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THE LENGTH OF THE PERIOD OF THE SIMPLE CONTINUED FRACTION OF $d^{1/2}$.

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Let $p(d)$ denote the length of the period of the simple continued fraction for $d^{1/2}$ and ε the fundamental unit in the ring $Z\,[d^{1/2}]$. We prove that as $d \to \infty$,

THEOREM 1. $p(d) \leqq 7/2\pi^{-2}d^{1/2}\log d + O(d^{1/2})$.

THEOREM 2. $\log \varepsilon \leqq 3\pi^{-2}d^{1/2}\log d + O(d^{1/2})$.

THEOREM 3. $p(d) \neq o(d^{1/2}/\log\log d)$.

THEOREM 4. **If** $\log \varepsilon \neq o(d^{1/2}\log d)$ **then also**

$$p(d) \neq o(d^{1/2}\log d)\ .$$

Recently Hickerson [1] has proved that $p(d) = O(d^{1/2+\delta})$ for every $\delta > 0$, and in fact a result somewhat more precise than this. Lehmer [2] has suggested that for arbitrarily large d, $p(d)$ might be as large as $0.30d^{1/2}\log d$, and if this is indeed the case then Theorem 1 is almost the best possible result. In fact it is easy to show that $p(d) = O(d^{1/2}\log d)$ using known results regarding $\log \varepsilon$, but the constant in Theorem 1 improves the best obtainable in this way.

Let ε_0 denote the fundamental unit in the field $Q(d^{1/2})$, $[a_0, \overline{a_1, a_2, \cdots a_{p(d)-1}, 2a_0}]$ the continued fraction for $d^{1/2}$ and P_r/Q_r its rth convergent. Then as is well known $\varepsilon = \varepsilon_0$ or ε_0^3. Thus by the result of Stephens [3],

$$\log \varepsilon \leqq 3\log \varepsilon_0 \leqq \frac{3}{2}(1 - e^{-1/2} + \delta)d^{1/2}\log d\ .$$

Now $Q_0 = 1$, $Q_1 = a_1 \geqq 1$ and $Q_{r+2} = a_{r+2}Q_{r+1} + Q_r \geqq Q_{r+1} + Q_r$ and so by induction $Q_r \geqq u_{r+1}$, the Fibonacci number, for $r \geqq 0$. Now

$$\begin{aligned}\varepsilon &= P_{p(d)-1} + Q_{p(d)-1}d^{1/2}\\ &> 2d^{1/2}Q_{p(d)-1} - 1\\ &\geqq 2d^{1/2}u_{p(d)} - 1\\ &> \left\{\frac{1+\sqrt{5}}{2}\right\}^{p(d)},\end{aligned}$$

and so $p(d) < Ad^{1/2}\log d$ where A is approximately $5/4$.

In exactly the same way, using $a_r < d^{1/2}$ for $0 \leqq r < p(d)$ it is possible to show that $p(d) \gg \log \varepsilon/\log d$. Since $d = 2^{2k+1}$ gives $\varepsilon = (1+\sqrt{2})^{2k}$, we find that for arbitrarily large d it is possible for $p(d) \gg d^{1/2}/\log d$, and it will be shown that this can be improved at least by replacing the $\log d$ by $\log\log d$. Theorems 1 and 3 together show that the scope for sharpening the results is somewhat limited; nevertheless the remaining problem is important and worthy of further study, for as we mention in the concluding remarks, if it could be proved that $p(d) = o(d^{1/2}\log d)$ this would imply also that $\log\varepsilon = o(d^{1/2}\log d)$ a result which has been sought in vain for many years.

Throughout we use ε_1 to denote the fundamental unit in $Z[d^{1/2}]$ with norm $+1$; then $\varepsilon_1 = \varepsilon$ or ε^2. In accordance with established practice, if for given integers d and N there exist integers X and Y with $X^2 - dY^2 = N$, then we say that $X + Yd^{1/2}$ is a solution of the equation $x^2 - dy^2 = N$. Given one such solution, all the members of the set $\pm(X + Yd^{1/2})\varepsilon_1^n$ are also solutions, and this set is called a *class* of solutions. A given equation may well have more than one such class of solutions, but it is well known that the number of such classes is finite.

LEMMA 1. *For each* r, $|P_r^2 - dQ_r^2| < 2d^{1/2}$.

This is well known.

LEMMA 2. *For a class* K *of solutions of* $x^2 - dy^2 = N$, *the g.c.d.,* (x, y) *depends only upon* K.

For if $x_1 + y_1d^{1/2}$ and $x_2 + y_2d^{1/2}$ belong to the same class, then for some integer n,

$$\begin{aligned} x_1 + y_1d^{1/2} &= \pm(x_2 + y_2d^{1/2})\varepsilon_1^n \\ &= \pm(x_2 + y_2d^{1/2})(a_n + b_nd^{1/2}) , \end{aligned}$$

say. Thus $(x_2, y_2)\,|\,(x_1, y_1)$ and similarly conversely.

A class K for which $(x, y) = 1$ is called a *primitive* class. The main result used in the proof of the theorems is

LEMMA 3. *The number of primitive classes,* $f(N; d)$, *of* $x^2 - dy^2 = N$ *does not exceed* $2^{\omega(|N|)}$. *In the special case* $2\,||\,N$, $f(N; d) \leqq 2^{\omega(|N|)-1}$. *Here* $\omega(N)$ *denotes the number of distinct prime factors of* N.

Proof. In the first place it suffices to consider the case in which (N, d) is square-free. For if $(N, d) = k_1^2k_2$ where k_2 is square-free, $(x, y) = 1$ and $x^2 - dy^2 = N$ then $k_1|x$ and so if $x_1 = x/k_1$, $N_1 = N/k_1^2$ and $d_1 = d/k_1^2$ then $x_1^2 - d_1y^2 = N_1$ with $(x_1, y) = 1$. For the latter equation we now have $(N_1, d_1) = k_2$ which is square-free and so the total number of classes of primitive solutions of the *given* equation does not exceed $2^{\omega(|N_1|)} \leqq 2^{\omega(|N|)}$ in the general case, or $2^{\omega(|N|_1)-1} \leqq 2^{\omega(|N|)-1}$ in the special case $2||N$ since in this case $2||N_1$ also. We suppose therefore from now on that (N, d) is square-free.

Let p denote any prime dividing N, and suppose that $p^s||N$;

(i) if $p|d$ then $p|x$, whence $p^2 \nmid dy^2$ otherwise we should find, since $p \nmid y$ that $p^2|d$ and $p^2|N$. Hence $s = 1$ and so $xy^{-1} \equiv 0(\text{mod } p^s)$.

(ii) if $p \nmid d$ then p can divide neither x nor y, otherwise it would have to divide them both. Thus $(xy^{-1})^2 \equiv d(\text{mod } p^s)$ and so if p is odd, $xy^{-1} \equiv \pm a_p(\text{mod } p^s)$.

(iii) if $p \nmid d$, $p = 2$ then $(xy^{-1})^2 \equiv d(\text{mod } p^s)$ gives

(a) if $s = 1$, $xy^{-1} \equiv d(\text{mod } 2)$, i.e., $xy^{-1} \equiv d(\text{mod } p^s)$

(b) if $s = 2$, since $x^2 - dy^2 \equiv 0(\text{mod } 4)$ and both x and y are odd, $d \equiv 1(\text{mod } 4)$ whence $(xy^{-1})^2 \equiv 1(\text{mod } 4)$, i.e., $xy^{-1} \equiv \pm 1(\text{mod } 4)$, i.e., $xy^{-1} \equiv \pm 1(\text{mod } p^s)$

(c) if $s \geqq 3$, then $d \equiv 1(\text{mod } 8)$ and now $(xy^{-1})^2 \equiv d(\text{mod } 2^s)$ gives $xy^{-1} \equiv \pm a(\text{mod } 2^{s-1})$.

Combining (i), (ii), and (iii) and using the Chinese Remainder Theorem, we see that xy^{-1} is congruent to one of at most

$2^{\omega(N)-1}$ residues modulo N if $2||N$

$2^{\omega(|N|)}$ residues modulo N unless $8|N$

$2^{\omega(|N|)}$ residues modulo $\frac{1}{2}N$ if $8|N$.

Next we prove that if $x^2 - dy^2 = X^2 - dY^2 = N$ and if $xy^{-1} \equiv XY^{-1}(\text{mod } N)$ then $x + yd^{1/2}$ and $X + Yd^{1/2}$ belong to the same class K. For

$$\frac{x + yd^{1/2}}{X + Yd^{1/2}} = \frac{(x + yd^{1/2})(X - Yd^{1/2})}{X^2 - dY^2} = \frac{xX - dyY}{N} + \frac{-xY + Xy}{N}d^{1/2}$$
$$= A + Bd^{1/2}, \quad \text{say}.$$

Now B is an integer and A rational, and since $A^2 - dB^2 = 1$ it follows that A too is an integer, and so that result of the lemma follows, except if $8|N$.

Finally, if $8|N$ then we find that if $xy^{-1} \equiv XY^{-1}(\text{mod } 1/2N)$ then $x + yd^{1/2}$ and $X + Yd^{1/2}$ belong to the same class; for if as above $A + Bd^{1/2}$ denote their quotient, we find that B equals either an integer or else half an odd integer. In the former case the result follows as above. In the latter case we find $(2A)^2 = d(2B)^2 + 4$ and since now $2B$ is an odd integer and $4 \nmid d$, $2A$ is also an odd integer, whence $d \equiv 5(\text{mod } 8)$. But this is inconsistent with $x^2 - dy^2 \equiv 0(\text{mod } 8)$ where $(x, y) = 1$ and so this latter case does not arise. This concludes the proof.

LEMMA 4. *If* $N(\varepsilon) = 1$, *then*

$$f(N, d) + f(-N, d) \leqq 2^{\omega(|N|)}$$

and

$$f(N, d) + f(-N, d) \leqq 2^{\omega(|N|)-1} \quad if \quad 2 \| N .$$

Proof. After Lemma 3, it merely remains to prove that $x^2 - dy^2 = N$ and $X^2 - dY^2 = -N$ with $xy^{-1} \equiv XY^{-1} (\bmod N)$, or even $(\bmod 1/2N)$ if $8 | N$, is impossible. For we should obtain if

$$A + Bd^{1/2} = (x + yd^{1/2})(X + Yd^{1/2})^{-1}$$

that $A^2 - dB^2 = -1$ with either A and B both integers, or else both half integers. Both cases are impossible if $N(\varepsilon) = +1$.

LEMMA 5. (1) *If* $N(\varepsilon) = 1$ *then*

$$p(d) \leqq \sum_{0<N<2d^{1/2}} \{f(N; d) + f(-N; d)\} .$$

(2) *If* $N(\varepsilon) = -1$ *then*

$$p(d) \leqq \sum_{0<N<2d^{1/2}} f(N; d) .$$

Proof. If $0 \leqq m < n \leqq p(d) - 1$ then $P_m + Q_m d^{1/2}$ and $P_n + Q_n d^{1/2}$ are primitive solutions in distint classes; they are primitive since $(P_r, Q_r) = 1$ and are in distinct classes since

$$1 < P_m + Q_m d^{1/2} < P_n + Q_n d^{1/2} \leqq \varepsilon_1 .$$

Hence using Lemma 1,

$$\begin{aligned} p(d) &\leqq \text{the number of distinct primitive classes of all} \\ &\qquad \text{equations } x^2 - dy^2 = N \text{ with } -2d^{1/2} < N < 2d^{1/2} \\ &= \sum_{-2d^{1/2}<N<2d^{1/2}} f(N; d), \quad \text{which gives (1)} . \end{aligned}$$

If $N(\varepsilon) = -1$ then the above reasoning applies if $0 \leqq m < n \leqq 2p(d) - 1$ and so (2) follows, since if $N(\varepsilon) = -1$, $f(N; d) = f(-N; d)$.

We remark that this result is best possible for example for the values $d = 7, 13$ respectively.

LEMMA 6. *As* $x \to \infty$

$$(1) \qquad F(x) = \sum_{1 \leqq N \leqq x} 2^{\omega(N)} = cx \log x + O(x) ,$$

$$(2) \qquad A(x) = \sum_{\substack{1<N\leqq x \\ 2|N}} 2^{\omega(N)} = \frac{2}{3} cx \log x + O(x) ,$$

$$(3) \qquad B(x) = \sum_{\substack{1\leqq N\leqq x \\ 2\nmid N}} 2^{\omega(N)} = \frac{1}{3} cx \log x + O(x) ,$$

$$(4) \qquad C(x) = \sum_{\substack{1<N\leqq x \\ 4|N}} 2^{\omega(N)} = \frac{1}{3} cx \log x + O(x) ,$$

$$(5) \qquad D(x) = \sum_{\substack{1<N\leqq x \\ 8|N}} 2^{\omega(N)} = \frac{1}{6} cx \log x + O(x) ,$$

$$(6) \quad E(x) = \sum_{\substack{1<N\leqq x \\ 16|N}} 2^{\omega(N)} = \frac{1}{12} cx \log x + O(x) , \quad \textit{where} \quad c = 6\pi^{-2} .$$

Proof. (1) The identity

$$2^{\omega(N)} = \sum_{k^2|N} d\left(\frac{N}{k^2}\right)\mu(k)$$

is easily proved by induction on the number of distinct prime factors of N. For if N is a prime or a prime power the result is immediate, and then the identity follows on observing that 2^{ω}, d and μ are all multiplicative. Thus

$$\begin{aligned} F(x) &= \sum_{1\leqq N\leqq x} \sum_{k^2|N} d\left(\frac{N}{k^2}\right)\mu(k) \\ &= \sum_{1\leqq k\leqq x^{1/2}} \sum_{1\leqq k_1\leqq xk^{-2}} d(k_1)\mu(k) \\ &= \sum_{1\leqq k\leqq x^{1/2}} \mu(k) \sum_{1\leqq k_1\leqq xk^{-2}} d(k_1) \\ &= \sum_{1\leqq k\leqq x^{1/2}} \mu(k) \left\{\frac{x}{k^2} \log \frac{x}{k^2} + O\left(\frac{x}{k^2}\right)\right\} \\ &= \sum_{1\leqq k\leqq x^{1/2}} \frac{x\mu(k) \log x}{k^2} + O(x) \\ &= \frac{x \log x}{\zeta(2)} + O(x) \\ &= cx \log x + O(x) . \end{aligned}$$

(2) We have

$$\begin{aligned} A(2x) &= \sum_{\substack{1<N\leqq 2x \\ 2|N}} 2^{\omega(N)} \\ &= \sum_{1\leqq 1/2N<x} 2^{\omega(2\cdot 1/2N)} \\ &= \sum_{\substack{1<1/2N\leqq x \\ 2|1/2N}} 2^{\omega(2\cdot 1/2N)} + \sum_{\substack{1\leqq 1/2N\leqq x \\ 2\nmid 1/2N}} 2^{\omega(2\cdot 1/2N)} \end{aligned}$$

$$= \sum_{\substack{1<1/2N\leqq x\\ 2|1/2N}} 2^{\omega(1/2N)} + \sum_{\substack{1\leqq 1/2N\leqq x\\ 2|1\nmid 2N}} 2^{1+\omega(1/2N)}$$
$$= A(x) + 2B(x)\ .$$

Thus $A(2x) + A(x) = 2A(x) + 2B(x) = 2F(x)$. We now prove by induction that

$$A(x) = 2\sum_{r=1}^{\infty}(-1)^{r-1}F(x\cdot 2^{-r})\ .$$

For, if $x = 1$, the result is clearly true since both sides vanish, and then if true for $x \leqq x_0$, we have for $x \leqq 2x_0$,

$$A(x) = 2F\left(\frac{1}{2}x\right) - A\left(\frac{1}{2}x\right)$$

which is again of the required form, and this completes the induction. Now $F(y) = 0$ if $y < 1$ and so we have

$$A(x) = 2\sum_{r=1}^{k}(-1)^{r-1}F(x\cdot 2^{-r})\ ,$$

where

$$k = \left[\frac{\log x}{\log 2}\right]\ .$$

Now by (1)

$$|F(y) - cy\log y| < Cy\ ,$$

for some constant C and all $y > 1$. Thus

$$\left|A(x) - 2c\sum_{r=1}^{k}(-1)^{r-1}\frac{x}{2^r}\cdot\log\frac{x}{2^r}\right| < 2C\sum_{r=1}^{k}\frac{x}{2^r} < 2Cx\ .$$

Hence

$$\left|A(x) - 2c\sum_{r=1}^{k}(-1)^{r-1}\frac{x}{2^r}\log x\right| < 2Cx + 2cx\log 2\cdot\sum_{r=1}^{k} r\cdot 2^{-r}$$
$$< C_1 x\ .$$

Finally,

$$\sum_{r=1}^{k}(-1)^{r-1}\frac{x}{2^r}\log x = \frac{1}{2}x\log x\cdot\frac{1-\left(-\frac{1}{2}\right)^k}{1-\left(-\frac{1}{2}\right)}$$
$$= \frac{1}{3}x\log x\{1 + O(x^{-1})\}$$
$$= \frac{1}{3}x\log x + O(\log x)\ ,$$

and so (2) follows.

(3) now follows since $B(x) = F(x) - A(x)$.

(4) follows since

$$C(x) = \sum_{\substack{1<1/2N\leqq 1/2x \\ 2|1/2N}} 2^{\omega(2\cdot 1/2N)} = A\left(\frac{1}{2}x\right).$$

(5) and (6) now follow similarly since $D(x) = C(1/2x)$ and $E(x) = D(1/2x)$.

Proof of Theorem 1. The idea of the proof is to combine the results of Lemmas 3–6. We have immediately that

$$p(d) \leqq \sum_{1\leqq N\leqq 2d^{1/2}} 2^{\omega(N)} = cd^{1/2}\log d + O(d^{1/2})$$

and the remainder of the proof deals with reducing the constant in the above. There are two ways of doing this; in the first place if $2||N$, then the upper bound $2^{\omega(N)}$ appearing above can immediately be halved in view of Lemmas 3 and 4; secondly depending upon the value of d, there are certain residue classes modulo 16 such that for any N belonging to one of them, the equation $x^2 - dy^2 = N$ cannot have any primitive solutions at all. In each case, it is not possible to dispose of all the odd values of N in this way, and corresponding to these we always obtain a term

$$\sum_{\substack{1\leqq N\leqq 2d^{1/2} \\ 2\nmid N}} 2^{\omega(N)} = B(2d^{1/2}).$$

There are various cases to consider.

(a) $d \equiv 1 \pmod 8$. In this case, since x and y cannot both be even, we find that $x^2 - dy^2 = N$ is either odd or divisible by 8. Thus we find that $p(d) \leqq B(2d^{1/2}) + D(2d^{1/2}) = 1/2cd^{1/2}\log d + O(d^{1/2})$, as required.

(b) $d \equiv 5 \pmod 8$. In this case, we find that if N is even, then $2^2||N$, and accordingly

$$p(d) \leqq B(2d^{1/2}) + C(2d^{1/2}) - D(2d^{1/2}) = \frac{1}{2}cd^{1/2}\log d + O(d^{1/2}).$$

(c) If $d \equiv 2$ or $3 \pmod 4$ then N can be even only if $2||N$ and we obtain

$$\begin{aligned} p(d) &\leqq B(2d^{1/2}) + \sum_{\substack{1<N\leqq 2d^{1/2} \\ 2||N}} 2^{\omega(N)-1} \\ &= B(2d^{1/2}) + \frac{1}{2}\{A(2d^{1/2}) - C(2d^{1/2})\} \\ &= \frac{1}{2}cd^{1/2}\log d + O(d^{1/2}). \end{aligned}$$

It is to be noted for future reference that if $4 \nmid d$, then the $7c/12$ of the theorem can be improved to $1/2c$.

(d) If $d \equiv 0 (\mathrm{mod}\ 4)$, then for a primitive solution of $x^2 - dy^2 = N$ we must have either that x is odd, in which case N is also odd, or else x is even, y odd and $4 | N$. In the latter case we find that $(1/2x)^2 - (1/4d)y^2 = 1/4N$ and so we obtain a primitive solution of the equation $X^2 - (1/4d)Y^2 = 1/4N$, in which moreover y is odd. Thus we have

either $1/4d \equiv 0$ or $1 (\mathrm{mod}\ 4)$ in which case $1/4N$ is odd or divisible by 4,

or $1/4d \equiv 2$ or $3 (\mathrm{mod}\ 4)$ in which case $1/4N$ is odd or $2 \| 1/4N$.

In the first case we obtain

$$\begin{aligned} p(d) &\leqq B(2d^{1/2}) + C(2d^{1/2}) - D(2d^{1/2}) + E(2d^{1/2}) \\ &= \frac{7}{12} c d^{1/2} \log d + O(d^{1/2}) , \end{aligned}$$

and in the second case we obtain similarly

$$\begin{aligned} p(d) &\leqq B(2d^{1/2}) + C(2d^{1/2}) - E(2d^{1/2}) \\ &= \frac{7}{12} c d^{1/2} \log d + O(d^{1/2}) , \end{aligned}$$

which concludes the proof.

LEMMA 7. *As* $x \longrightarrow \infty$,

$$F_1(x) = \sum_{1 \leqq N \leqq x} 2^{\omega(N)} \log \frac{x}{N} = cx \log x + O(x) .$$

Proof. Let $1 < \rho < x$; then

$$\begin{aligned} F_1(x) - F_1(x\rho^{-1}) &= \sum_{1 \leqq N \leqq x} 2^{\omega(N)} \log \frac{x}{N} - \sum_{1 \leqq N \leqq x\rho^{-1}} 2^{\omega(N)} \log \frac{x}{\rho N} \\ &= \sum_{1 \leqq N \leqq x\rho^{-1}} 2^{\omega(N)} \log \rho + \sum_{x\rho^{-1} < N \leqq x} 2^{\omega(N)} \log \frac{x}{N} \end{aligned}$$

and so

$$\log \rho \cdot F(x\rho^{-1}) \leqq F_1(x) - F_1(x\rho^{-1}) \leqq \log \rho \cdot F(x) ,$$

since $x/N < \rho$ for $N > x\rho^{-1}$.

Thus if $1 < \rho^n \leqq x < \rho^{n+1}$, we find that

$$\log \rho \cdot \sum_{r=1}^{n} F(x\rho^{-r}) \leqq F_1(x) - F_1(x\rho^{-n}) \leqq \log \rho \cdot \sum_{r=0}^{n-1} F(x\rho^{-r}) ,$$

and so to complete the proof it suffices to show that

$$\log\rho\cdot\sum_{0}^{n-1} F(x\rho^{-r}) \longrightarrow cx\log x + O(x) \quad \text{as} \quad \rho \longrightarrow 1+ ,$$

where $n = [(\log x/\log\rho)]$.

Now for all $y > 1$, we have for some constant A,

$$cy\log y - Ay < F(y) < cy\log y + Ay .$$

Thus

$$\begin{aligned}\log\rho\sum_{0}^{n-1} F(x\rho^{-r}) &< \log\rho\sum_{0}^{n-1}(cx\log x + Ax)\rho^{-1}\\ &< \rho\frac{\log\rho}{\rho-1}(cx\log x + Ax) \longrightarrow cx\log x + Ax ,\\ &\quad \text{as} \quad \rho \longrightarrow 1+ .\end{aligned}$$

On the other hand

$$\begin{aligned}\log\rho\sum_{0}^{n-1} F(x\rho^{-r}) &> \log\rho\sum_{0}^{n-1}(cx\log x - cxr\log\rho - Ax)\rho^{-r}\\ &= \log\rho\cdot(cx\log x - Ax)\sum_{0}^{n-1}\rho^{-r}\\ &\quad - cx(\log\rho)^2\sum_{0}^{n-1} r\rho^{-r}\\ &= X - Y , \quad \text{say} .\end{aligned}$$

Now

$$X = \frac{\rho(cx\log x - Ax)\log\rho}{\rho-1}\left\{1 - \frac{1}{\rho^n}\right\} \longrightarrow (cx\log x - Ax)(1 - x^{-1})$$

as $\rho \to 1$, since x lies between ρ^n and ρ^{n+1}. Also

$$Y < cx(\log\rho)^2\sum_{0}^{\infty} r\rho^{-r} = \rho^2 cx\left\{\frac{\log\rho}{\rho-1}\right\}^2 \longrightarrow cx \quad \text{as} \quad \rho \longrightarrow 1 +$$

and so the result follows.

LEMMA 8. *Let*

$$A_1(x) = \sum_{\substack{1<N\leqq x\\ 2|N}} 2^{\omega(N)}\log\frac{x}{N}$$

with analagous definitions for B_1, C_1, *and* D_1. *Then the results of Lemma* 6, (2)-(5) *hold also for the functions* A_1 *etc.*

Proof. These results follow from Lemma 7 in exactly the same way as the corresponding results follow from Lemma 6(1).

Proof of Theorem 2. We have for each convergent

$$\left| d^{1/2} - \frac{P_r}{Q_r} \right| < \frac{1}{Q_r Q_{r+1}} ,$$

whence

$$\frac{Q_{r+1}}{Q_r} < \frac{1}{Q_r |P_r - Q_r d^{1/2}|} = \frac{d^{1/2} + \dfrac{P_r}{Q_r}}{|P_r^2 - dQ_r^2|} < \frac{2d^{1/2} + 1}{N_r} ,$$

where

$$|P_r^2 - dQ_r^2| = N_r .$$

Consider first the case $N(\varepsilon) = -1$. Then

$$\begin{aligned} \varepsilon_1 = \varepsilon^2 &= P_{2p(d)-1} + Q_{2p(d)-1} d^{1/2} \\ &< (2d^{1/2} + 1) Q_{2p(d)-1} \\ &= (2d^{1/2} + 1) \prod_0^{2p(d)-2} \frac{Q_{r+1}}{Q_r} \\ &< (2d^{1/2} + 1) \prod_0^{2p(d)-2} \frac{2d^{1/2} + 1}{N_r} \\ &= \prod_0^{2p(d)-1} \frac{2d^{1/2} + 1}{N_r} . \end{aligned}$$

Thus

$$\begin{aligned} 2 \log \varepsilon &< \sum_0^{2p(d)-1} \log \frac{2d^{1/2} + 1}{N_r} \\ &\leqq \sum_{0<N<2d^{1/2}} \{f(N; d) + f(-N; d)\} \log \frac{2d^{1/2} + 1}{N} \\ &= \sum_{0<N<2d^{1/2}} \{f(N; d) + f(-N; d)\} \log \frac{2d^{1/2}}{N} + O\{d^{-1/2} F(2d^{1/2})\} \\ &= 2 \sum_{0<N<2d^{1/2}} f(N; d) \log \frac{2d^{1/2}}{N} + O(\log d) , \end{aligned}$$

since in this case $f(N; d) = f(-N; d)$.
Thus

$$\begin{aligned} \log \varepsilon &< \sum_{0<N<2d^{1/2}} f(N; d) \log \frac{2d^{1/2}}{N} + O(\log d) \\ &< \frac{1}{2} c d^{1/2} \log d + O(d^{1/2}) , \end{aligned}$$

as before, using Lemmas 7 and 8 in place of Lemma 6, since in this case $4 \nmid d$. In the case $N(\varepsilon) = +1$, we have

$$\begin{aligned}\varepsilon &= P_{p(d)-1} + Q_{p(d)-1}d^{1/2} \\ &< (2d^{1/2}+1)Q_{p(d)-1} \\ &< \prod_{0}^{p(d)-1} \frac{2d^{1/2}+1}{N_r}\,,\end{aligned}$$

as before.

Thus

$$\begin{aligned}\log\varepsilon &< \sum_{0}^{p(d)-1} \log\frac{2d^{1/2}+1}{N_r} \\ &\leqq \sum_{0<N<2d^{1/2}} \{f(N;\,d)+f(-N;\,d)\}\log\frac{2d^{1/2}+1}{N} \\ &= \sum_{0<N<2d^{1/2}} \{f(N;\,d)+f(-N;\,d)\}\log\frac{2d^{1/2}}{N} + O(\log d) \\ &\leqq \frac{1}{2}cd^{1/2}\log d + O(d^{1/2})\,,\end{aligned}$$

as before, provided $4 \nmid d$.
Finally if $4|d$ we observe that $\varepsilon = \eta$ or η^2 where η is the fundamental unit of the ring $Z[((1/4)d)^{1/2}]$. Then the result for this case follows by descent since now $\log\varepsilon \leqq 2\log\eta$.

This concludes the proof of Theorem 2.

Proof of Theorem 3. We have as before

$$\log\varepsilon < \sum_{r=0}^{p(d)-1} \log\frac{2d^{1/2}+1}{N_r}$$

and so for any K satisfying $1 < K < 2d^{1/2}$

$$\begin{aligned}\log\varepsilon &< \sum_{r=0}^{p(d)-1} \log\frac{2d^{1/2}}{N_r} + O(\log d) \\ &= \sum_{\substack{N_r\leqq K\\ 0\leqq r<p(d)}} \log\frac{2d^{1/2}}{N_r} + \sum_{\substack{N_r>K\\ 0\leqq r<p(d)}} \log\frac{2d^{1/2}}{N_r} + O(\log d) \\ &< \sum_{1\leqq N\leqq K} \{f(N;\,d)+f(-N;\,d)\}\log 2d^{1/2} \\ &\quad + p(d)\log\frac{2d^{1/2}}{K} + O(\log d) \\ &< A\log d\cdot K\log K + \frac{1}{2}p(d)\log(4dK^{-2}) + O(K\log d)\,.\end{aligned}$$

In particular taking $K = 2d^{1/2}(\log d)^{-3}$ we obtain

$$\log\varepsilon < 3p(d)\log\log d + o(d^{1/2})\,.$$

Now for $d = 2^{2k+1}$ we have $\varepsilon = (1+\sqrt{2})^{2k}$, i.e., $\log\varepsilon > Ad^{1/2}$ where $A > 0$ and so $p(d) \neq o(d^{1/2}/\log\log d)$, as required.

Proof of Theorem 4. If $\log \varepsilon \neq o(d^{1/2} \log d)$, then there exists a positive constant $c_1 < c$ so that for infinitely many values of d, $\log \varepsilon > c_1 d^{1/2} \log d$. Let $g(N; d)$ denote the number of distinct primitive classes of solutions of $x^2 - dy^2 = N$ for which x/y occurs as a convergent to the continued fraction for $d^{1/2}$. Then

$$2p(d) \geqq \sum_{-2d^{1/2}<N<2d^{1/2}} g(N; d)$$

and

$$\log \varepsilon < \sum_{-2d^{1/2}<N<2d^{1/2}} g(N; d) \log \frac{2d^{1/2}}{|N|} + O(\log d) .$$

Thus if $k \geqq 1$,

$$\begin{aligned}\log \varepsilon - 2p(d) \log k &< \sum_{-2d^{1/2}<N<2d^{1/2}} g(N; d) \log \frac{2d^{1/2}}{k|N|} + O(\log d) \\ &\leqq \sum_{0<|N|<2d^{1/2}k^{-1}} g(N; d) \log \frac{2d^{1/2}}{k|N|} + O(\log d) \\ &\leqq \sum_{0<N<2d^{1/2}k^{-1}} 2^{\omega(N)} \log \frac{2d^{1/2}k^{-1}}{N} + O(\log d)\end{aligned}$$

since $g(N; d) \leqq f(N; d)$. Thus

$$\begin{aligned}\log \varepsilon - 2p(d) \log k &< F_1(2d^{1/2}k^{-1}) + O(\log d) \\ &< cd^{1/2}k^{-1} \log d + O(d^{1/2}) .\end{aligned}$$

Thus if $k > c/c_1$, we have for infinitely many values of d,

$$p(d) > \frac{kc_1 - c}{2k \log k} d^{1/2} \log d + O(d^{1/2}) ,$$

as required.

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