PACIFIC JOURNAL OF MATHEMATICS
Vol. 74, No. 2, 1978

THE K-THEORY OF AN EQUICHARACTERISTIC DISCRETE VALUATION RING INJECTS INTO THE K-THEORY OF ITS FIELD OF QUOTIENTS

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Let A be an equicharacteristic discrete valuation ring with residue class field F and field of quotients K. The purpose of this note to prove that the transfer map $K_n(F) \to K_n(A)$ is zero ($n \geqq 0$).

By virtue of Quillen's localization sequence for A, this is equivalent to the statement that the map $K_n(A) \to K_n(K)$ is injective. This result has been conjectured by Gersten and proved by him in the case in which F is a finite separable extension of a field contained in A. We establish the general result by using a limit technique to reduce to this special case.

LEMMA. *Let A be a discrete valuation ring with maximal ideal m and residue class field $A/m = F$. Suppose that A contains a field L; suppose further that F' is a finite separable extension of L satisfying $L \subset F' \subset F$. Then there exists a subring A' of A such that:*

(a) *A' is a discrete valuation ring containing L;*

(b) *$A' \subset A$ is local and flat;*

(c) *if we denote by m' the maximal ideal of A, then $m = m'A$;*

(d) *the image of A' in F is F'; (since $m \cap A = m'$, this implies that we may identify the residue class field of A' with F').*

Proof. Let m be generated by the parameter π. Consider first the case in which A contains a field mapping isomorphically onto F'; let us denote this field also by F'. π is easily seen to be algebraically independent of F', so the subring $F'[\pi]$ of A is isomorphic to a polynomial ring in one variable over F', and π generates a maximal ideal m'. Then $A' = F'[\pi]_{m'}$ is a discrete valuation ring. Furthermore, elements of the complement of m' in $F'[\pi]$ are units in A, so $A' \subset A$. A is flat over A' since A' is Dedekind and A is torsion-free as an A'-module; the other conditions are clear.

Now suppose that A does not contain a field mapping isomorphically onto F'. F' is a simple extension of L, say $F' = L(\bar{\alpha})$; let $f \in L[X]$ be the minimal polynomial of $\bar{\alpha}$. Lift $\bar{\alpha}$ to $\alpha \in A$. If we denote by v the valuation on K, then $v(f(\alpha)) > 0$ since $f(\bar{\alpha}) = 0$

implies $f(\alpha) \in m$. If $v(f(\alpha)) > 1$, consider $\alpha+\pi$. We have $f(\alpha+\pi) \equiv f(\alpha) + \pi f'(\alpha) \equiv \pi f'(\alpha) \pmod{\pi^2}$. But $f'(\alpha)$ is a unit, for otherwise $f'(\bar{\alpha}) = 0$, contradicting separability. Thus $v(f(\alpha + \pi)) = 1$. By replacing α by $\alpha + \pi$, we may therefore assume without loss of generality that $v(f(\alpha)) = 1$.

Next we claim that α is transcendental over L. For, if not, let $g \in L[X]$ be the minimal polynomial of α. Then $g(\bar{\alpha}) = 0$ implies $f \mid g$, which forces $f = g$. But then $L[\alpha]$ is a field mapping isomorphically onto F', contradicting the assumption. Therefore $L[\alpha]$ is isomorphic to a polynomial ring, and $f(\alpha)$ generates a maximal ideal m'. If $h \in L[X]$ is such that $h(\alpha)$ is a nonunit in A, then $h(\bar{\alpha}) = 0$, which implies $f \mid h$; thus $h(\alpha) \in m'$, and it follows that the discrete valuation ring $A' = L[\alpha]_{m'}$ is a subring of A. $A' \subset A$ is local and flat, and A' projects onto F'. Since $v(f(\alpha)) = 1$, it follows also that $m'A = m$.

For any ring R, let $P(R)$ denote the category of finitely generated projective R-modules, and let $\mathrm{Mod}\, fg(R)$ denote the category of finitely generated R-modules. Then if R is a discrete valuation ring with residue class field F, restriction of scalars defines an exact functor $P(F) \to \mathrm{Mod}\, fg(R)$, which induces a map of K-groups $K_n(F) \to K_n\,(\mathrm{Mod}\, fg(R))$. Since R is a regular ring, the inclusion $P(R) \to \mathrm{Mod}\, fg(R)$ induces an isomorphism $K_n(R) \to K_n\,(\mathrm{Mod}\, fg(R))$ [2]. Quillen defines the transfer homomorphism tr: $K_n(F) \to K_n(R)$ to be the composition $K_n(F) \to K_n\;(\mathrm{Mod}\, fg(R)) \xrightarrow{\cong} K_n(R)$.

THEOREM. *Let A be an equicharacteristic discrete valuation ring with residue class field F. Then the transfer map* tr: $K_n(F) \to K_n(A)$ *is zero* ($n \geqq 0$).

Proof. Let us denote the maximal ideal of A by m. Let F_0 denote the prime field. Then we can write $F = \varinjlim F_i$, where F_i ranges over the subfields of F finitely generated over F_0. Since Quillen's K-groups commute with filtered inductive limits [2], we have $K_n(F) = \varinjlim K_n(F_i)$, and it suffices to prove that the composition $K_n(F_i) \to K_n(F) \to K_n(A)$ is zero for all i.

Since F_0 is perfect, F_i is separably generated over F_0; i.e., there exist elements $\bar{x}_1, \cdots, \bar{x}_t$ of F_i such that $L_i = F_0(\bar{x}_1, \cdots, \bar{x}_t)$ is purely transcendental over F_0, and F_i is finite separable over L_i. Lift $\{\bar{x}_1, \cdots, \bar{x}_t\}$ to $\{x_1, \cdots, x_t\}$ in A and consider the subring $F_0[x_1, \cdots, x_t]$ of A. $\{x_1, \cdots, x_t\}$ are clearly algebraically independent over F_0. Furthermore, all nonzero elements of this subring are units in A, so A contains the field of quotients of this subring. In other words,

A contains a field mapping isomorphically onto L_i. Then by the lemma we can find a discrete valuation ring $A_i \subset A$, with maximal ideal m_i, such that $L_i \subset A_i$, $A_i \subset A$ is local and flat, $m = m_i A$, and the diagram

$$\begin{array}{ccc} A & \longrightarrow & F \\ \cup & & \cup \\ A_i & \longrightarrow & F_i \end{array}$$

commutes.

Now consider the diagram of exact functors

$$\begin{array}{ccccc} P(F) & \longrightarrow & \operatorname{Mod} fg(A) & \longleftarrow & P(A) \\ \uparrow & & \uparrow & & \uparrow \\ P(F_i) & \longrightarrow & \operatorname{Mod} fg(A_i) & \longleftarrow & P(A_i) \end{array}$$

where the vertical arrows are induced by extension of scalars; the middle functor is exact bebause $A_i \subset A$ is flat.

The right-hand square clearly commutes. On the other hand, if V is a vector space over F_i, then the clockwise path of the left-hand square gives $V \to F \otimes_{F_i} V$, considered as an A-module. The other path gives $V \to A \otimes_{A_i} V \cong A \otimes_{A_i} (A_i/m_i) \otimes_{(A_i/m_i)} V \cong (A/m_i A) \otimes_{(A_i/m_i)} V = (A/m) \otimes_{(A_i/m_i)} V \cong (A/m) \otimes_{F_i} V = F \otimes_{F_i} V$, using the fact that $m_i A = m$. Thus the two paths agree up to natural isomorphism, and we have a commutative diagram of K-groups

$$\begin{array}{ccc} K_n(F) & \xrightarrow{\text{tr}} & K_n(A) \\ \uparrow & & \uparrow \\ K_n(F_i) & \xrightarrow{\text{tr}} & K_n(A_i) \end{array}$$

But the bottom map is zero by the result of Gersten alluded to above [1], so we have $K_n(F_i) \to K_n(F) \xrightarrow{\text{tr}} K_n(A)$ is zero, as required.

REFERENCES

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Received April 21, 1977.

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