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RINGS WHOSE PROPER CYCLIC MODULES ARE QUASI-INJECTIVE

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A ring R with identity is a right $PCQI$-ring (PCI-ring) if every cyclic right R-module $C \not\cong R$ is quasi-injective (injective). Left $PCQI$-rings (PCI-rings) are similarly defined. Among others the following results are proved: (1) A right $PCQI$-ring is either prime or semi-perfect. (2) A nonprime nonlocal ring is a right $PCQI$-ring iff every cyclic right R-module is quasi-injective or $R \cong \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$, where D is a division ring. In particular, a nonprime nonlocal right $PCQI$-ring is also a left $PCQI$-ring. (3) A local right $PCQI$-ring with maximal ideal M is a right valuation ring or $M^2 = (0)$. (4) A prime local right $PCQI$-ring is a right valuation domain. (5) A right $PCQI$-domain is a right Öre-domain. Faith proved (5) for right PCI-domains. If R is commutative then some of the main results of Klatt and Levy on pre-self-injective rings follow as a special case of these results.

Since, in a commutative Dedekind domain D, for each nonzero ideal A, D/A is a self-injective ring, or equivalently D/A is a quasi-injective D-module, every commutative Dedekind domain is a $PCQI$-ring. An example of a $PCQI$-ring which is not a Dedekind domain is given in Levy [14]. Commutative $PCQI$-rings are precisely the pre-self-injective rings characterized by Klatt and Levy [11]. PCI-rings have recently been investigated by Faith [4]. Right self-injective right $PCQI$-rings are qc-rings which have been studied by Ahsan [1] and Koehler [13].

1. Definitions and preliminaries.

Throughout all modules are unitary and right unless specified. An R-module X is called injective relative to an R-module M if for each short exact sequence $0 \to N \to M \to M/N \to 0$ the sequence $0 \to \mathrm{Hom}_R(M/N, X) \to \mathrm{Hom}_R(M, X) \to \mathrm{Hom}_R(N, X) \to 0$ is exact. X is called quasi-injective if X is injective relative to itself. Any R-module injective relative to all R-modules is called injective. Relative projectivity is defined dually.

A ring R is called a right q-ring if each of its right ideals is quasi-injective (see Jain, Mohamed, and Singh [9]). For more results, see [7], [8], [13], [15]. Dually, a ring R is called a right q^*-ring if each cyclic right R-module is quasi-projective (see Koehler [12]).

A ring R is right qc-ring if each cyclic right R-module is quasi-injective (see Ahsan [1]). A well-known result of Osofsky [16] states that R is semisimple artinian iff each cyclic R-module is injective. Koehler [13] showed that R is a right qc-ring iff R is a finite direct sum of rings each of which is semisimple artinian or a rank o duo maximal valuation ring. As a consequence, every qc-ring is both a q-ring and q^*-ring.

In this paper the classes of rings initially called q-rings, q^*-rings, and qc-rings have been called Q-rings, Q^*-rings, and QC-rings respectively.

Let $J(R)$ denote the radical of a ring R. R is called semiperfect if $R/J(R)$ is semisimple artinian and idempotents modulo $J(R)$ can be lifted to R. If R is semiperfect, then there exists a finite maximal family of primitive orthogonal idempotents $\{e_i\}_{1\leqq i\leqq n}$ such that $R = \bigoplus \sum_{i=1}^{n} e_iR$.

R is called a local ring if it has a unique maximal right ideal which must be the radical $J(R)$.

R is a right valuation ring if the set of all right ideals is linearly ordered. R is a maximal valuation ring if every family of pairwise solvable congruences of the form $x \equiv x_\alpha(\text{mod } A_\alpha)$ has a simultaneous solution where $x_\alpha \in R$ and each A_α is an ideal in R. R is called an almost maximal valuation ring if each of its proper homomorphic images is a maximal valuation ring.

A ring is right duo if every right ideal is two-sided. A ring R has rank O if every prime ideal is a maximal ideal. By duo rings or valuation rings, we shall mean both right and left.

3. General results.

SUBLEMMA 1. *Let I be a right ideal in a ring R such that $R/I \cong R$. Then $R = I \oplus J$, where J is a right ideal, and thus $I = eR$, $e = e^2 \in R$.*

Proof. $R/I \cong R$ implies R/I is projective, and hence I is a direct summand of R.

PROPOSITION 2. *Let R be a right PCQI-ring. If I is a right ideal of R such that $R/I \cong R$, then I is contained in every nonzero two-sided ideal of R.*

Proof. Let S be a nonzero two-sided ideal of R. Then R/S is a qc-ring, hence is semiperfect. Let $f: R/I \to R$ be an isomorphism. Since $1 + I$ generates R/I, $\bar{R} = xR$, where $x = f(1 + I)$. Then $I = \operatorname{ann} x = \{r \in R \mid xr = 0\}$. So there exists $y \in R$ such that $xy = 1$. Since R/S is semiperfect, $(x + S)(y + S) = 1 + S = (y + S)(x + S)$. Then $1 - yx \in S$. Let $a \in I$, i.e., $xa = 0$. Then $(1 - yx)a = a - yxa = a$, hence $a \in S$. So $I \subseteq S$.

PROPOSITION 3. *Let R be a right PCQI-ring. Then either R is a prime ring or R is semiperfect with nil radical.*

Proof. Suppose R is not prime, and $P \neq 0$ is a prime ideal. Then R/P is a qc-ring, and hence a q-ring. So R/P is simple artinian [9]. Thus P is maximal, hence primitive. So the Jacobson radical is nil.

Since R is not prime, there exist nonzero ideals A, B such that $AB = 0$. Since R is a right $PCQI$-ring, R/A and R/B are semiperfect, hence each of them has finitely many prime ideals. Since every prime ideal of R contains A or B, it follows that R has finitely many prime ideals as well. Thus $R/J(R)$ is semisimple artinian, and since $J(R)$ is nil, R is semiperfect.

4. Nonlocal semiperfect $PCQI$-rings.

By Proposition 3, all nonprime right $PCQI$-rings are semiperfect, so the results of this section hold for the class of nonprime nonlocal right $PCQI$-rings. The case of local right $PCQI$-rings is discussed in the next section.

LEMMA 4. *Let R be a semiperfect ring. Then R/A is a proper cyclic right R-module, for all nonzero right ideals A.*

Proof. There exists a positive integer n such that R is a direct sum of n indecomposable right R-modules, and R cannot be expressed as a direct sum of more than n right R-modules. Now, if $R/A \cong R$, then, by Lemma 1, $R = A \oplus B$ and $B \cong R$. So $A = (0)$, proving the lemma.

Let R be a nonlocal semiperfect ring, and let $\{e_i\}_{1 \leqq i \leqq n}$ be a maximal set of primitive orthogonal idempotents in R. Then $R = \bigoplus \sum_{i=1}^{n} e_i R$ and $n \geqq 2$. Throughout this section, e_i's will denote primitive idempotents. We shall often use a well-known fact that if $A \oplus B$ is a quasi-injective module then any monomorphism $A \to B$ splits.

LEMMA 5. *Let R be a semiperfect nonlocal right PCQI-ring. If $\sigma \in \mathrm{Hom}_R(e_i R, e_j R)$ such that $\sigma \neq 0$, where $i \neq j$, then* $\ker \sigma = (0)$.

Proof. Suppose $\ker \sigma \neq (0)$, where $0 \neq \sigma \in \mathrm{Hom}_R(e_i R, e_j R)$, $i \neq j$. Then $R/\ker \sigma \cong \bigoplus \sum_{k=1, k \neq i}^{n} e_k R \times \mathrm{Im}\, \sigma$, and $R/\ker \sigma$ is quasi-injective. Since $\mathrm{Im}\, \sigma \subseteqq e_j R$, the inclusion map $i \colon \mathrm{Im}\, \sigma \to \bigoplus \sum_{k=1, k \neq i}^{n} e_k R$ is a monomorphism. Since $R/\ker \sigma$ is quasi-injective, the inclusion map splits. So $\mathrm{Im}\, \sigma$ is a direct summand of $e_j R$, hence $\mathrm{Im}\, \sigma = e_j R$. Since $e_j R$ is projective, $\sigma \colon e_i R \to e_j R$ splits. Thus $\ker \sigma = (0)$.

LEMMA 6. *Let R be a semiperfect nonlocal right PCQI-ring with decomposition $\bigoplus\sum_{i=1}^{n} e_iR$, where $n > 2$. Then* $\mathrm{Hom}_R(e_iR, e_jR) \neq 0$ *iff* $e_iR \cong e_jR$, i.e., $e_jRe_i \neq 0$ *iff* $e_iR \cong e_jR$.

Proof. Let $\sigma \in \mathrm{Hom}_R(e_iR, e_jR)$ such that $\sigma \neq 0$. By Lemma 5, $\ker\sigma = 0$. Since $n > 2$, $e_iR \bigoplus e_jR \cong R/\bigoplus\sum_{\substack{k=1\\k\neq i,j}}^{n} e_kR$ is quasi-injective. Then σ splits, and $0 \neq \mathrm{Im}\,\sigma$ is a direct summand of e_jR. So $\mathrm{Im}\,\sigma = e_jR$, and σ is an isomorphism. The converse is trivial.

PROPOSITION 7. *Let R be a semiperfect nonlocal right PCQI-ring with decomposition $R = \bigoplus\sum_{i=1}^{n} e_iR$, where $n > 2$. Then R is a qc-ring.*

Proof. For each i, $e_iR \cong R/\bigoplus\sum_{\substack{k=1\\k\neq i}}^{n} e_kR$. So e_iR is quasi-injective, for each i. Let A_i be the sum of all those e_iR which are isomorphic to each other. Then $R = \bigoplus\sum_{i=1}^{p} A_i$. We claim that A_i is a two-sided ideal of R, for each i. Clearly A_i is a right ideal. Consider e_jR such that $e_jR \nsubseteq A_i$. Define $f\colon e_iR \to e_jR$, where $e_iR \subseteq A_i$, by $f(e_ir) = e_jxe_ir$, for $x \in R$. Then $f \in \mathrm{Hom}_R(e_iR, e_jR)$. Since e_iR and e_jR are not isomorphic, $f = 0$ by Lemma 6. So, for $e_jR \subsetneq A_i$, $e_jRA_i = 0$. So $RA_i \subset A_i$. Since A_i is a finite direct sum of isomorphic quasi-injective right ideals, A_i is quasi-injective, hence a *qc*-ring. Thus, by Koehler [13], R is a *qc*-ring.

PROPOSITION 8. *Let R be a semiperfect right PCQI-ring such that $R = e_1R \bigoplus e_2R$. If $e_1R \cong e_2R$, then R is a qc-ring.*

Proof. Now $e_1R \cong e_2R$ and $R/e_2R \cong R/e_1R$, hence e_2R and e_1R are quasi-injective. Since $e_1R \cong e_2R$, $R = e_1R \bigoplus e_2R$ is quasi-injective, hence right self-injective. So R is a *qc*-ring.

PROPOSITION 9. *Let R be a semiperfect right PCQI-ring such that $R = e_1R \bigoplus e_2R$. If $e_1Re_2 = 0$ and $e_2Re_1 = 0$, then R is a qc-ring.*

Proof. If $e_1Re_2 = 0$ and $e_2Re_1 = 0$, then e_1R and e_2R are two-sided ideals of R. Thus $e_1R \cong R/e_2R$ and $e_2R \cong R/e_1R$ are *qc*-rings. Then $R = e_1R \bigoplus e_2R$ is a *qc*-ring.

PROPOSITION 10. *Let R be a semiperfect right PCQI-ring such that $R = e_1R \bigoplus e_2R$. If $e_1Re_2 \neq 0$ and $e_2Re_1 \neq 0$, then R is a qc-ring.*

Proof. $e_1Re_2 \neq 0$ and $e_2Re_1 \neq 0$ imply that there exist nonzero homomorphisms, hence monomorphisms by Lemma 5, from e_1R to e_2R and from e_2R to e_1R. Thus, by Bumby [2], $e_1R \cong e_2R$, and Proposition 8 yields the result.

PROPOSITION 11. *Let $R = e_1R \oplus e_2R$ be a semiperfect right PCQI-ring where $e_1R \not\cong e_2R$ and exactly one of e_1Re_2 or e_2Re_1 is zero. Then R is nonprime with nil radical.*

Proof. It follows from that the fact that if $e_1Re_2 \neq 0$, then e_1Re_2 is a nilpotent ideal.

THEOREM 12. *Let R be a nonlocal right PCQI-ring. Then R is semiperfect iff R is nonprime or simple artinian.*

Proof. Necessity follows by Proposition 3, and sufficiency follows from Proposition 7–11 and Koehler's characterization of qc-rings [13] (cf. definitions and preliminaries).

THEOREM 13. *Let R be a semiperfect nonlocal ring. Then R is a right PCQI-ring iff either* (i) $R = \bigoplus \sum_{i=1}^{n} R_i$, *where R_i is semisimple artinian or a rank o duo maximal valuation ring or* (ii) $R = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$, *where D is a division ring.*

Proof. Let R be a right $PCQI$-ring. By Propositions 7–10, R is a qc-ring unless $R = e_1R \oplus e_2R$, where e_1R and e_2R are not isomorphic and exactly one of e_1Re_2 or e_2Re_1 is zero, say $e_1Re_2 \neq 0$ and $e_2Re_1 = 0$. If R is a QC-ring, we get (i) by Koehler [13]. Otherwise, we have $R \cong \begin{pmatrix} e_1Re_1 & e_1Re_2 \\ 0 & e_2Re_2 \end{pmatrix}$. We claim that e_1Re_1 and e_2Re_2 are isomorphic division rings and $M = e_1Re_2$ is a (D, D)-bimodule such that $\dim_D M = 1 = \dim M_D$, where $D \cong e_1Re_1 \cong e_2Re_2$. Clearly e_1Re_2 is nilpotent ideal and since it is nonzero, R is not prime. So, by Proposition 3, the radical N of R is a nil ideal. Thus e_2Ne_2 is nil. We claim that $e_2Ne_2 = 0$. Let $e_2xe_2 \in e_2Ne_2$. Define $\sigma: e_2R \to e_2R$ by $\sigma(e_2y) = e_2xe_2y$. Then $\sigma \in \mathrm{Hom}_R(e_2R, e_2R)$, and since e_2xe_2 is nilpotent, σ is not a monomorphism. So $\ker \sigma \neq (0)$. Since $\mathrm{Hom}_R(e_2R, e_1R) \neq 0$, there exists an embedding $\eta: e_2R \to e_1R$. Now $\eta\sigma: e_2R \to e_1R$, and since $\ker \sigma \neq (0)$, $\ker \eta\sigma \neq (0)$. By Lemma 5, $\eta\sigma = 0$. Since η is a monomorphism, we have $\sigma = 0$. Thus $e_2xe_2 = 0$, and $e_2Ne_2 = 0$. So e_2Re_2 is a division ring. Further $e_2Re_2 = e_2R$ since $e_2Re_1 = (0)$. Thus $e_2N = 0$, and e_2R is a minimal right ideal. Now e_1R is uniform because it is quasi-injective and indecomposable. Since $0 \neq e_1Re_2R$ is the sum of the images of all R-homomorphisms of e_2R into e_1R, the fact that e_2R is minimal and e_1R is uniform yields that e_1Re_2R itself is the unique minimal right subideal of e_1R, is isomorphic to e_2R, and is contained in every nonzero right subideal of e_1R. We claim that $e_1Ne_1 = 0$. Let $0 \neq e_1xe_1 \in e_1Ne_1$. Since N is nil, e_1xe_1 is nilpotent. Then $\sigma: e_1R \to e_1R$ defined by $\sigma(e_1r) = e_1xe_1r$ is an endomorphism of e_1R with $\ker\sigma \neq (0)$. Let $A = \ker\sigma$. Then $e_1Re_2R \subset A$, and we have $e_1xe_1Re_2 = (0)$. On the other hand, $e_1Re_2R \subseteqq e_1xe_1R$ yields that $e_1xe_1Re_2 \neq (0)$. This is a contradiction. Hence $e_1Ne_1 = (0)$, and e_1Re_1 is a division ring. Now using the fact that $\mathrm{Hom}_R(e_1R, e_1R)$ is a division ring and that e_1R is quasi-injective, it follows that every member of $\mathrm{Hom}(e_1Re_2R, e_1Re_2R)$ admits a unique extension to an endomorphism of e_1R. Further, every endomorphism of e_1R maps e_1Re_2R into itself since e_1Re_2R is the unique minimal subideal of e_1R. Thus $\mathrm{Hom}(e_1Re_2R, e_1Re_2R) \cong \mathrm{Hom}(e_1R, e_1R)$. Since $e_1Re_2R \cong e_2R$, we obtain $e_1Re_1 \cong e_2Re_2$.

Now $e_1N = e_1Ne_2$ because $e_1Ne_1 = (0)$. Since $e_1Re_2R \subseteqq e_1N$, we get $e_1N = e_1Re_2 = e_1Re_2R$. Thus $M = e_1Re_2$ is a one-dimensional right vector space over $D = e_2Re_2$. We show that M is also a one-dimensional left e_1Re_1-space. Let $X = \begin{pmatrix} e_1Re_1 & M \\ 0 & 0 \end{pmatrix} \cong R/A$, where $A = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$. Then X is quasi-injective. Let $0 \neq x \in M$, and let $y \in M$. Consider $\sigma\colon \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ defined by $\sigma\begin{pmatrix} 0 & xc \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & yc \\ 0 & 0 \end{pmatrix}$, for $c \in D$. Then σ is an R-endomorphism, so it can be extended to an endomorphism η of X. Let $\eta\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. Then we have $\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = \sigma\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \eta\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ax \\ 0 & 0 \end{pmatrix}$. Thus $y = ax$, so $M = e_1Re_1x$. So M is a one-dimensional left vector space over e_1Re_1. Thus, for each $d \in e_1Re_1$, there exists a unique $d' \in e_2Re_2$ such that $dx = xd'$. Define $\theta\colon e_1Re_1 \to e_2Re_2$ by $\theta(d) = d'$. Then θ is an isomorphism, and we may identify d and d'. Then $\eta\colon \begin{pmatrix} D & D \\ 0 & D \end{pmatrix} \to \begin{pmatrix} D & M \\ 0 & D \end{pmatrix}$ defined by $\eta\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & bx \\ 0 & c \end{pmatrix}$ is an isomorphism.

Conversely, if R satisfies (i), then, by Koehler [13], R is a QC-ring, hence a $PCQI$-ring. If R satisfies (ii), then straightforward computation shows that R is a right $PCQI$-ring.

Since every right QC-ring is a left QC-ring and $\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ is also a left $PCQI$-ring, we get the following corollary.

COROLLARY. *A nonlocal semiperfect right PCQI-ring is also a left PCQI-ring.*

5. Local $PCQI$-rings. Theorem 13 and Theorems 14, 15, and 16 which follow generalize Klatt and Levy's [11] theorems for commutative pre-self-injective rings which are not domains. Throughout this section M will denote the unique maximal right ideal of a local ring R. M is then the Jacobson radical of R, and R/M is a division ring.

THEOREM 14. *Let R be a local right PCQI-ring with maximal ideal M. Then either R is a right valuation ring or $M^2 = (0)$ and M_R has composition length* 2.

Proof. First note that for all nonzero right ideals A, R/A is indecomposable quasi-injective and hence uniform. Now we show that all nonzero right ideals are either minimal or essential. Let A, B be nonzero right ideals such that $A \cap B = (0)$. We claim that A is minimal. Let C be a nonzero right ideal properly contained in A. Then R/C is quasi-injective and not uniform since $A/C \cap (B + C)/C = 0$. This is a contradiction, so A is minimal. Similarly, B is minimal. In particular, it follows that any maximal independent family of minimal right ideals can contain at most two members.

If $\operatorname{Soc} R_R = (0)$, then all nonzero right ideals are essential. Let A, B be two nonzero right ideals. If neither $A \subsetneqq B$ nor $B \subsetneqq A$, then $R/A \cap B$ is quasi-injective but not uniform since $A/(A \cap B) \cap B/(A \cap B) = (\bar{0})$. As before, this is a contradiction. So either $A \subseteqq B$ or $B \subseteqq A$.

If $\operatorname{Soc} R_R$ consists of a unique minimal right ideal then it is clear that R is a right valuation ring.

Finally, suppose $\operatorname{Soc} R_R = A \oplus B$, where A, B are minimal right ideals. Then R cannot be prime. Let $x \in M$, and consider xR. If xR is not minimal, then xR is quasi-injective and decomposable. Then $xR = A \oplus B$. In any case, for all $x \in M$, $x \in \operatorname{Soc} R_R$. This implies that $M^2 = (0)$, and the composition length of M is 2, completing the proof.

The next two theorems give the structure of non-prime local right $PCQI$-rings. Prime local $PCQI$-rings are discussed in the next section.

THEOREM 15. *For a nonprime right valuation ring R, the following are equivalent*:

(i) *R is a right $PCQI$-ring.*

(ii) *R is a right duo almost maximal valuation ring of rank 0 such that any left ideal containing a nonzero right ideal is two-sided.*

Proof. (i) $\Rightarrow$ (ii). Since R is not prime, M is nil by Proposition 3. So, if xR is a nontrivial principal right ideal of R, xR is quasi-injective. Since xR is essential in R, the injective hull of xR is the same as that of R. Hence, by Johnson and Wong [10], $RxR \subseteqq xR$. So xR is a two-sided ideal of R. Thus R is a right duo ring. Since each proper homomorphic image of a $PCQI$-ring is a QC-ring, the proof of (i) $\Rightarrow$ (ii) as well as that of (ii) $\Rightarrow$ (i) is completed by a theorem of Koehler [13].

THEOREM 16. *For a local ring R with $M^2 = (0)$ and the composition length of M_R equal to 2, the following are equivalent*:

(i) *R is a right PCQI-ring.*

(ii) *For each nonzero right ideal A in R and for each $m_1, m_2 \notin A$, the congruence $xm_1 \equiv m_2(\text{mod } A)$ has a solution, $x = \alpha$, such that $\alpha A \subset A$.*

Proof. Under the hypothesis the only nonzero right ideals A of R different from M and R are minimal right ideals, and M/A is a simple right R-module.

(i) $\Rightarrow$ (2) Let A be a nontrivial right ideal in R, and let $m_1, m_2 \in R$ such that $m_1, m_2 \notin A$. Then $\bar{m}_1R = M/A = \bar{m}_2R$, and the mapping $\sigma: M/A \to M/A$ which sends $\bar{m}_1r$ to $\bar{m}_2r$ is a well-defined R-homomorphism. Since R/A is quasi-injective, σ can be lifted to $\sigma^* \in \text{Hom}_R(R/A, R/A)$. Let $\sigma^*(\bar{1}) = \bar{\alpha}$. Then $\bar{\alpha}m_1 = \bar{m}_2$. Hence $xm_1 \equiv m_2(\text{mod } A)$ has a solution $x = \alpha$. Clearly $\alpha A \subset A$.

(ii) $\Rightarrow$ (i) We only need to prove that if A is a nontrivial right ideal of R and $\sigma: M/A \to R/A$, is a nonzero R-homomorphism, then σ can be extended to an R-homomorphism $\sigma^*: R/A \to R/A$. Let $m \in M$, where $m \notin A$. Then $M/A = \bar{m}R$. Also, $\sigma(M/A) = M/A$. Let $\sigma(\bar{m}) = \bar{m}r$. Since $M^2 = (0)$, $r \notin M$. So r is invertible, and $mr \notin A$. Let $\alpha \in R$ be chosen such that $\alpha m \equiv mr(\text{mod } A)$, and $\alpha A \subseteqq A$. Then $\sigma^*(\bar{r}) = \bar{\alpha}R$ is well-defined, and it extends σ, completing the proof.

The example which follows shows that a local right *PCQI*-ring is not necessarily a left *PCQI*-ring.

EXAMPLE. Let F be a field which has a monomorphism $\rho: F \to F$ such that $[F: \rho(F)] > 2$. Take x to be an indeterminate over F. Make $V = xF$ into a right vector space over F in a natural way. Let $R = \{(\alpha, x\beta) \mid \alpha, \beta \in F\}$. Define

$$(\alpha_1, x\beta_1) + (\alpha_2, x\beta_2) = (\alpha_1 + \alpha_1, x\beta_1 + x\beta_2)$$

and

$$(\alpha_1, x\beta_1)(\alpha_2, x\beta_2) = (\alpha_1\alpha_2, x(\rho(\alpha_1)\beta_2 + \beta_1\alpha_2)) .$$

Then R is a local ring with identity with the maximal ideal

$$M = \{(0, x\alpha) \mid \alpha \in F\} .$$

In fact, M is also a minimal right ideal and $M^2 = (0)$. Thus R is a right *PCQI*-ring. Further, if $\{\alpha_i\}_{i \in I}$ is a basis of F as a vector space over $\rho(F)$ then straightforward computations yield that $M = \bigoplus \sum R(0, x\alpha_1)$ as a direct sum of irreducible left R-modules $R(0, x\alpha_i)$. Since card $I > 2$, it follows by Theorem 14 that R is not a left *PCQI*-ring.

6. Prime local *PCQI*-rings.

THEOREM 17. *Let R be a prime local right PCQI-ring. Then R is a right valuation domain, hence right semihereditary.*

Proof. By Theorem 14, R is a right valuation ring. Let A denote the intersection of all nonzero two-sided ideals of R. The proof that R is a domain falls into three cases.

(i) $A = (0)$.

Let $x, y \in R$ such that $xy = 0$. Suppose $y \neq 0$. Then yR is a nonzero right ideal of R. Since R is right valuation and $A = (0)$, yR must contain a nonzero two-sided ideal of R. Further, each proper homomorphic image of R is a local QC-ring, hence a duo ring [13]. This implies that yR is two-sided. Hence $x = 0$, and R is an integral domain.

(ii) $A \neq (0)$ and $A \neq M$.

Under these hypotheses, A cannot be a prime ideal. So there exist $x, y \in R$ such that $xRy \subseteqq A$, $x \notin A$ and $y \notin A$. Since R is right valuation, $A \subseteqq xR$ and $A \subseteqq yR$. So both xR and yR are two-sided ideals. For definiteness, let $xR \subseteqq yR$. Then $(xR)^2 \subseteqq (xR)(yR) \subseteqq AR = A$ gives that $(xR)^2 = A$ by the minimality of A. Also $A = A^2$, hence $(xR)^2 = (xR)^4$. It follows that $x^2R = x^4R$. Then $x^2 = x^4r$, for some $r \in R$, and $x^2(1 - x^2r) = 0$. So $x^2 = 0$. Thus $A = (0)$, and this case cannot occur.

(iii) $A = M$.

Let $S \subset R$, and let $r(S)$ denote the right annihilator of S in R. Let $Z(R) = \{x \in R \mid r(x)$ is an essential right ideal$\}$. Then $Z(R)$ is an ideal in R called the right singular ideal.

Since R is a right valuation ring, R is immediately a domain if $Z(R) = (0)$.

So assume that $Z(R) \neq (0)$. Then $Z(R) = M$, and each element in M is a right zero divisor. So $x \in M$ implies that xR is proper cyclic, hence quasi-injective. Also xR is an essential right ideal in R. By Johnson and Wong [10], $RxR \subseteqq xR$. Hence xR is two-sided. So R is a prime right duo ring, and it follows that R is a domain.

7. *PCQI*-domains.

In this section we discuss right *PCQI*-rings which are integral domains and prove that these are right Öre-domains. This generalizes the result of Faith [4]. Our proof, in this case, though it runs on the same lines as that of Faith, does not use Faith's result.

PROPOSITION 18. *Let R be a right PCQI-domain, and let I be a nonessential right ideal of R. Then R/I is an injective right R-*

module containing a copy of R.

Proof. Since I is nonessential, there exists a nonzero right ideal J in R such that $I \cap J = 0$. Let $a \in J$ such that $a \neq 0$. Then $aR \cap I \subseteqq J \cap I = 0$. Consider $r(a + I) = \{x \in R \mid ax \in I\}$. Clearly $r(a + I) = 0$. So R/I contains a copy of R. Since R/I is also quasi-injective, this implies that R/I is injective by [17].

For a right R-module A, let $\hat{A}$ denote the injective hull of A.

PROPOSITION 19. *Let R be a right PCQI-domain which is not a right Öre-domain. Then $\hat{R}$ is finitely presented.*

Proof. Let $a \in R$ such that $a \neq 0$ and aR is not essential. Then R/aR is injective. Since R/aR contains a copy of R and is injective, R/aR contains a copy of $\hat{R}$. Then $R/aR = Y/aR \oplus X/aR$, where $X/aR \cong \hat{R}$. Now Y/aR is cyclic. So $Y = aR + bR$, for some $b \in R$, and the short exact sequence $0 \to Y \to R \to R/Y \cong X/aR \cong \hat{R} \to 0$ shows that $\hat{R}$ is finitely presented.

THEOREM 20. *A right PCQI-domain R is a right Öre-domain.*

Proof. Let R be a right $PCQI$-domain. Suppose R is not a right Öre-domain. Then, as in Proposition 19, there exists $a \in R$ such that $R/aR = Y/aR \oplus X/aR$, where $X/aR \cong \hat{R} \cong R/Y$ and $Y = aR + bR$. We also get that $R = X + Y$, where $X \cap Y = aR$. This yields an exact sequence $0 \to aR \to X \times Y \to R \to 0$ which splits. So $X \times Y \cong aR \times R \cong R \times R$. This implies that $Y = aR + bR$ is a finitely generated projective right ideal. Since $\hat{R} \cong R/Y$, $0 \to Y \to R \to \hat{R} \to 0$ is exact. Then $Y \otimes_R \hat{R} \to R \otimes_R \hat{R} \to \hat{R} \otimes_R \hat{R} \to 0$ is exact. Also, a finitely generated projective R-module is essentially finitely related. So, by Cateforis ([3], Proposition 1.7), $(aR + bR) \otimes_R \hat{R}$ is projective as an $\hat{R}$-module. Then $Y \otimes_R \hat{R}$ is a direct summand of a free $\hat{R}$-module. Now $Z(\hat{R}_{\hat{R}}) = 0$, hence $Z(Y \otimes_R \hat{R}) = 0$ because $Y \otimes_R \hat{R}$ is a direct summand of a free $\hat{R}$-module. Now consider $Y \otimes_R \hat{R} \xrightarrow{i} R \otimes_R \hat{R} \to \hat{R} \otimes_R \hat{R} \to 0$. Again, by Cateforis ([3], Lemma 1.8), $\ker i = Z(Y \otimes_R \hat{R}) = 0$. So $0 \to Y \otimes_R \hat{R} \xrightarrow{i} R \otimes_R \hat{R} \to \hat{R} \otimes_R \hat{R} \to 0$ is exact. Since $R \otimes_R \hat{R} \cong \hat{R}$, let $f: R \otimes_R \hat{R} \to \hat{R}$ be the canonical isomorphism. Then $fi: Y \otimes_R \hat{R} \to \hat{R}$ is a monomorphism, and $Y \otimes_R \hat{R} \cong Y\hat{R}$. Since Y is finitely generated, $Y\hat{R}$ is a finitely generated right ideal of $\hat{R}$. So $Y\hat{R} = e\hat{R}$, where $e^2 = e$. Thus we have the following exact sequence: $0 \to e\hat{R} \to \hat{R} \to \hat{R} \otimes_R \hat{R} \to 0$, and $\hat{R} \otimes_R \hat{R} \cong \hat{R}/e\hat{R} = (1 - e)\hat{R}$. Hence $\hat{R} \otimes_R \hat{R}$ is isomorphic to a direct summand of $\hat{R}$. Since $Z(\hat{K}_R) = 0$, $Z(\hat{R} \otimes_R \hat{R}) = 0$. Since $\hat{R} = xR$, for some $x \in \hat{R}$, the kernel of the canonical map $f: \hat{R} \otimes_R \hat{R} \to \hat{R}$ defined by $f(a \otimes b) = ab$ is contained in $Z(\hat{R} \otimes_R \hat{R})$ and hence must be zero. Since f is surjective, f is an isomorphism. By Silver ([18], Proposition 1.1), there exists an epimorphism in the category of rings from R to $\hat{R}$.

Let M be a right $\hat{R}$-module which is quasi-injective as a right R-module. We claim that M is quasi-injective as a right $\hat{R}$-module. Let $0 \to A_{\hat{R}} \to M_{\hat{R}} \to B_{\hat{R}} \to 0$ be exact. Consider $0 \to \mathrm{Hom}_{\hat{R}}(B_{\hat{R}}, M_{\hat{R}}) \to \mathrm{Hom}_{\hat{R}}(M_{\hat{R}}, M_{\hat{R}}) \to \mathrm{Hom}_{\hat{R}}(A_{\hat{R}}, M_{\hat{R}})$. By Silver ([18], Corollary 1.3), $\mathrm{Hom}_{\hat{R}}(N, N^1) \cong \mathrm{Hom}_R(N, N^1)$, where N, N^1 are right $\hat{R}$-modules. Also $0 \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(A, M) \to 0$ is exact since M_R is quasi-injective. Thus $0 \to \mathrm{Hom}_{\hat{R}}(B, M) \to \mathrm{Hom}_{\hat{R}}(M, M) \to \mathrm{Hom}_R(A, M) \to 0$ is exact. So $M_{\hat{R}}$ is quasi-injective. Let K be a cyclic right R-module. Then K is a cyclic right R-module. Since R is a right $PCQI$-domain, K_R is quasi-injective. Thus $K_{\hat{R}}$ is quasi-injective. Since $\hat{R}$ is right self-injective, $\hat{R}$ is a QC-ring. So $\hat{R}$ is semiperfect and simple, hence simple artinian. Thus $\hat{R}$ is a division ring. This proves that R is a right Öre-domain.

We conclude by a remark that we have not studied arbitrary prime right $PCQI$-rings. This case remains open. Indeed, a characterization of right $PCQI$-domains has not yet been obtained.

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