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GENERALIZED PRIMITIVE ELEMENTS FOR TRANSCENDENTAL FIELD EXTENSIONS

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Let L be a finitely generated separable extension of a field K of characteristic $p \neq 0$. Artin's theorem of a primitive element states that if L is algebraic over K, then L is a simple extension of K. If L is non-algebraic over K, then an element $\theta \in L$ with the property $L = L'(\theta)$ for every L', $L \supseteq L' \supseteq K$, such that L is separable algebraic over L' is called a generalized primitive element for L over K. The main result states that if $[K : K^p] > p$, then there exists a generalized primitive element for L over K. An example is given showing that if $[K : K^p] \leqq p$, then L need not have a generalized primitive element over K.

I. Introduction. Let L be a finitely generated extension of a field K of characteristic $p \neq 0$. Artin's theorem of the primitive element states that if L is separable algebraic over K, then L is a simple extension of K. In this paper we examine the following analogue of Artin's theorem in the case where L is a separable non-algebraic extension of K. Does there exist an element $\theta \in L$ with the property that θ is a primitive element for L over every intermediate field L' such that L is separable algebraic over L'? The main result states that if K has at least two elements in a p-basis, then there does exist such a generalized primitive element (Theorem 4). Such elements θ are characterized by the condition that L is reliable over $K(\theta)$ (Theorem 1). As a corollary, it follows that automorphisms of L over K are uniquely determined by their action on a generalized primitive element θ. Other results which indicate the essential nature of a generalized primitive element include the following. If L_1 and L_2 are intermediate fields of L/K where L is separable over L_1 and L_2, then $L_2 \supseteq L_1$ if and only if some generalized primitive element for L_1 is in L_2 (Theorem 6).

II. Generalized primitive elements. Throughout we assume L is a finitely generated extension of a field K of characteristic $p \neq 0$. As usual, a relative p-basis for L over K is a minimal generating set for L over $K(L^p)$.

DEFINITION. L is a reliable extension of K if $L = K(M)$ for every relative p-basis M of L over K.

In the case where L is finitely generated over K, L is reliable over K if and only if there does not exist a proper intermediate field L' with L separable over L' [**5**, Theorem 1, p. 524]. Using this result it follows that if L is reliable over K, then L is reliable over any intermediate field M.

THEOREM 1 [**1**, Theorem 1.9]. *If L is finitely generated over K, then there exists a unique intermediate field C with the property L/C is separable and C/K is reliable.*

In fact, C is the intersection of all subfields L' such that L/L' is separable. If L is separable over K, then an element θ in L is a generalized primitive element for L over K if $L = L'(\theta)$ for any L' such that L is separable algebraic over L'. Henceforth, L will be a finitely generated separable (non-algebraic) extension of K.

THEOREM 2. *An element θ in L is a generalized primitive element for L over K if and only if L is reliable over $K(\theta)$.*

Proof. Assume θ is a generalized primitive element. It suffices to show there are no proper intermediate fields L', $L \supset L' \supsetneq K(\theta)$, over which L is separable. Since θ is a generalized primitive element, there are no proper fields over which L is separable algebraic. But in any finitely generated separable extension L/L' there exist subfields over which L is separable algebraic (by applying Luroth's Theorem). Thus $L/K(\theta)$ is reliable.

Conversely, assume there exists an element θ such that L is reliable over $K(\theta)$ and let L' be any intermediate field such that L/L' is separable algebraic. Then $L/L'(\theta)$ is also separable. Since $L/K(\theta)$ is reliable and $L' \supseteq K$, $L/L'(\theta)$ is reliable and hence $L = L'(\theta)$.

The following result of Mordeson and Vinograde is essential to this paper.

THEOREM 3 [**4**, Theorem 2]. *Assume L is a finitely generated separable extension of K, $L \neq K$, and assume $[K: K^p] > p$. Then there exists a field $M = L(\alpha)$ where M is reliable over K and α^p is in L.*

THEOREM 4. *Let L be a finitely generated separable extension of K and assume $[K: K^p] > p$. Then there exists a generalized primitive element for L over K.*

Proof. By Theorem 3, there exists a field $M = L(\alpha)$ which is reliable over K and $\alpha^p \in L$. Let $\theta = \alpha^p$ and we show θ is the desired element. By Theorem 2, it suffices to show L is reliable over $K(\theta)$. Assume there exists an intermediate field L', $L \supseteq L' \supseteq K(\theta)$

where L is separable over L'. Since $\alpha^p \in K(\theta)$, $\alpha^p \in L'$. Thus $L'(\alpha)$ is purely inseparable over L'. Thus L and $L'(\alpha)$ are linearly disjoint over L'. By [3, Corollary 4, p. 265], $L'(\alpha)(L) = M$ is separable over $L'(\alpha)$. As M is reliable over $L'(\alpha)$, $M = L'(\alpha)$ and since L and $L'(\alpha)$ are linear disjoint over L', we must have $L = L'$ and L is reliable over $K(\theta)$.

COROLLARY 1. *If L/K is nonalgebraic, then any generalized primitive element is transcendental over K.*

Proof. Let θ be a generalized primitive element. If θ were algebraic over K, then $L/K(\theta)$ would be separable and hence $L = K(\theta)$.

The following corollary is a direct result of a calculation in [4]. For completeness, it is presented here.

COROLLARY 2. *Assume $L = K(z_1, \cdots, z_{n-1}, z_n)$ where $z_1, \cdots, z_{n-1}$ are algebraically independent over K and $F/K(z_1, \cdots, z_{n-1})$ is nontrivial separable. Let $\{x, y\}$ be p-independent in K. Then $\theta = \alpha^p$ is a generalized primitive element for L/K where*

$$\alpha = \sum_{1}^{n-1} k_j z_j^{p^j} + k_n z_n^{p^{n-1}}$$

and

$$k_1 = y^{-1}$$

$$k_j = (-1)^{j-1} \frac{x^{p^0 + \cdots + p^{j-2}}}{y^{p^0 + \cdots + p^{j-1}}} \quad \text{for} \quad j = 2, \cdots, n-1$$

$$k_n = (-1)^{n-1} \left(\frac{x}{y}\right)^{p^0 + \cdots + p^{n-2}}.$$

Proof. This follows from Theorem 4 and the proof of [4, Theorem 1, p. 44].

COROLLARY 3. *Let θ be a generalized primitive element for L over K. Then any automorphism of L over K is uniquely determined by its action on θ.*

Proof. Let σ, τ be automorphisms of L/K and assume $\sigma(\theta) = \tau(\theta)$. Then $\sigma\tau^{-1}(\theta) = \theta$ and $K(\theta)$ is contained in the fixed field L' of $\sigma\tau^{-1}$. Since L is separable over L', $L = L'$ and $\sigma = \tau$.

LEMMA 1. *Let θ be a generalized primitive element for L over K, and let F be an intermediate field such that L is separable nonalgebraic over F. Then F is free from $K(\theta)$ and $F(\theta)$ is separable over $K(\theta)$.*

Proof. If θ were algebraic over F, then L would be separable over $F(\theta)$, a contradiction to L being reliable over $F(\theta)$. Thus $K(\theta)$ is free from F. The remainder of the Lemma follows from [**3**, Corollary 4, p. 265].

A generalized primitive element for L over K will generate L over any subfield L' such that L is separable algebraic over L'. The following theorem shows that with one exception these are the only subfields with this property.

THEOREM 5. *Let θ be a generalized primitive element for L over K, and let L' be a subfield of L containing K. Then $L = L'(\theta)$ if and only if either L/L' is separable algebraic or $L = K(\theta)$.*

Proof. Assume $L = L'(\theta)$. If L/L' are not algebraic, then L/L' would be pure transcendental and hence separable. But then by Lemma 1, $L/K(\theta)$ would be separable, and hence $L = K(\theta)$. Thus we may assume L/L' is algebraic and $L \neq K(\theta)$. Since $L/K(\theta)$ is not separable and L/K is, $\theta \in K(L^p)$ [**1**, Proposition 1.3]. Thus $L = L'(L^p)$ and L is relatively perfect over L'. Since L/L' is also finitely generated, L/L' is separable algebraic [**6**, Theorem 2, p. 419]. The converse is Theorem 2.

If L is a finitely generated separable extension of K, then any intermediate field L' is also finitely generated and separable over L. If $[K:K^p] > p$, then L' will also have a generalized primitive element θ' over K. Moreover, each element of L will be a generalized primitive element for a unique subfield L' where L/L' is separable. For if $\theta \in L$, let L' be the unique intermediate field of $L/K(\theta)$ such that L is separable over L' and L' is reliable over $K(\theta)$. Then θ is a generalized primitive element for L'. Thus any intermediate field L' where L is separable over L' is uniquely determined by any of its generalized primitive elements. The following theorem and corollary indicate how a generalized primitive element is basic in the structure of an intermediate field.

THEOREM 6. *Assume L is a finitely generated separable extension of K and let L_1 and L_2 be two intermediate fields over which L is separable. Then the following are equivalent.*

(1) $L_1 \subseteq L_2$

(2) *Every generalized primitive element for L_1 is in L_2*

(3) *Some generalized primitive element for L_1 is in L_2.*

Proof. We show (3) implies (1). Let θ_1 be a generalized primitive element for L_1/K and assume $\theta_1 \in L_2$. If L_2 is separable over $K(\theta_1)$, then L is separable over $K(\theta_1)$ and L_1 is separable over $K(\theta_1)$. Since L_1 is reliable over $K(\theta_1)$, $L_1 = K(\theta_1)$ and $L_1 \subseteq L_2$. If L_2 is inseparable over $K(\theta_1)$, then there is a unique field C_2, $L_2 \supseteq C_2 \supseteq K(\theta_1)$ where L_2 is separable over C_2 and C_2 is reliable over $K(\theta_1)$. Thus L is separable over C_2 and C_2 is reliable over $K(\theta_1)$. But L_1 is uniquely determined by these properties and hence $C_2 = L_1$ and $L_1 \subseteq L_2$.

COROLLARY 4. *Assume L is separable over L_1, $L \supseteq L_1 \supseteq K$, and θ_1 is a generalized primitive element for L_1 over K. If L_2 is any intermediate field of L over K such that L is separable algebraic over L_2, then $L_2(L_1) = L_2(\theta_1)$.*

Proof. Since L is separable algebraic over L_2, L is separable over $L_2(\theta_1)$. By Theorem 6, $L_2(\theta_1) \supseteq L_1$ and hence $L_2(L_1)$. Obviously $L_2(\theta_1) \subseteq L_1(L_2)$ and thus $L_2(L_1) = L_2(\theta_1)$.

EXAMPLE 1. If $[K: K^p] \leqq p$, then L may not have a generalized primitive element over K. Let K be a perfect field and let $L = K(x, y, z)$ where $\{x, y, z\}$ is algebraically independent over K. We claim there is no generalized primitive element for L over K. Assume θ is one. Then $L/K(\theta)$ is reliable. However $K(\theta)$ has one element in a relative p-basis and hence by [**2**, Theorem 7 (iv)] L is separable over $(K(\theta))* \cap L$, where $(K(\theta))*$ is the perfect closure of $K(\theta)$. But $(K(\theta))* \cap L$ is of transcendence degree at most 1 over K, and hence $(K(\theta))* \cap L \neq L$. This contradicts L being reliable over $K(\theta)$.

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