

The topology in $\mathscr{K}_p$ is defined by the family of semi-norms $v_k$. Then $\mathscr{K}_p$ becomes a Frechet space.

The dual $\mathscr{K}_p'$ of $\mathscr{K}_p$ is a space of distributions. A distribution $u$ is in $\mathscr{K}_p'$ if and only if there exists a multi-index $\alpha$, an integer $k \geqq 0$ and a bounded, continuous function $f$ on $R^n$ such that

$$u = D^\alpha [e^{k|x|} f^p(x)] .$$

If $u \in \mathscr{K}_p'$ and $\varphi \in \mathscr{K}_p$, then the convolution $u * \varphi$ is a function in $C^\infty(R^n)$ defined by

$$u * \varphi(x) = \langle u_y, \varphi(x - y) \rangle ,$$

where $\langle u, \varphi \rangle = u(\varphi)$.

The space $\mathscr{O}_C'(\mathscr{K}_p' : \mathscr{K}_p')$ of convolution operators in $\mathscr{K}_p'$ consists of distributions $S \in \mathscr{K}_p'$ satisfying one of the equivalent conditions:

( i ) The products $S_x \exp [k(1 + |x|^2)^{p/2}]$, $k = 0, 1, \cdots$, are tempered distributions

(ii) For every $k \geqq 0$ there exists an integer $m \geqq 0$ such that

$$S = \sum_{|\alpha| \leqq m} D^\alpha f_\alpha ,$$

where $f_\alpha$, $|\alpha| \leqq m$, are continuous functions in $R^n$ whose products with $\exp(k|x|^p)$ are bounded

(iii) For every $\varphi \in \mathscr{K}_p$, the convolution $S * \varphi$ is in $\mathscr{K}_p$; moreover, the mapping $\varphi \to S * \varphi$ of $\mathscr{K}_p$ into $\mathscr{K}_p$ is continuous.

If $S \in \mathscr{O}_C'(\mathscr{K}_p' : \mathscr{K}_p')$ and $\check{S}$ is the distribution in $\mathscr{K}_p'$ defined by $\langle \check{S}, \varphi \rangle = \langle S_x, \varphi(-x) \rangle$, $\varphi \in \mathscr{K}_p$, then $\check{S}$ is also in $\mathscr{O}_C'(\mathscr{K}_p' : \mathscr{K}_p')$. The convolution of $S$ with $u \in \mathscr{K}_p'$ is then defined by

$$(1) \qquad \langle S * u, \varphi \rangle = \langle u * S, \varphi \rangle = \langle u, \check{S} * \varphi \rangle, \ \varphi \in \mathscr{K}_p .$$

For a function $\varphi \in \mathscr{K}_p$, the Fourier transform

$$\hat{\varphi}(\xi) = \int_{R^n} e^{-i\langle \xi, x \rangle} \varphi(x) dx$$

can be continued in $C^n$ as an entire function such that

$$w_k(\hat{\varphi}) = \sup_{\zeta \in C^n} (1 + |\xi|)^k e^{-|\eta|^{q}/k} |\hat{\varphi}(\zeta)| < \infty\ , \quad k = 1, 2, \cdots ,$$

where $\zeta = \xi + i\eta$. We denote by $K_p$ the space of Fourier transforms of functions in $\mathscr{K}_p$. If the topology in $K_p$ is defined by the family of semi-norms $w_k$, then the Fourier transformation is an isomorphism of $\mathscr{K}_p$ onto $K_p$.

The dual $K'_p$ of $K_p$ is the space of Fourier transforms of distributions in $\mathscr{K}'_p$. The Fourier transform $\hat{u}$ of a distribution $u \in \mathscr{K}'_p$ is defined by the Parseval formula

$$\langle \hat{u}, \hat{\varphi} \rangle = (2\pi)^n \langle u_x, \varphi(-x) \rangle\ .$$

For $S \in \mathscr{O}'_C(\mathscr{K}'_p : \mathscr{K}'_p)$, the Fourier transform $\hat{S}$ is a function which can be continued in $C^n$ as an entire function having the following property: For every $k > 0$ there exist constants $C''$ and $N''$ such that

$$|\hat{S}(\xi + i\eta)| \leqq C''(1 + |\xi|)^{N''} e^{|\eta|^{q}/k}\ . \tag{2}$$

Furthermore, if $S \in \mathscr{O}'_C(\mathscr{K}'_p : \mathscr{K}'_p)$ and $u \in \mathscr{K}'_p$, we have the formula

$$\widehat{S*u} = \hat{S}\hat{u}\ , \tag{3}$$

where the product on the right-hand side is defined in $K'_p$ by $\langle \hat{S}\hat{u}, \psi \rangle = \langle \hat{u}, \hat{S}\psi \rangle$, $\psi \in K_p$.

In the proof of our theorem we shall make use of the following lemma of L. Hörmander (see [3], Lemma 3.2):

If $F$, $G$ and $F/G$ are entire functions and $\rho$ is an arbitrary positive number, then

$$|F(\zeta)/G(\zeta)| \leqq \sup_{|\zeta - z| < 4\rho} |F(z)| \sup_{|\zeta - z| < 4\rho} |G(z)| \Big/ \Big( \sup_{|\zeta - z| < \rho} |G(z)| \Big)^2$$

where $\zeta$, $z \in C^n$.

*Proof of the theorem.* It is obvious that (a) $\Rightarrow$ ($a'$) and (c) $\Rightarrow$ (b). The implication (b) $\Rightarrow$ (a) was proved in [1] for $S \in \mathscr{E}'$. If $S \in \mathscr{O}'_C(\mathscr{K}'_p : \mathscr{K}'_p)$ the proof is the same and therefore we omit it. Our only task is to prove that ($a'$) $\Rightarrow$ ($c$).

Let $S$ be a distribution in $\mathscr{O}'_C(\mathscr{K}'_p : \mathscr{K}'_p)$ whose Fourier transform satisfies condition ($a'$), and let $T = \check{S}$. Then the Fourier transform of $T$ also satisfies condition ($a'$). We consider the mapping $S^* : u \longrightarrow S*u$ of $\mathscr{K}'_p$ into $\mathscr{K}'_p$. By (1), it is the transpose of the mapping $T^* : \varphi \longrightarrow T*\varphi$ of $\mathscr{K}_p$ into $\mathscr{K}_p$. In order to prove (c) it suffices to show that $T^*$ is an isomorphism of $\mathscr{K}_p$ onto $T^*\mathscr{K}_p$ (see [2], Corollary on p. 92).

Since $T$ is in $\mathscr{O}'_C(\mathscr{K}'_p : \mathscr{K}'_p)$, the mapping $T^*$ is continuous, by (iii). Also, using Fourier transforms and formula (3), it is easy to see that $T^*$ is injective. We now prove that the inverse of $T^*$, i.e. the mapping $T^*\varphi \to \varphi$, is continuous. Since the Fourier transformation is an isomorphism from $\mathscr{K}_p$ onto $K_p$, it suffices to prove the equivalent statement that the mapping $\hat{T}\hat{\varphi} \to \hat{\varphi}$ is continuous.

Suppose that

$$\hat{T}\hat{\varphi} = \hat{\psi} ,$$

where $\hat{\varphi}, \hat{\psi} \in K_p$. We recall that $\hat{T}$ is an entire function satisfying condition (a′) and estimates of the form (2). Given an arbitrary integer $k > 0$, we pick an integer $k'$ such that

$$k' > (10^q + 1)k . \tag{4}$$

In view of (2), for $k'$ there exist constants $N'', C'' > 0$ such that

$$|\hat{T}(\zeta)| \leqq C''(1 + |\xi|)^{N''} e^{|\eta|^q/k'}, \ \zeta = \xi + i\eta \in C^n .$$

Hence, setting

$$\rho = |\eta| + A'[\log (2 + |\xi|)]^{1/q} \tag{5}$$

and making use of the inequality

$$(a + b)^q \leqq 2^q(a^q + b^q), \ a, b \geqq 0 ,$$

we obtain

$$\begin{aligned} \sup_{|\zeta - z| < 4\rho} |\hat{T}(z)| &= \sup_{|z| < 4\rho} |\hat{T}(\zeta + z)| \\ &\leqq C''(1 + |\xi| + 4\rho)^{N''} e^{(|\eta| + 4\rho)^q/k'} \\ &\leqq C_1(1 + |\xi|)^{N''}(1 + |\eta|)^{N''} e^{[(10|\eta|)^q + (8A')^q \log(2+|\xi|)]/k'} \\ &\leqq C'_1(1 + |\xi|)^{N'' + (8A')^q/k'} e^{(10^q+1)|\eta|^q/k'} \end{aligned} \tag{6}$$

where $z \in C^n$ and $C_1, C'_1$ are constants.

On the other hand

$$\begin{aligned} \sup_{|\zeta - z| < \rho} |\hat{T}(z)| = \sup_{|z| < \rho} |\hat{T}(\zeta + z)| &\geqq \sup_{|z| < A'[\log(2+|\xi|)]^{1/q}} |\hat{T}(\xi + z)| \\ &\geqq \frac{C'}{(1 + |\xi|)^{N'}} , \end{aligned} \tag{7}$$

by condition (a′).

Applying now to the functions $\hat{\psi}$, $\hat{T}$ and $\hat{\psi}/\hat{T} = \hat{\varphi}$ Hörmander's lemma with $\rho$ given by (5) and making use of the estimates (6) and (7), we obtain

$$
\begin{aligned}
(8) \qquad |\hat{\varphi}(\zeta)| &\leqq \sup_{|\xi - z| < 4\rho} |\hat{\psi}(z)| \sup_{|\zeta - z| < 4\rho} |T(z)| \Big/ \Big( \sup_{|\zeta - z| < \rho} |T(z)| \Big)^2 \\
&\leqq C_2(1 + |\xi|)^{2N' + N'' + (8A')^q/k'} e^{(10q+1)|\eta|^q/k'} \sup_{|z| < 4\rho} |\hat{\psi}(\zeta + z)| ,
\end{aligned}
$$

where $C_2$ is another constant. But, for any integer $l > 0$ and all $z = x + iy \in C^n$ with $|z| < 4\rho$, we have

$$
\begin{aligned}
|\hat{\psi}(\zeta + z)| &\leqq w_l(\hat{\psi})(1 + |\xi + x|)^{-l} e^{|\eta + y|^q/l} \\
&\leqq w_l(\hat{\psi})(1 + |x|)^l (1 + |\xi|)^{-l} e^{(|\eta| + |y|)^q/l} \\
(9) \qquad &\leqq w_l(\hat{\psi})(1 + 4\rho)^l (1 + |\xi|)^{-l} e^{(|\eta| + 4\rho)^q/l} \\
&\leqq C_3 w_l(\hat{\psi})(1 + |\eta|)^l (1 + |\xi|)^{1-l} e^{[(10|\eta|)^q + (8A')^q \log(2 + |\xi|)]/l} \\
&\leqq C_3' w_l(\hat{\psi})(1 + |\xi|)^{1-l+(8A')^q/l} e^{(10^q+1)|\eta|^q/l} ,
\end{aligned}
$$

where $C_3$ and $C_3'$ depend only on $l$ and $q$. We choose the integer $l$ so that

$$
l > \max \left\{ k + 1 + 2N' + N'' + 2(8A')^q, (10^q + 1) \Big/ \left( \frac{1}{k} - \frac{10^q + 1}{k'} \right) \right\} ,
$$

which is possible because of (4). Then

$$
k + 1 + 2N' + N'' + (8A')^q \left( \frac{1}{k'} + \frac{1}{l} \right) - l < 0
$$

and

$$
(10^q + 1)\left( \frac{1}{k'} + \frac{1}{l} \right) - \frac{1}{k} < 0 .
$$

Consequently from (8) and (9) it follows that

$$
w_k(\hat{\varphi}) \leqq C_4 w_l(\hat{\psi}) = C_4 w_l(\hat{T}\hat{\varphi}) ,
$$

for some $C_4$ independent of $\hat{\varphi}$. This proves the continuity of the mapping $\hat{T}\hat{\varphi} \to \hat{\varphi}$ and thus completes the proof of the implication $(\mathrm{a}') \Rightarrow (\mathrm{c})$.

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STATE UNIVERSITY OF NEW YORK AT BUFFALO